On Dialectics of Physical Systems, Schrodinger Equation and Collapse of the Wave Function: Critical Behaviour for Quantum Computing

Armando V.D.B. Assis
Departamento de Física,
Universidade Federal de Santa Catarina,
88040-900, Florianópolis, SC, Brazil

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This paper is intended to show the Schrodinger equation, within its structure, allows the manifestation of the wave function collapse within a very natural way of reasoning. In fact, as we will see, nothing new must be inserted to the classical quantum mechanics, viz., only the dialectics of the physical world must be interpreted under a correct manner. We know the nature of a physical system turns out to be quantical or classical, and, once under the validity of the Schrodinger equation to provide the evolution of this physical system, the dialectics, quantum or classical, mutually exclusive, must also be under context through the Schrodinger equation, issues within the main scope of this paper. We will show a classical measure, the obtention of a classical result, emerges from the structure of the Schrodinger equation, once one demands the possibility that, over a chronological domain, the system begins to provide a classical dialectic, showing the collapse may be understood from both: the structure of the Schrodinger equation as well as from the general solution to this equation. The general solution, even with a dialectical change of description, leads to the conservation of probability, obeying the Schrodinger equation. These issues will turn out to be a consequence of a general potential energy operator, obtained in this paper, including the possibility of the classical description of the physical system, including the possibility of interpretation of the collapse of the quantum mechanical state vector within the Schrodinger equation scope.

Keywords: Quantum mechanics, Schrodinger equation, collapse of the wave function

INITIAL POSITION OF THE PROBLEM

For the moment, we will be interested in an one particle physical system under a non-relativistic scope. We are interested in the general potential energy operator for this physical system, $V(x,t)$, such that the Schrodinger equation in $(1+1)$-dimensions, in two classical spacetime dimensions:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = E\Psi(x,t) = \left[V(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\Psi(x,t), \tag{1}$$

becomes complete for a quantical or a classical characterization of the system represented by its wave function $\Psi(x,t)$, where m is the [constant] mass of our physical system, $\hbar \equiv h/(2\pi)$, where h is the Planck constant, E is the total energy of our physical system, x a spatial location for our physical system at an instant t.

Considering the system such that a physical potential energy $V_p(x,t)$ that characterizes the interaction for the physical problem under consideration is known, in this case, one canonically writes:

$$V(x,t) \equiv V_p(x,t), \tag{2}$$

within the Schrodinger equation [within the Eq. (1)]. E.g., one canonically writes, if the physical problem is the one of a simply harmonic oscillator:

$$V(x,t) \equiv V_p(x,t) = \frac{1}{2}m\omega^2 x^2, \tag{3}$$

with nothing else related to this physical interaction, with nothing else related to a possible change of characterization of the physical system, e.g., when an observer turns out to describe the system under its own [classical] dialetic. For this, the axiomatic structure of the quantum mechanics counts with the Born rule.

There is a fundamental difference between a system that evolves under the Schrodinger equation with its quantical [say a q-dialectical] characterization *ad infinitum*, viz., such that:

$$V(x,t) \equiv V_p(x,t)\ \forall\, t \in (-\infty, +\infty), \tag{4}$$

and a system that does not have such q-dialectical characterization *ad infinitum*, viz., such that:

$$V(x,t) \equiv V_p(x,t)\ \forall\, t \in (-\infty, \tau), \tag{5}$$

i.e., having got a classical [say a c-dialectical] characterization from $t = \tau$:

$$V(x,t) = V_c(x,t)\ \forall\, t \in [\tau, +\infty), \tag{6}$$

where $V_c(x,t)$ is unknown at our stage of reasoning. $V_c(x,t)$ seems to require a quality to characterize the collapse of the wave function, once one has, from a collapsed state beginning at τ, a classical c-dialectic of characterization for the system.

These dialectical characterizations turn out to be mutually exclusive, once the chronological domains for these characterizations are mutually exclusive, viz.:

$$\text{for } \mathcal{I}_q \equiv (-\infty, \tau), \tag{7}$$

and:

$$\text{for } \mathcal{I}_c \equiv [\tau, +\infty), \tag{8}$$

one has got:

$$\mathcal{I}_q \cap \mathcal{I}_c = (-\infty, \tau) \cap [\tau, +\infty) = \emptyset. \tag{9}$$

In relation to the chronological domain, once one is interested in the solution for the Eq. (1):

$$\forall t \in \mathbb{R}, \tag{10}$$

i.e.:

$$\forall t \in \mathcal{I}_q \cup \mathcal{I}_c, \tag{11}$$

one may write for the potential energy operator:

$$V(x,t) \equiv (1 - \delta_{t\bar{t}}) V_p(x,t) + \delta_{t\bar{t}} V_c(x,t), \tag{12}$$

where:

$$\delta_{t\bar{t}} = 0 \ \forall t \in \mathcal{I}_q; \tag{13}$$
$$\delta_{t\bar{t}} = 1 \ \forall t \in \mathcal{I}_c, \tag{14}$$

where the intervals $\mathcal{I}_q$ and $\mathcal{I}_c$ were defined, respectively, by the Eqs. (7) and (8).

The potential energy operator $V_q(x,t)$ is the one, as discussed above, one considers for the physical problem under consideration, e.g., the one given by the Eq. (3) for the simply harmonic oscillator, over the chronological domain $\mathcal{I}_q \equiv (-\infty, \tau)$. Here is instructive to assert one may consider a q-dialectic *ad infinitum*, i.e., $\mathcal{I}_q = \mathbb{R}$: taking $\tau \to +\infty$, avoiding a c-dialectic, viz., with $\mathcal{I}_c = \emptyset$. In this latter case [provided $\tau \to +\infty$], the Eqs. (12), (13) and (14) lead to the Eq. (4), and the c-dialectic turns out to be void via the Schrodinger equation, being accomplished *ad hoc*, axiomatically, via Born rule. Otherwise, to have a c-dialectic *ad infinitum*, under the considerations we are developing here, i.e., to prescribe $\mathcal{I}_c = \mathbb{R}$, it is sufficient to consider: $\tau \to -\infty$, avoiding a q-dialectic, viz., with $\mathcal{I}_q = \emptyset$, for which the Eqs. (12), (13) and (14) lead to:

$$V(x,t) \equiv V_c(x,t) \ \forall\, t \in (-\infty, +\infty), \tag{15}$$

and the q-dialectic turns out to be void via the Schrodinger equation, i.e., cannot be achieved via the axiomatic structure of quantum mechanics in the sense the quantum mechanical q-dialectic for the system emerging from the Schrodinger equation [via its solution] would be banned, so that the solution would be providing a c-dialectical solution *ad infinitum*; in other words: the system would be collapsed from the very beginning.

In virtue of these considerations, the position of the problem is the complete determination of the Eq. (12), for which one needs to determine $V_c(x,t)$. This done, we must turn back to the Eq. (1), to obtain its solution under the action of the complete potential energy operator given by the Eq. (12). These issues are to be considered in the next sections.

POSITION OF THE PROBLEM: $V_c(x,t)$?

The potential energy operator to generate our c-dialectic must be given by:

$$V_c(x,t) \equiv [E - K](x,t), \tag{16}$$

as demanded by the Physics, where E is the constant mechanical energy of the system, being K its kinetic energy [operator]. Once the very nature of a system, essentially, even for a c-dialectically described, is quantum mechanical: albeit under a c-dialectical [observer's] perception, this perception, within the reasonings we are developing, is to arise as solution of the Eq. (1), provided the Eq. (12), from which one turns out to simply write:

$$V_c(x,t) \equiv E + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}. \tag{17}$$

The Eq. (17) seems to have been obtained through an obvious reasoning, but there is subtle point that must be pointed out. In fact, the dynamics of the system, over the $\mathcal{I}_c$ domain, is to be understood as:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = E\Psi(x,t) = \left[V_c(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\Psi(x,t), \tag{18}$$

in virtue of the Eq. (1). The essential characteristic of a classical [Newtonian] description [of a c-dialectical one] of a system is the condition, theoretical and instrumentally verified:

- *Classical conservative systems have got a constant [scalar] energy E.*

Hence, a quantum mechanical operational cause, via the Hamiltonian operator:

$$\overbrace{\left[i\hbar\frac{\partial}{\partial t}\right]}^{\text{q-dialectics}}\Psi(x,t) \equiv \overbrace{\left[V_c - \frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\right](x,t)}^{\text{q-dialectics}}\Psi(x,t), \tag{19}$$

acting on the system $\Psi(x,t)$, leads, under a chronological evolution starting at an instant τ, to a classical effect, viz.:

$$\overbrace{[\cdots]}^{\text{q-dialectics, Eq. (19), the cause}} = \underbrace{E}_{\text{c-dialectics, the effect}} \Psi(x,t). \tag{20}$$

The imposition of a necessary constant, a constant scalar, not an operator, for E, in virtue of the preceeding reasonings through the march that led to the Eq. (20), characterizes E in the Eq. (17) as a classical quantity in the sense of its intrinsically classical [c-dialectic] dialectic. The Eq. (17) follows from the Eqs. (19) and (20). One should infer, from the Eq. (19), that the operator:

$$K \equiv -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}, \tag{21}$$

appearing in the Eq. (16), is essentially q-dialectical. Hence, since E is, in the context of $V_c(x,t)$ [Eq. (20)], c-dialectical:

- *The potential energy operator $V_c(x,t)$ given by the Eq. (17) has not a purely quantum-mechanical dialectic.*

We will see this requisite will lead to a necessary reduction of the state vector, will consequently destroy the q-dialectical characteristic of superposition for the system $\Psi(x,t)$ to a localized $\Psi(x,t)$ over the chronological domain $\mathcal{I}_c$.

