Special properties of the first absolute Fermat pseudoprime, the number 561

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Abstract. Though is the first Carmichael number, the number 561 doesn't have the same fame as the third absolute Fermat pseudoprime, the Hardy-Ramanujan number, 1729. I try here to repair this injustice showing few special properties of the number 561.

I will just list (not in the order that I value them, because there is not such an order, I value them all equally as a result of my more or less inspired work, though they may or not "open a path") the interesting properties that I found regarding the number 561, in relation with other Carmichael numbers, other Fermat pseudoprimes to base 2, with primes or other integers.

1. The number 2*(3 + 1)*(11 + 1)*(17 + 1) + 1, where 3, 11 and 17 are the prime factors of the number 561, is equal to 1729. On the other side, the number 2*lcm((7 + 1),(13 + 1),(19 + 1)) + 1, where 7, 13 and 19 are the prime factors of the number 1729, is equal to 561. We have so a function on the prime factors of 561 from which we obtain 1729 and a function on the prime factors of 1729 from which we obtain 561.

Note: *The formula N = 2*(d_1 + 1)*...*(d_n + 1) + 1, where d_1, d_2, ...,d_n are the prime divisors of a Carmichael number, leads to interesting results (see the sequence A216646 in OEIS); the formula M = 2*lcm((d_1 + 1),...,(d_n + 1)) + 1 also leads to interesting results (see the sequence A216404 in OEIS). But we didn't obtained anymore through one of these two formulas a Carmichael number from another, so this bivalent realtion might only exist between the numbers 561 and 1729.*

2. The number 561 can be expressed as C = a*b + b – a, where b is prime and a can be any prime factor of the number 1729: 561 = 7*71 + 71 – 7 = 13*41 + 41 – 13 = 19*29 + 29 = 19 (even more than that, for those that consider that 1 is a prime number, so a prime factor of 1729, 561 = 1*281 + 281 - 1).

Note: *The formula (a + 1)*(b + 1)*(b – a + 1) + 1 seems to lead to interesting results: for instance, (19 + 1)*(29 + 1)*(29 – 19 + 1) = 6601, also a Carmichael number and for the pairs [a, b] = [7, 71] and [a, b] = [13, 41] we obtain through this formula primes, which make us think that this formula deserves further study. Also the triplets [a, b, a*b + b – a], where a, b and a*b + b – a are all three primes might lead to interesting results.*

Note: *I can't, unfortunately, to state that 561 is the first integer that can be written in three (or even four, if we consider that 1 is prime) distinct ways as a*b + b – a, where a and b are primes, because there is a smaller number that has this property: 505 = 3*127 + 124 = 11*43 + 32 = 13*37 + 24 = 17*29 + 12. I yet assert that Carmichael numbers (probably the Fermat pseudoprimes to base 2 also) and the squares of primes can be written in many ways as such.*

3. Another interesting formula inspired by the number 561: we have the expression (2*3 + 3)*(2*11 + 3)*(2*17 + 3) – 4, where 3, 11 and 17 are the prime factors of 561, egual to 8321, a Fermat pseudoprime to base 2.

Note: *If we apply this formula to the prime factors of another Carmichael number, 2821 = 7*13*31, we obtain 32041 = 179^2, an interesting result.*

4. We consider the triplets of primes of the form [p, p + 560, p + 1728]. The first triplet of such primes, [59, 619, 1787], we notice that has the following property: 59 + 619 + 1787 = 2465, a Carmichael number.

Note: *For the next two such triplets, [83, 643, 1811] and [149, 709, 1877] we didn't obtain convincing results.*

5. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form 3*(4*n – 1)*(6*n – 2), where n is integer different from 0.

Note: *See the sequence A210993 in OEIS.*

6. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form 3*n*(9*n + 2)*(18*n – 1), where n is an odd number.

Note: *See the sequence A213071 in OEIS.*

7. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form 8*p*n + p^2, where p is prime and n is integer (for n = 0 we include in this sequence the squares of the only two Wieferich primes known).

Note: *See the sequence A218483 in OEIS.*

8. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form 5*p^2 – 4*p, where p is prime.

Note: *See the sequence A213812 in OEIS.*

9. The numbers obtained through the method of concatenation from reversible primes and the number 561 are often primes.

Note: *We obtain 11 primes from the first 20 reversible primes concatenated with the number 561; these primes are: 37561, 73561, 79561, 97561, 149561, 157561, 311561, 337561, 347561, 359561, 389561.*

10. The numbers obtained through the method of concatenation from palindromic primes and the number 561 are often primes.

Note: *We obtain 9 primes from the first 20 palindromic primes concatenated with the number 561; these primes are: 101561, 131561, 151561, 191561, 313561, 373561, 727561, 797561, 929561.*

11. The numbers obtained through the method of concatenation from the powers of 2 and the number 561 are often primes or products of few primes.

Note: *The numbers 4561, 16561, 32561, 256561 are primes.*

12. Yet another relation between the numbers 561 and 1729: the numbers obtained through the method of concatenation from the prime factors of 1729 raised to the third power and the number 561 are primes.

Note: *These are the numbers: 343561 (where 7^3 = 343); 2197561 (where 13^3 = 2197) and 6859561 (where 19^3 = 6859).*

13. The number (561*n – 1)/(n – 1), where n is integer different from 1, is often integer; more than that, is often prime.