To a better understanding of the essential difference in defining E as essentially constant, not an operator, consider, e.g., for purposes of brevity and clarity, a case in which the potential energy does not depend on t. For the general solution of the Eq. (1), with $V(x,t) \equiv V_p(x)$, with the initial and boundary conditions for the physical problem properly condidered, one reaches:

$$\Psi(x,t) = \sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) , \tag{22}$$

with: E_k being the k-eigenvalue of $i\hbar\partial/\partial t$ operating on its respective k-eigenvector $\Psi_k(x,t)$, $\phi_k(x)$ being the [orthonormalized] k-eigenvector of the time independent Schrodinger equation [we will be back to these issues later; for now, the Eq. (1) and the Eq. (22) are sufficient for the argument we are raising here], a_k being the k-coefficient:

$$\begin{aligned} a_k &= \int_{-\infty}^{\infty} \phi_k^*(x) \Psi(x,0)\, dx \qquad (23) \\ &= \int_{-\infty}^{\infty} \phi_k^*(x) \left[\sum_{\forall p} a_p \phi_p(x) \right] dx \\ &= \sum_{\forall p} a_p \int_{-\infty}^{\infty} \phi_k^*(x) \phi_p(x)\, dx \\ &= \sum_{\forall p} a_p \delta_k^p \\ &= a_k, \end{aligned} \tag{24}$$

where the initial condition reads:

$$\Psi(x,0) = \sum_{\forall k} a_k \phi_k(x) . \tag{25}$$

The Eq. (22) may be written:

$$\Psi(x,t) = \sum_{\forall k} a_k \Psi_k(x,t) , \tag{26}$$

i.e., as a superposition of [orthonormalized] eigenstates:

$$\Psi_k(x,t) = e^{-iE_k t/\hbar} \phi_k(x) , \tag{27}$$

eigenstates of the operator $i\hbar\partial/\partial t$. The eigenvalue problem:

$$i\hbar \frac{\partial}{\partial t} \Psi_k(x,t) = E_k \Psi_k(x,t) , \tag{28}$$

is fully obeyed, for E_k constant, as one may verify from the Eq. (27):

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi_k(x,t) &= i\hbar \frac{\partial}{\partial t} \left[e^{-iE_k t/\hbar} \phi_k(x) \right] \\ &= i\hbar \phi_k(x) \frac{\partial}{\partial t} \left[e^{-iE_k t/\hbar} \right] \\ &= i\hbar \phi_k(x) e^{-iE_k t/\hbar} \left(-i \frac{E_k}{\hbar} \right) \\ &= E_k e^{-iE_k t/\hbar} \phi_k(x) \\ &= E_k \Psi_k(x,t) . \end{aligned} \tag{29}$$

But, once under superposition, a necessary property of q-dialectic, the eigenvalue problem:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = E \Psi(x,t) , \tag{30}$$

with E constant, not an operator, does not hold, once a pure quantum mechanical state written as a superposition of the energy operator eigenstates, as in the Eq. (22), has not got a defined energy. In fact, from the Eqs. (22) and (30):

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi(x,t) &= i\hbar \frac{\partial}{\partial t} \left[\sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) \right] \\ &= i\hbar \sum_{\forall k} a_k \phi_k(x) \frac{\partial}{\partial t} \left(e^{-iE_k t/\hbar} \right) \\ &= i\hbar \sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) \left(-i \frac{E_k}{\hbar} \right) \\ &= \sum_{\forall k} E_k a_k e^{-iE_k t/\hbar} \phi_k(x) \\ &\overset{!}{\neq} E \Psi(x,t) = E \sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) . \end{aligned} \tag{31}$$

with E being a constant (not an operator).

The right-hand side of the Eq. (31) is a necessary condition for a c-dialectical description for the system, as one infers from our considerations leading to the Eq. (20). Thus, the q-dialectic is incompatible with the c-dialectic, mutually exclusive, the first chronologically holding over $\mathcal{I}_q$, the latter over $\mathcal{I}_c$, as discussed before. Hence, the march that led us to the obtention of the Eqs. (16), (17) and (18), as well as its subsequent considerations, to define the c-dialectical potential energy operator $V_c(x,t)$, is not a triviality. Concluding this section, we have got our problem in position: with the obtention we have got carried out through this section for the c-dialectical description of the system $\Psi(x,t)$ at any instant $t \in \mathcal{I}_c$ via the potential energy operator $V_c(x,t)$ acting on $\Psi(x,t)$ over $\mathcal{I}_c$:

$$V_c(x,t) \equiv E + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2},\ E \text{ constant (not an operator)}, \tag{32}$$

the complete potential energy operator for the entire chronology $\mathbb{R} = \mathcal{I}_q \cup \mathcal{I}_c$ of the system, Eq. (12), reads:

$$V(x,t) = (1-\delta_{t\bar{t}}) V_p(x,t) + \delta_{t\bar{t}}\left(E + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right). \tag{33}$$

Now, with the Eq. (33), we are in position to solve the Eq. (1), the Schrodinger equation, an issue to be solved in the next section.

SOLVING THE SCHRODINGER EQUATION

The Schrodinger equation, Eq. (1), with the insertion of the potential energy operator, Eq. (33), reads:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(x,t) &= (1-\delta_{t\bar{t}}) V_p(x,t)\Psi(x,t) + \\ &+ \delta_{t\bar{t}}\left(E + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Psi(x,t) + \\ &- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t). \end{aligned} \tag{34}$$

Since $t \in \mathcal{I}_q$ or $t \in \mathcal{I}_c$, depending on the description, on the dialectic that is being manifested to infer the reality of the physical system $\Psi(x,t)$, as discussed before, we analyse the solution for the Eq. (34) through two successive but mutually exclusive parts: $t \in \mathcal{I}_q$ or $t \in \mathcal{I}_c$ [cf. the Eqs. (7) and (8)].

For $t \in \mathcal{I}_q = (-\infty,\tau) = \mathbb{R} - \mathcal{I}_c = \mathbb{R} - [\tau,+\infty)$:

In this case, $\delta_{t\bar{t}} = 0$, in virtue of the Eq. (13). Hence, the Eq. (34) reads:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = V_p(x,t)\Psi(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t). \tag{35}$$

In virtue of linearity, one may firstly solve the Eq. (35) for the eigenstates $\Psi_k(x,t)$, with $k \in \{1,\cdots,d\}$, being d the number of linearly independent eigenvectors of the $i\hbar\partial/\partial t$ operator, i.e., in other words:

$$k \in \{1,\cdots,\mathbf{dim}(\mathcal{H}_\Psi|\Psi_k(x,t)\in\mathcal{H}_\Psi)\}, \tag{36}$$

being $\mathcal{H}_\Psi$ the Hilbert vector space to which $\Psi(x,t)$ belongs. Thus:

$$i\hbar\frac{\partial}{\partial t}\Psi_k(x,t) = V_p(x,t)\Psi_k(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_k(x,t). \tag{37}$$

Since the Eq. (34) is developed here as being for general scope under the $(1+1)$-dimensional spacetime [2], once we are interested in its consequences, the consideration of a general physical potential $V_p(x)$ [instead of $V_p(x,t)$] energy operator to be acting on $\Psi(x,t)$ over $\mathcal{I}_q$ is irrelevant, i.e., we will suppose, with no loss of generality [in fact, the process of solution we are accomplishing here for $t \in \mathcal{I}_q$, could be accomplished, once demanded, for a general $V_p(x,t)$; this does not change the necessity of reasoning related to the solution of the Eq. (34) for $t \in \mathcal{I}_c$, this latter being mutually exclusive to the interval $\mathcal{I}_q$ over which $V_p(x,t)$ [or $V_p(x)$] operates on $\Psi(x,t)$], the potential energy of the physical case under consideration obeys:

$$V_p(x,t) = V_p(x). \tag{38}$$

Back to the Eq. (37):

$$i\hbar\frac{\partial}{\partial t}\Psi_k(x,t) = V_p(x)\Psi_k(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_k(x,t), \tag{39}$$

one canonically proposes a solution under a separable form:

$$\Psi_k(x,t) \equiv \varphi_k(t)\phi(x), \tag{40}$$

leading, by substitution within the Eq. (39):

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}[\varphi_k(t)\phi_k(x)] &= V_p(x)\varphi_k(t)\phi_k(x) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\varphi_k(t)\phi_k(x)] \Rightarrow \\ i\hbar\phi_k(x)\frac{\partial}{\partial t}\varphi_k(t) &= V_p(x)\varphi_k(t)\phi_k(x) - \frac{\hbar^2}{2m}\varphi_k(t)\frac{\partial^2}{\partial x^2}\phi_k(x) \Rightarrow \\ i\hbar\phi_k(x)\frac{d}{dt}\varphi_k(t) &= V_p(x)\varphi_k(t)\phi_k(x) - \frac{\hbar^2}{2m}\varphi_k(t)\frac{d^2}{dx^2}\phi_k(x) \overset{\times[\Psi_k(x,t)]^{-1}\neq\infty}{\Rightarrow} \end{aligned}$$

$$i\hbar\frac{\phi_k(x)}{\Psi_k(x,t)}\frac{d}{dt}\varphi_k(t) = V_p(x)\frac{\varphi_k(t)\phi_k(x)}{\Psi_k(x,t)} - \frac{\hbar^2}{2m}\frac{\varphi_k(t)}{\Psi_k(x,t)}\frac{d^2}{dx^2}\phi_k(x) \overset{\text{Eq. (40)}}{\Rightarrow}$$
$$i\hbar\frac{\phi_k(x)}{\varphi_k(t)\phi_k(x)}\frac{d}{dt}\varphi_k(t) = V_p(x)\frac{\varphi_k(t)\phi_k(x)}{\varphi_k(t)\phi_k(x)} - \frac{\hbar^2}{2m}\frac{\varphi_k(t)}{\varphi_k(t)\phi_k(x)}\frac{d^2}{dx^2}\phi_k(x) \therefore$$

$$\frac{i\hbar}{\varphi_k(t)}\frac{d}{dt}\varphi_k(t) = V_p(x) - \frac{\hbar^2}{2m}\frac{1}{\phi_k(x)}\frac{d^2}{dx^2}\phi_k(x). \quad (41)$$

In relation to the Eq. (41), its left-hand side solely depends on t and its right-hand side solely depends on x, from which, to accomodate both the necessities, *a fortiori*, one writes:

$$\frac{i\hbar}{\varphi_k(t)}\frac{d}{dt}\varphi_k(t) = V_p(x) - \frac{\hbar^2}{2m}\frac{1}{\phi_k(x)}\frac{d^2}{dx^2}\phi_k(x) = E_k, \quad (42)$$

where E_k is a constant of context, viz.: for each k, since each k generates an equation identical to the Eq. (41).

Hence, we have generated two equations within each k-eigenvector context, both coupled, via Eq. (42), by the context constant E_k [this constant will turn out to be the energy associated to each eigenstate $\Psi_k(x,t)$, as we will develop, but, this is a very well known fact within the context we are developing here]:

$$\frac{i\hbar}{\varphi_k(t)}\frac{d}{dt}\varphi_k(t) = E_k; \quad (43)$$
$$V_p(x) - \frac{\hbar^2}{2m}\frac{1}{\phi_k(x)}\frac{d^2}{dx^2}\phi_k(x) = E_k. \quad (44)$$

To solve the Eq. (43) we multiply both the sides of the Eq. (43) by $\Delta t \to 0$, and by $1/(i\hbar)$:

$$\frac{d\varphi_k(t)}{\varphi_k(t)} = \frac{E_k}{i\hbar}dt. \quad (45)$$

Integrating both the sides of the Eq. (45) from $t_0 < t < \tau$ to t:

$$\begin{aligned}
\int_{t_0}^{t}\frac{d\varphi_k(t)}{\varphi_k(t)} &= \int_{t_0}^{t}\frac{E_k}{i\hbar}dt \Rightarrow \quad (46)\\
\ln[\varphi_k(t)]\big|_{t_0}^{t} &= \frac{E_k}{i\hbar}t\Big|_{t_0}^{t} \Rightarrow\\
\ln[\varphi_k(t)] - \ln[\varphi_k(t_0)] &= \frac{E_k}{i\hbar}t - \frac{E_k}{i\hbar}t_0 = \frac{E_k}{i\hbar}(t-t_0) \Rightarrow\\
\ln\left[\frac{\varphi_k(t)}{\varphi_k(t_0)}\right] &= \frac{E_k}{i\hbar}(t-t_0) \Rightarrow\\
\frac{\varphi_k(t)}{\varphi_k(t_0)} &= \exp\left[\frac{E_k}{i\hbar}(t-t_0)\right] \Rightarrow\\
\varphi_k(t) &= \varphi_k(t_0)\exp\left[\frac{E_k}{i\hbar}(t-t_0)\right] \Rightarrow\\
\varphi_k(t) &= \left[\varphi_k(t_0)e^{iE_kt_0/\hbar}\right]e^{-iE_kt/\hbar}. \quad (47)
\end{aligned}$$

Hence, for purposes of superposition, one has got:

$$\varphi_k(t) \propto e^{-iE_kt/\hbar}, \quad (48)$$

where the proporcionality is accomplished by a k-context constant λ_k given by:

$$\lambda_k = \varphi_k(t_0)e^{iE_kt_0/\hbar}. \quad (49)$$

To solve the Eq. (44), one multiplies both the sides of this equation by $\phi_k(x) \neq 0$, by hypothesis, and reaches the so-called time-independent Schrodinger equation:

$$V_p(x)\phi_k(x) - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi_k(x) = E_k\phi_k(x). \quad (50)$$

The solution for the time-independent Schrodinger equation, Eq. (50), depends on the specific potential energy operator $V_p(x)$ under consideration, but it is clear the solution for this equation reads $\phi_k(x)$.

In virtue of the Eqs. (48), [(49)], and (50), the orthonormal members $\Psi_k(x,t)$ of the basis $\mathcal{B} = \{\Psi_k(x,t)\}$ are given by:

$$\Psi_k(x,t) = e^{-iE_kt/\hbar}\phi_k(x), \quad (51)$$

a k-solution for the eigenvalue problem given by the Eq. (39).

To obtain the general solution for the Eq. (35) [we are analyzing the cases in which $V_p(x,t) = V_p(x)$, as discussed above, with no restriction to our fundamental purpose of this paper]:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = V_p(x)\Psi(x,t) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t), \quad (52)$$

we use the superposition principle [of course, the main principle that distinguishes a quantum mechanical object from a classically newtonian one, regarding the object *per se*]:

$$\Psi(x,t) = \sum_{\forall k}a_k\Psi_k(x,t) = \sum_{\forall k}a_ke^{-iE_kt/\hbar}\phi_k(x), \quad (53)$$

in virtue of the Eq. (51), members of the basis $\mathcal{B} = \{\Psi_k(x,t)\}$. Of course, the solution given by the Eq. (53) must satisfy the initial and boundary conditions for the physical problem established by $V_p(x,t) = V_p(x)$, this latter acting on the system $\Psi(x,t)$ over the chronological domain $\mathcal{I}_q$ throughout which a q-dialectical description holds, the main purpose of this subsection. Hence, the following condition must hold:

$$\text{for } t_0 \in \mathcal{I}_q: \left. \sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) \right|_{t=t_0} = \Psi(x, t_0), \tag{54}$$

where $\Psi(x, t_0)$ is initially given, prepared. This discussion was accomplished through the march that led from the Eq. (23) to the Eq. (24), although, there, we had adopted $t_0 \equiv 0$. Analogous reasoning will lead to:

$$a_k e^{-iE_k t_0/\hbar} = \int_{-\infty}^{\infty} \phi_k^*(x) \Psi(x, t_0) \, dx \tag{55}$$

$$= \int_{-\infty}^{\infty} \phi_k^*(x) \left[\sum_{\forall l} a_l e^{-iE_l t_0/\hbar} \phi_l(x) \right] dx$$

$$= \sum_{\forall l} a_l e^{-iE_l t_0/\hbar} \int_{-\infty}^{\infty} \phi_k^*(x) \phi_l(x) \, dx$$

$$= \sum_{\forall l} a_l e^{-iE_l t_0/\hbar} \delta_k^l$$

$$= a_k e^{-iE_k t_0/\hbar}, \tag{56}$$

from which:

$$a_k = e^{iE_k t_0/\hbar} \int_{-\infty}^{\infty} \phi_k^*(x) \Psi(x, t_0) \, dx. \tag{57}$$

Regarding the boundary conditions [at given spatial positions], supposing the canonical cases in which one has got fixed ones $\forall t$ [here, with $t \in \mathcal{I}_q$], the consideration of these boundary conditions within the context of the time-independent Schrodinger equation, Eq. (50), is to accomplish the job related to them.

Thus, we accomplish this subsection, obtaining the general solution, given by the Eq. (53) and its subsidiary conditions, Eqs. (50) and (57), these for the Eq. (34) $\forall\ t \in \mathcal{I}_q$ [considering $V_p(x,t) = V_p(x)$]. Now, we pass to the next subsection, concerned with the chronological domain $t \in \mathcal{I}_c$ over which a classical description of the system $\Psi(x,t)$ turns out to be the inferred one, as the Nature may dictate.