Note: *We obtained the following primes (in the brackets is the corresponding value of n): 701(5), 673(6), 641(8),* *631(9), 617(11), 601(14), 421(-3), 449(-4), 491(-7), 521(-13) etc. I assert that for a Carmichael number C the number (C*n – 1)/(n – 1), where n is integer different from 1, is often an integer (comparing to other integers beside C). In fact the primes appear so often that I will risk a conjecture.*

Conjecture: *Any prime number p can be written as p = (C*q – 1)/(q – 1), where C is a Carmichael number and q is a prime.*

14. The number 561 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form (n*109^2 – n)/360, where n is integer (561 is obtained for n = 17).

Note: *Another term of this sequence, obtained for n = 19897, is the Carmichael number 656601.*

Note: *The number 1729 is the first term in the sequence of Fermat pseudoprimes to base 2 of the form (n*181^2 – n)/360, where n is integer (1729 is obtained for n = 19). The next terms of the sequence, obtained for n = 31, is the Carmichael number 2821.*

Note: *Because the numbers 561 and 1729 have both three prime factors, the sequences from above can be eventually translated into the property of the numbers of the form 360*(a*b) + 1, where a and b are primes, to generate squares of primes. Corresponding to the sequences above, for [a, b] = [3, 11] we obtain 109^2 and for [a, b] = [7, 13] we obtain 181^2.*

Conjecture: *If the number 360*(a*b) + 1, where a and b are primes, is equal to c^2, where c is prime, then exists an infinite series of Carmichael numbers of the form a*b*d, where d is a natural number (obviously odd, but not necessarily prime).*

Note: *The numbers of the form 360*(a*b) + c, where a, b and c are primes, seems to have also the property to generate primes. Indeed, if we take for instance [a, c] = [3, 7], we obtain primes for b = 5, 11, 17, 23, 29, 31, 43, 47, 59, 67 etc. (note the chain of 5 consecutive primes of the form 6*k – 1).*

15. The number 561 is the first term of the sequence of Carmichael numbers that can be written as 2^m + n^2, where m and n are integers (561 is obtained for m = 5 and n = 23).

Note: *The next few terms of this sequence are: 1105 = sqrt(2^4 + 33^2), 2465 = sqrt(2^6 + 49^2) etc.*

16. Some Carmichael numbers are also Harshad numbers but the most of them aren't. The number 561 has yet another interesting related property; if we note with s(n) the iterated sum of the digits of a number n that not goes until the digital root but stops to the last odd prime obtained before this, than 561 is divisible by s((561 + 1)/2) equivalent to s(281) equivalent to 11. Also other Carmichael numbers have this property: 1105 is divisible by s(1105) = 13 and 6601 is divisible by s(6601) = 7.

17. For the randomly chosen, but consecutive, 7 primes (129689, 1299709, 1299721, 1299743, 1299763, 1299791 and 1299811) we obtained 3 primes and 3 semiprimes when introduced them in the formula 2*561 + p^2 – 360.

18. Another relation between 561 and Hardy Ramanujan number: (62745 + 24) mod 1728 = 561 (where 24 is, *e.g.*, the sum of the digits of the Carmichael number 62745 or a constant and 1728 is, obviously, one less than Hardy-Ramanujan number).

19. Yet another relation between 561 and Hardy Ramanujan number: 561 mod 73 = 1729 mod 73 = 50. The formula 73*n + 50, from which we obtain 561 and 1729 for n = 7 and n = 23, leads to other interesting results for n of the form 7 + 16*k: we obtain primes for n = 39, 71, 119, 167 etc.

20. A formula that generating primes: 561^2 – 561 – 1 = 314159 is prime; 561^4 – 561^3 – 561^2 – 561 – 1 = 98872434077 is prime. Also for other Carmichael number the formula C^2 – C – 1 conducts to: 1105^2 – 1105 – 1 = 1219919 prime, 6601^2 – 6601 – 1 = 43566599 prime (semiprimes were obtained for the numbers 1729, 2465, 2821 and so on). Yet the number 2465^4 – 2465^3 – 1 = 36905532355999 is prime and the number 15841^4 – 15841^3 – 1 = 62965543898204639 is prime.

21. The formula N = d_1^2 + d_2^2 + d_3^2 – 560, where d_1, d_2 and d_3 are the only prime factors of a Carmichael number, and they are all three of the form 6*k + 1, seems to generate an interesting class of primes:
: for C = 1729 = 7*13*19 we have N = 19 prime;
: for C = 2821 = 7*13*31 we have N = 619 prime;
: for C = 8911 = 7*19*67 we have N = 4339 prime;
: for C = 15841 = 7*31*73 we have N = 5779 prime.

22. The number 544, obtained as the difference between the first two Carmichael numbers, 1105 and 561, has also a notable property: the relation n^C mod 544 = n seems to be verified for a lot of natural numbers n and a lot of Carmichael numbers C, especially when C is also an Euler pseudoprime.

Conjecture: *The expression n^E mod 544 = n, where n is any natural number, is true if E is an Euler pseudoprime.*

23. The difference between the squares of the first two Carmichael numbers, 1105 and 561, has also the notable property that results in a square of an integer: 952^2 = 1105^2 – 561^2.

Conclusion: I am aware of the excessive use of the word "interesting" in this article, but this was the purpose of it: to show how many "interesting" paths can be opened just studying the number 561, not to follow until the last consequences one of these paths. I didn't succed to show that the properties of the number 561 eclipses the ones of the number 1729 (very present in this article) but hopefully I succeded to show that they are both a pair of extraordinary numbers (and that the number 561 deserves his place on the license plate of a taxi-cab).