For $t \in \mathcal{I}_c = [\tau, +\infty) = \mathbb{R} - \mathcal{I}_q = \mathbb{R} - (-\infty, \tau)$:

Throughout the initial considerations of this paper, mainly within the discussion that led to the obtention of the Eq. (31), we pointed out the apparent incompatibility between the complete Schrodinger equation, with its intrinsical dependence on t, and a characterization that is necessary to assert an unique an well defined energy, E, for the physical system under consideration, $\Psi(x,t)$, when the system turns out to exhibit a description that is not quantum mechanical. In other words: when the system is under the description given by the Eq. (22), fully quantum mechanical in the sense of the system *per se*, once it would be under a superposition of states, such [superposition of] state[s] is not compatible with a purely classical [Newtonian system]. Taking an instrumentalist point-of-view, one is constrained to descriptions to physical systems that emerge from the interactions the designed apparata provide, and a pure quantum mechanical system turns out to have got, intrinsically, by construction, an objective sense, once the description must be provided by an external axiom having got this special function within the theory: Born's rule. But, if the Schrodinger equation is a complete description of the system dynamics, and one infers that under a description by superposition, a system have not got a well defined energy state, [considering only the Schrodinger equation, we are not arguing: well, if one wants an energy, one should accomplish a measure, and the Born rule shuts up the question; under the same reasoning, one may argue the Schrondiger equation provides well defined eigenstates, without the necessity of an external axiom to infer the mathematical existence of such energy eigenvalues] which one is the more natural manner to handle with the very fact the Nature dictates there exist well defined states of energy:

- *To create a separated axiom to handle with this necessity, Born rule?*

or:

- *To accept the possibility the well defined energy states should also be described by the Schrodinger equation as well as the eigenstates are mathematically inferred within the very structure of the Schrodinger equation, concluding, a fortiori, once the Nature dictates such c-dialectical description and once a c-dialectic should follow [remembering the quantum mechanics turns out cover the classical physics when one is faced with the necessity of description of the Nature from the microscopic picture, whose constituints are the building blocks of the macroscopic world] from the quantum-mechanical description, that the c-dialectical issue may not be well stated within the dynamical structure [context of the Schrodinger equation] of the classical quantum mechanics?*

We will accept the second possibility, not as a point-of-view, but in virtue of the facts the Nature provides in relation to the c-dialectic for the physical world, with which one also interprets the Nature, with which one also constructs physical theories within specific domains of instrumental validity. These issues are the main purpose of this subsection. We are to deal with the chronological domain, $\mathcal{I}_c$, here, the main purpose of this subsection, but, prior to this, we will consider the fundamental issues that will lead to the main purpose.

Firstly, before something else, one should define the very meaning of a system having got a *well defined en-*

ergy. Once with this accomplished, then, one should try, if possible, to accomodate this very meaning within the context of the Schrodinger equation and, of course, since the Nature is the very main object of the Physics, within the physical world, the razor of plausibilities.

In spite of a quantum mechanical description, instrumentally, a system exhibits a very well defined physically valued characteristic *iff* the very same value for this characteristic is obtained at each instrumentally accomplished measure. This proccess turns out to be a statistical one, once a miriad of measures [or a satisfactory quantity of measures to provide a sufficiently rich statistics] turns out to be the necessary method to collect the data necessary to infer, or not, the data are providing the same value for the measure that is being putted under the instrumental scrutinity [within a error gap/bar]. In other words, the value being scrutinized must exhibit no fluctuations around the mean value, viz., that the standard deviation from the mean value turns out to statistically vanish:

$$\sqrt{\sigma_E^2} = 0 \Leftrightarrow \sigma_E^2 = 0, \tag{58}$$

where E is the value of a physical system being putted under scrutinity, $\sqrt{\sigma_E^2}$ the standard deviation from the mean value $\langle E \rangle$ and, σ_E^2, is the variance of E:

$$\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2 . \tag{59}$$

The statistical characteristic of the quantum mechanics is putted within the theory via an extra axiom, independent of the Schrodinger equation over the axiomatic structure of the quantum mechanics. The correlation between statistics and Schroedinger equation borns within the axiomatic structure of the quantum mechanics via this very extra axiom: Born's rule. We will take a position regarding the Schrodinger equation and its correlation to statistics via the wave function $\Psi(x,t)$ that represents a quantum mechanical system, not from an extra axiom, and, emerging the statistics from the very structure of the wave function $\Psi(x,t)$, the correlation to the Schrodinger equation turns out to be automatic, since the very evolution of the system, of the wave function $\Psi(x,t)$ representing the physical system in question, is governed by the very Schrodinger equation. A question must be answered here:

- *But, how can one establish the statistical characteristic of the quantum mechanics is intrinsical to the wave function in spite of that extra axiom, viz., in spite of the Born rule?*

We will adopt the position in [1], from which the utility in obtaining the eigenvalue E_k, e.g., from a quantum mechanical system mathematically represented by:

$$\Psi(x,t) = \sum_{\forall k} a_k \Psi_k(x,t), \tag{60}$$

where $\mathcal{B} = \{\Psi_k\}$ is an orthonormal basis that consists of eigenstates of a physical quantity, say $i\hbar\partial/\partial t$ [the considerations in [1] are quite general ones, not constrained to the Hamiltonian operator, but we are putting our attention, pragmatically, to the representation we are working here], is given by:

$$a_k^* a_k = p_k, \tag{61}$$

where p_k is the probability of obtention of E_k, the probability of obtention of $\Psi_k(x,t)$ [we will consider the eigenvalues are not degenerated, viz., that our system consists of a superposition of eigenstates that will not present a same value of eigenvalue E_k, for purposes of simplicity, once the cases with degenerated states is handled with an attention on the subspace, a degenerated subspace for short, that is generated by those eigenstates that present a same eigenvalue, which is trivial to handle and to generalize, once the utility of obtention of a given degenerated eigenvalue turns out to be the probability of obtention of eigenvectors spanning the degenerated subspace].

Under the utility picture as in [1], the maximal utility is obtained when there exists the whole set of eigenvalues E_k available for obtention, when the information is maximally available within the physical system mathematically represented by the superposition given by the Eq. (60). But the maximal utility cannot exceed the utility encapsulated within the physical system mathematically represented by the superposition given by the Eq. (60), at least if this system is to be the unique relevant system under scrutinity, once the maximal utility is the utility of the whole system under scrutinity. When a measure is accomplished on a system represented by the Eq. (60), some utility is provided from the system, and a question emerges:

- *The provided utility from the system is an utility of what?*

As said, it is the utility of an eigenvalue obtention, but, once obtained, the utility of the obtained eigenvalue, from which the obtained eigenvalue carries with it the obtained utility. And another question emerges:

- *Since an eigenvalue was obtained by the measure, has got the system, hence, an unique state defined by this eigenvalue? In other words, the state of the system after the measure carries the eigenvalue with it, from which will the system turn out to necessarily carry the utility that was obtained from the measure?*

To answer this question, one must recognize there exists just one system being measured, from which the system is the very same. What may change is the state inferred from the obtained eigenvalue. If this eigenvalue characterizes the state of the system, in virtue of the discussion that was carried out leading to the Eq. (31) in the previous subsection, the system cannot be under a superposition of eigenstates after the measure, being this latter case the one to which the obtained eigenvalue would not suffice to characterize the system after the measure. Furthermore, since the system is the very same prior to the measure and after it, the system, being the unique relevant system under scrutinity, must preserve its utility, and the entire maximal utility must be carried by the system after the measure. Prior to the measure, a given eigenstate E_k had an utility associated to its obtention, but this utility cannot be the system utility prior to the measure, since a superposition of eigenstates cannot be defined by an unique eigenvalue. Hence, the maximal utility, the utility of a whole system must be conserved. In other words, as obtained in [1], Eq. (61) here, the probability must be conserved. But one may argue the probability of obtention of E_k is not conserved, since it would jump to 1 after a measure that uniquely characterized the system after the measure. The probability of obtention of an eigenstate may change in virtue of the very fact such eigenstate would not be uniquely characterizing the system $\forall\, t \in \mathbb{R} = \mathcal{I}_q \cup \mathcal{I}_c$. The q-dialectic needs superposition, since the system is not uniquely characterized by an unique energy state. The c-dialectic does not need superposition, once the system is uniquely characterized by an unique energy state. But the system is the very same, only the dialectical characterization changes.

Prior to the chronological domain $\mathcal{I}_c$, a given eigenstate, say $\Psi_k(x,t)$, carries an utility related to the obtention of its eigenvalue E_k, at any instant $t \in \mathcal{I}_q$, which is obtained from the Eq. (61), where:

$$\begin{aligned} a_k &= \int_{-\infty}^{\infty} \Psi_k^*(x,t)\, \Psi(x,t)\, dx \qquad (62)\\ &= \int_{-\infty}^{\infty} \Psi_k^*(x,t) \left[\sum_{\forall l} a_l \Psi_l(x,t)\right] dx \\ &= \int_{-\infty}^{\infty} \Psi_k^*(x,t) \sum_{\forall l} a_l \Psi_l(x,t)\, dx \\ &= \sum_{\forall l} a_l \int_{-\infty}^{\infty} \Psi_k^*(x,t)\, \Psi_l(x,t)\, dx \\ &= \sum_{\forall l} a_l \delta_k^l \\ &= a_k. \qquad (63) \end{aligned}$$

But, once the q-dialectical description holds, one has got:

$$\Psi_k(x,t) \neq \Psi(x,t)\,, \qquad (64)$$

once the superposition is necessary by a q-dilalectic, for $t \in \mathcal{I}_q$, and the maximal utility is not obtained from $a_k^* a_k$, since the maximal utility is related to the whole system, and the whole system is not $\Psi_k(x,t)$ in virtue of the Eq. (64) $\forall\, t \in \mathcal{I}_q$, being $a_k^* a_k$ the utility related to the eigenstate $\Psi_k(x,t)$. Conversely, since $\mathcal{I}_q \cap \mathcal{I}_c = \emptyset$, once the c-dialectic holds, and the system becomes to be descripted in terms of an unique energy state, one simply has got the negative of the Eq. (64):

$$\Psi_k(x,t) = \Psi(x,t)\,, \qquad (65)$$

$\forall\, t \in \mathcal{I}_c$, which is a dialectical razor dictated by Nature.

The utility of the whole system S, $a_S^* a_S$, under scrutinity, $\forall\, t \in \mathcal{I}_q \cup \mathcal{I}_c$, once the system is the very same, prior to the measure and after it, follows from:

$$a_S = \int_{-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx, \qquad (66)$$

i.e.:

$$a_S^* a_S = |a_S|^2 = \left| \int_{-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx \right|^2, \qquad (67)$$

and must be conserved $\forall\, t \in \mathcal{I}_q \cup \mathcal{I}_c$. Once:

$$\Psi^*(x,t)\, \Psi(x,t) > 0, \qquad (68)$$

The Eq. (67) may read:

$$a_S^* a_S = \left[\int_{-\infty}^{\infty} |\Psi^*(x,t)\, \Psi(x,t)|\, dx \right]^2. \qquad (69)$$

But:

$$\begin{aligned} a_S^* a_S &= \left[\int_{-\infty}^{\infty} |\Psi^*(x,t)\, \Psi(x,t)|\, dx \right]^2 \\ &= \left[\int_{-\infty}^{\infty} \left| |\Psi(x,t)|^2 \right| dx \right]^2 \\ &= \left[\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx \right]^2 \\ &= \left[\int_{-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx \right]^2 \\ &= \left\{ \int_{-\infty}^{\infty} \left[\sum_{\forall k} a_k \Psi_k(x,t) \right]^* \left[\sum_{\forall l} a_l \Psi_l(x,t) \right] dx \right\}^2 \\ &= \left[\sum_{\forall k} \sum_{\forall l} a_k^* a_l \int_{-\infty}^{\infty} \Psi_k^*(x,t)\, \Psi_l(x,t)\, dx \right]^2 \\ &= \left[\sum_{\forall k} a_k^* a_k \right]^2. \qquad (70) \end{aligned}$$

But, as showed in [1]:

$$\sum_{\forall k} a_k^* a_k = 1, \qquad (71)$$

the maximal utility of the quantum system under scrutinity [1]. Hence, the Eq. (70) reads:

$$\begin{aligned} a_S^* a_S &= \left[\int_{-\infty}^{\infty} \left| \Psi^* (x,t) \Psi (x,t) \right| dx \right]^2 \\ &= \left[\sum_{\forall k} a_k^* a_k \right]^2 \\ &\overset{\text{Eq. (71)}}{=} [1]^2 \\ &= 1 \\ &= \sum_{\forall k} a_k^* a_k. \end{aligned} \tag{72}$$

Furthermore:

$$\begin{aligned} a_S^* a_S &= \int_{-\infty}^{\infty} \Psi^* (x,t) \Psi (x,t) \, dx \\ &= \int_{-\infty}^{\infty} \left| \Psi (x,t) \right|^2 dx. \end{aligned} \tag{73}$$

In fact:

$$\begin{aligned} a_S^* a_S &= \int_{-\infty}^{\infty} \Psi^* (x,t) \Psi (x,t) \, dx \\ &= \int_{-\infty}^{\infty} \left| \Psi (x,t) \right|^2 dx \\ &= \int_{-\infty}^{\infty} \left[\sum_{\forall k} a_k \Psi_k (x,t) \right]^* \left[\sum_{\forall l} a_l \Psi_l (x,t) \right] dx \\ &= \int_{-\infty}^{\infty} \sum_{\forall k} a_k^* \Psi_k^* (x,t) \sum_{\forall l} a_l \Psi_l (x,t) \, dx \\ &= \int_{-\infty}^{\infty} \sum_{\forall k} \sum_{\forall l} a_k^* a_l \Psi_k^* (x,t) \Psi_l (x,t) \, dx \\ &= \sum_{\forall k} \sum_{\forall l} a_k^* a_l \int_{-\infty}^{\infty} \Psi_k^* (x,t) \Psi_l (x,t) \, dx \\ &= \sum_{\forall k} \sum_{\forall l} a_k^* a_l \delta_{kl} \\ &= \sum_{\forall k} a_k^* a_k \\ &\overset{\text{Eq. (71)}}{=} 1 \\ &= [1]^2 \\ &= \left[\sum_{\forall k} a_k^* a_k \right]^2 \\ &\overset{\text{Eq. (70)}}{=} a_S^* a_S. \end{aligned} \tag{74}$$

Hence we have concluded, in virtue of the conservation of the maximal utility, viz., in virtue of the conservation of the utility of the system S under scrutinity that:

$$\begin{aligned} \forall t \in \mathcal{I}_q \cup \mathcal{I}_c: \; a_S^* a_S &= \sum_{\forall k} a_k^* a_k = \sum_{\forall k} \left| a_k \right|^2 \\ &= \int_{-\infty}^{\infty} \Psi^* (x,t) \Psi (x,t) \, dx \\ &= \int_{-\infty}^{\infty} \left| \Psi (x,t) \right|^2 dx = 1. \end{aligned} \tag{75}$$

The result obtained and stated by the Eq. (75) will be shown from the general solution for the Eq. (34) we are to complete within this subsection [with $V_p(x,t) = V_p(x)$]. One should infer the Eq. (34) has two chronological domains that are mutually exclusive due to a possible change of dialectical description, and reflect on the very meaning of the Eqs. (64) and (65). The change is dialectical, the system under scrutinity is the very same.

Now, we are in position to analyze the condition stated by the Eq. (58) for a system that c-dialectically presents a well defined energy when scrutinized. In fact, the Eq. (75) turns out to state the maximal utility, viz., the utility of the whole system being scrutinized, $a_S^* a_S$, is given by an utility density $\left|\Psi(x,t)\right|^2$, at a given instant t, $\forall\, t \in \mathcal{I}_q \cup \mathcal{I}_c$, over the spatial positions x. Since, as derived in [1], the utility is probability, one may infer from this and from the Eq. (75), a probability density, $\rho(x,t)$, given by:

$$\rho(x,t) = \left| \Psi (x,t) \right|^2, \tag{76}$$

emerges.

We have seen that under a q-dialectical description that requires a superposition of eigenstates to represent a physical system, through the march that led to the Eq. (31), one cannot state the system has got a well defined energy. Also, under a q-dialectic, the operational description holds, and the energy of an eigenstate $\Psi_k(x,t)$ follows from the Schrodinger eigenvalue equation, Eq. (29). Hence, for eigenstates, both the sides of the following equation are the very same:

$$\begin{aligned} &\int_{-\infty}^{\infty} \Psi_k^* \; (x,t) \left[i\hbar \frac{\partial}{\partial t} \right] \Psi_k (x,t) \, dx \\ &\quad = \int_{-\infty}^{\infty} \Psi_k^* (x,t) \, E_k \Psi_k (x,t) \, dx \\ &\quad = E_k, \end{aligned} \tag{77}$$

in virtue of the Eq. (75) with $\Psi(x,t) \to \Psi_k(x,t)$, also by orthonormalization, if one prefers. Furthermore, both the sides of the following equation are, also, the very same, for eigenstates:

$$\begin{aligned} &\int_{-\infty}^{\infty} \Psi_k^* \; (x,t) \left[i\hbar \frac{\partial}{\partial t} \right]^2 \Psi_k (x,t) \, dx \\ &\quad = \int_{-\infty}^{\infty} \Psi_k^* (x,t) \, E_k^2 \Psi_k (x,t) \, dx \\ &\quad = E_k^2. \end{aligned} \tag{78}$$

It is trivial to infer from the Eqs. (76), (77) and (78) that eigenstates obey the condition given by the Eq. (58) for well defined energy states. In fact, from the Eqs. (77) and (78), with the Eq. (76), i.e., if the state of the system is an eigenstate, from which $E = E_k$, where E is the energy of the system, the Eq. (58) follows. But the converse is a little bit tricky to prove. One writes the state as a superposition by hypothesis:

$$\Psi(x,t) = \sum_{\forall k} a_k \Psi_k(x,t). \tag{79}$$

Hence:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(x,t) &= i\hbar\frac{\partial}{\partial t}\sum_{\forall k} a_k \Psi_k(x,t) \\ &= \sum_{\forall k} a_k i\hbar\frac{\partial}{\partial t}\Psi_k(x,t) \\ &= \sum_{\forall k} a_k E_k \Psi_k(x,t), \end{aligned} \tag{80}$$

and to the following integral, to be:

$$\begin{aligned} \langle E\,\rangle &\overset{!}{=} \int_{-\infty}^{\infty} \Psi^*(x,t)\, i\hbar\frac{\partial}{\partial t}\Psi(x,t)\,dx \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t)\, E\Psi(x,t)\,dx, \end{aligned} \tag{81}$$

it requires, in virtue of the Eqs. (79) and (80):

$$\begin{aligned} \langle E\,\rangle &= \int_{-\infty}^{\infty}\left[\sum_{\forall k} a_k\Psi_k(x,t)\right]^*\left[\sum_{\forall l} a_l E_l \Psi_l(x,t)\right]dx \\ &= \int_{-\infty}^{\infty}\sum_{\forall k} a_k^*\Psi_k^*(x,t)\left[\sum_{\forall l} a_l E_l\Psi_l(x,t)\right]dx \\ &= \int_{-\infty}^{\infty}\sum_{\forall k}\sum_{\forall l} a_k^* a_l E_l \Psi_k^*(x,t)\,\Psi_l(x,t)\,dx \\ &= \sum_{\forall k}\sum_{\forall l} a_k^* a_l E_l \int_{-\infty}^{\infty}\Psi_k^*(x,t)\,\Psi_l(x,t)\,dx \\ &= \sum_{\forall k}\sum_{\forall l} a_k^* a_l E_l \delta_{kl} \\ &= \sum_{\forall k} a_k^* a_k E_k. \end{aligned} \tag{82}$$

Furthermore, in virtue of the Eq. (80):

$$\begin{aligned} \left[i\hbar\frac{\partial}{\partial t}\right]^2\Psi(x,t) &= i\hbar\frac{\partial}{\partial t}\sum_{\forall k} a_k E_k\Psi_k(x,t) \\ &= \sum_{\forall k} a_k E_k i\hbar\frac{\partial}{\partial t}\Psi_k(x,t) \\ &= \sum_{\forall k} a_k E_k E_k \Psi_k(x,t) \\ &= \sum_{\forall k} a_k E_k^2 \Psi_k(x,t), \end{aligned} \tag{83}$$

and to the following integral to be:

$$\begin{aligned} \langle E^2\,\rangle &\overset{!}{=} \int_{-\infty}^{\infty}\Psi^*(x,t)\left[i\hbar\frac{\partial}{\partial t}\right]^2\Psi(x,t)\,dx \\ &= \int_{-\infty}^{\infty}\Psi^*(x,t)\,E^2\Psi(x,t)\,dx, \end{aligned} \tag{84}$$

it requires, in virtue of the Eqs. (79) and (83):

$$\begin{aligned} \langle E^2\,\rangle &= \int_{-\infty}^{\infty}\left[\sum_{\forall k} a_k\Psi_k(x,t)\right]^*\left[\sum_{\forall l} a_l E_l^2\Psi_l(x,t)\right]dx \\ &= \int_{-\infty}^{\infty}\sum_{\forall k} a_k^*\Psi_k^*(x,t)\left[\sum_{\forall l} a_l E_l^2\Psi_l(x,t)\right]dx \\ &= \int_{-\infty}^{\infty}\sum_{\forall k}\sum_{\forall l} a_k^* a_l E_l^2\Psi_k^*(x,t)\,\Psi_l(x,t)\,dx \\ &= \sum_{\forall k}\sum_{\forall l} a_k^* a_l E_l^2\int_{-\infty}^{\infty}\Psi_k^*(x,t)\,\Psi_l(x,t)\,dx \\ &= \sum_{\forall k}\sum_{\forall l} a_k^* a_l E_l^2\delta_{kl} \\ &= \sum_{\forall k} a_k^* a_k E_k^2. \end{aligned} \tag{85}$$

From the Eqs. (82) and (85), one reaches:

$$\langle E^2\,\rangle - \langle E\,\rangle^2 = \sigma_E^2 = \sum_{\forall k} a_k^* a_k E_k^2 - \left(\sum_{\forall k} a_k^* a_k E_k\right)^2, \tag{86}$$

which may be written as:

$$\sigma_E^2 = \sum_{\forall k} a_k^* a_k E_k^2 - \sum_{\forall k} a_k^* a_k \langle E\,\rangle E_k. \tag{87}$$

Now, one imposes the condition given by the Eq. (58) to obtain its necessary condition, once $E = E_k$ was already shown as being a sufficient condition to obey the Eq. (58). Hence, in virtue of the Eq. (87):

$$\sigma_E^2 = \sum_{\forall k} a_k^* a_k E_k\left(E_k - \langle E\,\rangle\right) = 0. \tag{88}$$

Since we are imposing $\sigma_E^2 = 0$, the Eq. (88):

$$\sum_{\forall k} a_k^* a_k E_k E_k = \sum_{\forall k} a_k^* a_k E_k \langle E\,\rangle, \tag{89}$$

must hold in any situation by hypothesis. This imply the Eq. (88) must identically vanish, viz., the left-hand side and the right-hand side of the Eq. (89) are to be identical ones. Otherwise, one would have to choose a value for $\langle E\,\rangle$ at each specific situation to satisfy the Eq. (89) [Eq. (88), equivalently], which is an absurd once the $\langle E\rangle$ is not constrained by the Eq. (88), but a consequence of the Eq. (82). Hence, to the Eq. (88) identically vanish:

$$\forall k:\ E_k = \langle E\rangle, \tag{90}$$

from which, in virtue of these conclusions regarding the necessity and the sufficiency for the Eq. (58):

$$\sigma_E^2 = 0 \Leftrightarrow \forall k: \; E_k = \langle E \rangle = E. \tag{91}$$

The Eq. (91) translates the characteristic of a c-dialectic regarding a well defined energy over the chronological domain $t \in \mathcal{I}_c$. Back to the Eq. (34), in virtue of the Eq. (14), one needs to solve:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = E\Psi(x,t), \tag{92}$$

where E is constant. One should grasp the meaning of the Eq. (91) is not that one cannot consider different eigenvalues *over the entire chronology of the Schrodinger equation*, but, once the physical system has got its description under a c-dialectic, viz.: $\forall t \in \mathcal{I}_c$, this latter being *just a part of the entire chronology*, the c-dialectical description implies a well defined energy $E = E_k$, in spite of which k c-dialectically holds, since $E = E_k$ must be the same despite of k over $\mathcal{I}_c$. Over the mutually exclusive q-dialectical chronology, $\forall\ t \in \mathcal{I}_q$, the history is mutually exclusive to the one over $\mathcal{I}_c$, and different eigenvalues may be very well putted under superposition, as discussed before and obtained in the previous section for the q-dialectic.

In virtue of the Eq. (20), one should solve the Eq. (92) via the canonical way, viz.:

$$i\hbar\frac{\partial}{\partial t}\Psi_k(x,t) = E_k\Psi_k(x,t), \tag{93}$$

also noting that the Eq. (91) will turn out to impose a chronological boundary condition, since the Eq. (91) holds $\forall\ t \in \mathcal{I}_c$, in spite of the spatial location x. This latter assertion will follow from separability, as we will infer below.

In fact, imposing, again, now $\forall\ t \in \mathcal{I}_c$, emphasizing:

$$\Psi_k(x,t) = f_k(t)\,\chi_k(x), \tag{94}$$

$\forall\ t \in \mathcal{I}_c = [\tau, +\infty)$, one has got, in virtue of the Eqs. (93) and (94):

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left[f_k(t)\,\chi_k(x)\right] &= E_k f_k(t)\,\chi_k(x) \Rightarrow \\
i\hbar\chi_k(x)\frac{\partial}{\partial t}f_k(t) &= E_k f_k(t)\,\chi_k(x) \Rightarrow \\
\chi_k(x)\left[i\hbar\frac{\partial}{\partial t}f_k(t) - E_k f_k(t)\right] &= 0 \quad \overset{\forall\chi_k(x)\neq 0}{\therefore} \\
i\hbar\frac{\partial}{\partial t}f_k(t) = i\hbar\frac{d}{dt}f_k(t) &= E_k f_k(t),
\end{aligned} \tag{95}$$

from which one infers $\forall\chi_k(x) \neq 0 \in \mathbb{C}$, the solution of the relevant differential equation that turns out to emerge to be solved, Eq. (95), did not require any correlation to a solution related to a differential equation for $\chi_k(x)$, as for the q-dialectical domain, this latter related to the Eqs. within the Eq. (42), viz.: Eqs. (43) and (44) correlated by the separation constant E_k, as discussed throughout the previous section. Hence, one trivially solves the Eq. (95), i.e.:

$$\begin{aligned}
i\hbar\frac{df_k(t)}{f_k(t)} &= E_k dt \Rightarrow \\
i\hbar\int_\tau^t\frac{df_k(t)}{f_k(t)} &= E_k\int_\tau^t dt \Rightarrow \\
i\hbar\ln\left[f_k(t)\right]\Big|_\tau^t &= E_k t\Big|_\tau^t \Rightarrow \\
i\hbar\ln\left[f_k(t)\right] - i\hbar\ln\left[f_k(\tau)\right] &= E_k t - E_k\tau \Rightarrow \\
i\hbar\left\{\ln\left[f_k(t)\right] - \ln\left[f_k(\tau)\right]\right\} &= E_k t - E_k\tau \Rightarrow \\
\ln\left[f_k(t)\right] - \ln\left[f_k(\tau)\right] &= \frac{E_k t}{i\hbar} - \frac{E_k\tau}{i\hbar} \Rightarrow \\
\ln\left[\frac{f_k(t)}{f_k(\tau)}\right] &= \frac{E_k t}{i^2\hbar}i - \frac{E_k\tau}{i^2\hbar}i \Rightarrow \\
\ln\left[\frac{f_k(t)}{f_k(\tau)}\right] &= -\frac{iE_k t}{\hbar} + \frac{iE_k\tau}{\hbar} \Rightarrow \\
\frac{f_k(t)}{f_k(\tau)} &= e^{iE_k\tau/\hbar}e^{-iE_k t/\hbar} \quad \therefore
\end{aligned} \tag{96}$$

$$f_k(t) = f_k(\tau)\,e^{iE_k\tau/\hbar}e^{-iE_k t/\hbar}, \tag{97}$$

from which and from the Eq. (94):

$$\Psi_k(x,t) = f_k(\tau)\,e^{iE_k\tau/\hbar}e^{-iE_k t/\hbar}\chi_k(x), \tag{98}$$

$\forall\chi_k(x) \neq 0 \in \mathbb{C}$. Now, also in virtue of the Eq. (20), one applies a superposition as valid for a quantical cause, asseverating the condition given by the boundary classicality that will be instrumentally imposed, given by the Eq. (91), will turn out to accomplish the effect given by the right-hand side of the Eq. (20); dialectically, the effect, given by the razor for dialectics: Nature, from which, firstly:

$$\begin{aligned}
\Psi(x,t) &= \sum_{\forall k}\gamma_k\Psi_k(x,t) \\
&\overset{\text{Eq. (98)}}{=} \sum_{\forall k}\gamma_k f_k(\tau)\,e^{iE_k\tau/\hbar}e^{-iE_k t/\hbar}\chi_k(x),
\end{aligned} \tag{99}$$

where the γ_k's, $\forall\ k$, are the coefficients of the superposition. Now, defining, with no loss of generality:

$$\gamma_k f_k(\tau)\,e^{iE_k\tau/\hbar} \equiv a_k^+ \in \mathbb{C}, \tag{100}$$

one simply rewrites the Eq. (99), for the solution of the Eq. (34) $\forall\ t \in \mathcal{I}_c$:

$$\Psi(x,t) = \sum_{\forall k} a_k^+ e^{-iE_k t/\hbar}\chi_k(x), \tag{101}$$

with no imposition, yet, of the condition given by the Eq. (91). Now, in virtue of the Eqs. (91) and (101), a c-dialectical description $\forall\ t \in \mathcal{I}_c = [\tau, +\infty)$ implies:

$$\sum_{\forall k} a_k^+ e^{-iE_k t/\hbar} \chi_k(x) = e^{-iE_c t/\hbar} \chi_c(x), \quad (102)$$

$\forall\ c \in \{k\}$, being c c-classically fixed *ex post*, from which one concludes that:

$$a_k^+ e^{-iE_k t/\hbar} = \delta_k^c e^{-iE_k t/\hbar}, \quad (103)$$

where δ_k^c is the Kronecker delta: $\delta_k^c = 1$ for $c = k$, $\delta_k^c = 0$ otherwise, from which $\forall\ t \in \mathcal{I}_c$:

$$a_k^+ = \delta_k^c. \quad (104)$$

Back to the Eq. (101), one obtains the solution for the Eq. (34) over the c-dialectical domain $\mathcal{I}_c$:

$$\begin{aligned} \Psi(x,t) &= \sum_{\forall k} \delta_k^c e^{-iE_k t/\hbar} \chi_k(x) \\ &= \delta_c^c e^{-iE_c t/\hbar} \chi_c(x) = e^{-iE_c t/\hbar} \chi_c(x). \end{aligned} \quad (105)$$

Hence, one reaches the general solution of the Eq. (34), for $V_p(x,t) = V_p(x)$, as discussed before, in virtue of the Eqs. (50), (53), (57) [these over the chronology $\mathcal{I}_q$] and (105) [this over the chronology $\mathcal{I}_c$], i.e.:

$$\forall t \in \mathcal{I}_q \cup \mathcal{I}_c = (-\infty, \tau) \cup [\tau, +\infty) = \mathbb{R}: \ \Psi(x,t) = (1 - \delta_{t\bar{t}}) \sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) + \delta_{t\bar{t}} \sum_{\forall k} \delta_k^c e^{-iE_k t/\hbar} \chi_k(x), \quad (106)$$

given the Eqs. (50), (53), (54), (57) [these over $\mathcal{I}_q$, to establish the physical problem generating the a_k coefficients], and the Eqs. (13) and (14). In the next section, we will verify the validity of the probability conservation for the entire chronology.

ON THE PROBABILITY CONSERVATION OVER $t \in \mathcal{I}_q \cup \mathcal{I}_c$

Putting the solution for the Eq. (34), given by the Eq. (106), within the Eq. (75), one has got:

$$\begin{aligned} a_S^* a_S = \int_{-\infty}^{\infty} &\left[(1 - \delta_{t\bar{t}}) \sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) + \delta_{t\bar{t}} \sum_{\forall k} \delta_k^c e^{-iE_k t/\hbar} \chi_k(x)\right]^* \times \\ &\times \left[(1 - \delta_{t\bar{t}}) \sum_{\forall l} a_l e^{-iE_l t/\hbar} \phi_l(x) + \delta_{t\bar{t}} \sum_{\forall l} \delta_l^c e^{-iE_l t/\hbar} \chi_l(x)\right] dx = 1 \end{aligned} \quad (107)$$

For $t \in \mathcal{I}_q$, one has, in virtue of the Eq. (13), that $\delta_{t\bar{t}} = 0$, from which the Eq. (107) reads:

$$\begin{aligned}
a_S^* a_S &= \int_{-\infty}^{\infty} \left[\sum_{\forall k} a_k e^{-iE_k t/\hbar} \phi_k(x) \right]^* \sum_{\forall l} a_l e^{-iE_l t/\hbar} \phi_l(x)\, dx \\
&= \int_{-\infty}^{\infty} \sum_{\forall k} \sum_{\forall l} a_k^* a_l e^{-i(E_l - E_k)t/\hbar} \phi_k^*(x)\, \phi_l(x)\, dx \\
&= \sum_{\forall k} \sum_{\forall l} a_k^* a_l e^{-i(E_l - E_k)t/\hbar} \int_{-\infty}^{\infty} \phi_k^*(x)\, \phi_l(x)\, dx \\
&= \sum_{\forall k} \sum_{\forall l} a_k^* a_l e^{-i(E_l - E_k)t/\hbar} \delta_{kl} \\
&= \sum_{\forall k} a_k^* a_k e^{-i(E_k - E_k)t/\hbar} \delta_{kk} \\
&= \sum_{\forall k} a_k^* a_k \overset{\text{Eq. (71)}}{=} 1. \qquad (108)
\end{aligned}$$

Hence:

$$\forall t \in \mathcal{I}_q : \int_{-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx = \sum_{\forall k} a_k^* a_k = 1. \qquad (109)$$

For $t \in \mathcal{I}_c$, one has, in virtue of the Eq. (14), that $\delta_{t\bar{t}} = 1$, from which the Eq. (107) reads:

$$\begin{aligned}
a_S^* a_S &= \int_{-\infty}^{\infty} \left[\sum_{\forall k} \delta_k^c e^{-iE_k t/\hbar} \chi_k(x) \right]^* \sum_{\forall l} \delta_l^c e^{-iE_l t/\hbar} \chi_l(x)\, dx \\
&= \int_{-\infty}^{\infty} \sum_{\forall k} \sum_{\forall l} \delta_k^c \delta_l^c e^{-i(E_l - E_k)t/\hbar} \chi_k^*(x)\, \chi_l(x)\, dx \\
&= \sum_{\forall k} \sum_{\forall l} \delta_k^c \delta_l^c e^{-i(E_l - E_k)t/\hbar} \int_{-\infty}^{\infty} \chi_k^*(x)\, \chi_l(x)\, dx \\
&= \sum_{\forall k} \sum_{\forall l} \delta_k^c \delta_l^c e^{-i(E_l - E_k)t/\hbar} \delta_{kl} \\
&= \sum_{\forall k} \delta_k^c \delta_c^c e^{-i(E_c - E_k)t/\hbar} \delta_{kc} \\
&= \sum_{\forall k} \left(\delta_k^c\right)^2 e^{-i(E_c - E_k)t/\hbar} \\
&= \left(\delta_c^c\right)^2 e^{-i(E_c - E_c)t/\hbar} \\
&= \left(\delta_c^c\right)^2 = (1)^2 = 1, \qquad (110)
\end{aligned}$$

where the separated solutions $\chi_k(x)$, for $t \in \mathcal{I}_c$, are orthonormalized taken:

$$\int_{-\infty}^{\infty} \chi_k^*(x)\, \chi_l(x)\, dx = \delta_{kl}. \qquad (111)$$

From the Eqs. (108) and (110), the probability is conserved $\forall\ t$. Also, one may take:

$$\chi_k(x) \equiv \phi_k(x), \qquad (112)$$

within the Eq. (106), in virtue of the arbitrarity related to the separated functions $\chi_k(x)$, as discussed in the previous section, once the eigenfunctions $\phi_k(x)$ are orhonormalized, obviously satisfying the Eq. (111), from which the general solution of the Eq. (34), Eq. (106), turns out to read:

$$\forall t \in \mathcal{I}_q \cup \mathcal{I}_c = (-\infty, \tau) \cup [\tau, +\infty) = \mathbb{R}: \; \Psi(x,t) = \sum_{\forall k} \left[(1-\delta_{t\bar{t}})\, a_k + \delta_{t\bar{t}} \delta_k^c\right] e^{-iE_k t/\hbar} \phi_k(x). \qquad (113)$$

From the Eq. (113), one takes the general coefficient of superposition $\forall\ t$:

$$\beta_k = \left[(1-\delta_{t\bar{t}})\, a_k + \delta_{t\bar{t}} \delta_k^c\right], \qquad (114)$$

to which one also infers:

$$\begin{aligned}
\sum_{\forall k} \beta_k^* \beta_k &= \sum_{\forall k} \left[(1-\delta_{t\bar{t}})\, a_k + \delta_{t\bar{t}} \delta_k^c\right]^* \left[(1-\delta_{t\bar{t}})\, a_k + \delta_{t\bar{t}} \delta_k^c\right] = \sum_{\forall k} \left[(1-\delta_{t\bar{t}})\, a_k^* + \delta_{t\bar{t}} \delta_k^c\right] \left[(1-\delta_{t\bar{t}})\, a_k + \delta_{t\bar{t}} \delta_k^c\right] \\
&= \sum_{\forall k} \left[(1-\delta_{t\bar{t}})^2 a_k^* a_k + \delta_{t\bar{t}} (1-\delta_{t\bar{t}})\, a_k^* \delta_k^c + \delta_{t\bar{t}} (1-\delta_{t\bar{t}})\, a_k \delta_k^c + (\delta_{t\bar{t}})^2 (\delta_k^c)^2\right] \\
&= (1-\delta_{t\bar{t}})^2 \sum_{\forall k} a_k^* a_k + \delta_{t\bar{t}} (1-\delta_{t\bar{t}}) \sum_{\forall k} \delta_k^c (a_k^* + a_k) + (\delta_{t\bar{t}})^2 \sum_{\forall k} (\delta_k^c)^2 \\
&= (1-\delta_{t\bar{t}})^2 + 2\delta_{t\bar{t}} (1-\delta_{t\bar{t}}) \sum_{\forall k} \delta_k^c \mathbf{Re}[a_k] + (\delta_{t\bar{t}})^2, \qquad (115)
\end{aligned}$$

where we have used the Eq. (71) [remembering the a_k's are the coefficients of superposition over the q-dialectical domain, i.e., $\forall\ t \in \mathcal{I}_q$] and the fact that δ_k^c is the Kronecker delta [from which $\delta_k^c = 1$ for $c = k$, $\delta_k^c = 0$ otherwise]. The second term in the right-hand side of the Eq. (115) vanishes, once $\mathcal{I}_q \cap \mathcal{I}_c = \emptyset$ [cf. the Eqs. (13) and (14)] in spite of $\mathbf{Re}[a_k]$, the real part of a_k [3]. Hence, the Eq. (115) reads:

$$\sum_{\forall k} \beta_k^* \beta_k = (1-\delta_{t\bar{t}})^2 + (\delta_{t\bar{t}})^2 = \left|(1-\delta_{t\bar{t}}) + i\,(\delta_{t\bar{t}})\right|^2. \qquad (116)$$

Of course the Eq. (116) is coherent, since, in virtue of the Eqs. (13) and (14):

$$\forall t \in \mathcal{I}_q = (-\infty, \tau): \; \delta_{t\bar{t}} = 0 \overset{\text{Eq. (116)}}{\Rightarrow} \sum_{\forall k} \beta_k^* \beta_k = 1, \qquad (117)$$

or:

$$\forall t \in \mathcal{I}_c = [\tau, +\infty): \; \delta_{t\bar{t}} = 1 \overset{\text{Eq. (116)}}{\Rightarrow} \sum_{\forall k} \beta_k^* \beta_k = 1, \qquad (118)$$

i.e., the probability is conserved, as previous and equivalently demonstred through the marches that led to the Eqs. (108) and (110). Furthermore, it is instructive to infer the coherence of the Eq. (116) via the imposition:

$$\sum_{\forall k} \beta_k^* \beta_k = (1-\delta_{t\bar{t}})^2 + (\delta_{t\bar{t}})^2 = 1, \qquad (119)$$

from which:

$$\begin{aligned}
1 - 2\delta_{t\bar{t}} + (\delta_{t\bar{t}})^2 + (\delta_{t\bar{t}})^2 &= 1 \Rightarrow \\
2(\delta_{t\bar{t}})^2 - 2\delta_{t\bar{t}} &= 0 \Rightarrow \\
2\delta_{t\bar{t}} (\delta_{t\bar{t}} - 1) &= 0 \Rightarrow \\
\delta_{t\bar{t}} (\delta_{t\bar{t}} - 1) &= 0. \qquad (120)
\end{aligned}$$

Hence, the Eq. (120) has the solution-set:

$$\delta_{t\bar{t}} \in \{0, 1\}, \qquad (121)$$

in accordance with the Eqs. (13) and (14), also asseverating the mutual exclusivity between the chronological sets: $\mathcal{I}_q \cap \mathcal{I}_c = \emptyset$, in that sense that was so well emphasized through a chapter by the Prof. Dr. Niels Bohr in [4], profoundly connected to the [Natural] Complementarity Principle of Bohr: the corpuscular and wavelike aspects are complementary ones, so that both are necessary ones, but they cannot be simultaneously observed.

CONSIDERATIONS AT $t = \tau$

Under a distribution context, throughout this section, we will investigate the structure of the Schrodinger equation, within the resonings of this paper, around $t = \tau$, which we have got considered as being the instant from which the physical system under scrutinity turns out to present a c-dialectical description.

Firstly, the solution we have obtained, given by the Eq. (113), reads:

$$\forall t \in \mathcal{I}_q \cup \mathcal{I}_c = (-\infty, \tau) \cup [\tau, +\infty) = \mathbb{R} : \ \Psi(x,t) = \sum_{\forall k} \left[(1-\delta_{t\bar{t}}) a_k + \delta_{t\bar{t}} \delta_k^c \right] e^{-iE_k t/\hbar} \phi_k(x) , \tag{122}$$

from which, for the energy operator $i\hbar\partial/\partial t$, we are firstly interested in the quantity:

$$\begin{aligned} & i\hbar \ \left[\Psi(x, \tau + \theta/2) - \Psi(x, \tau - \theta/2) \right] (1/\theta) \overset{!}{\cong} \\ & \overset{!}{\cong} \ i\hbar \frac{\partial}{\partial t} \Psi(x,t) \bigg|_{t=\tau} , \end{aligned} \tag{123}$$

at $t = \tau$, with $\overset{!}{\cong}$ meaning $0 < \theta \to 0$. Hence, from the Eqs. (122) and (123), one has got:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi(x,t) \bigg|_{t=\tau} & \overset{!}{\cong} \ i\hbar \left\{ \left\{ \sum_{\forall k} \left[\left(1 - \delta_{(\tau+\theta/2)\bar{t}} \right) a_k + \delta_{(\tau+\theta/2)\bar{t}} \delta_k^c \right] e^{-iE_k(\tau+\theta/2)/\hbar} \phi_k(x) + \right. \right. \\ & - \left. \left. \sum_{\forall k} \left[\left(1 - \delta_{(\tau-\theta/2)\bar{t}} \right) a_k + \delta_{(\tau-\theta/2)\bar{t}} \delta_k^c \right] e^{-iE_k(\tau-\theta/2)/\hbar} \phi_k(x) \right\} \frac{1}{\theta} \right\} \\ & = \ i\hbar \left\{ \left\{ \sum_{\forall k} \delta_k^c e^{-iE_k(\tau+\theta/2)/\hbar} \phi_k(x) - \sum_{\forall k} a_k e^{-iE_k(\tau-\theta/2)/\hbar} \phi_k(x) \right\} \frac{1}{\theta} \right\} , \end{aligned} \tag{124}$$

where we have used the Eqs. (13) and (14). Now, applying Taylor series to represent:

$$\begin{aligned} e^{-iE_k(\tau+\theta/2)/\hbar} & = \ e^{-iE_k\tau/\hbar} e^{-iE_k\theta/(2\hbar)} \\ & = \ e^{-iE_k\tau/\hbar} \left[\sum_{\forall l} \frac{1}{l!} \left(-\frac{iE_k\theta}{2\hbar} \right)^l \right] ; \end{aligned} \tag{125}$$

$$\begin{aligned} e^{-iE_k(\tau-\theta/2)/\hbar} & = \ e^{-iE_k\tau/\hbar} e^{iE_k\theta/(2\hbar)} \\ & = \ e^{-iE_k\tau/\hbar} \left[\sum_{\forall l} \frac{1}{l!} \left(\frac{iE_k\theta}{2\hbar} \right)^l \right] , \end{aligned} \tag{126}$$

one reaches, in virtue of the Eq. (124):

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) \bigg|_{t=\tau} \overset{!}{\cong} i\hbar \left\{ \left\{ \sum_{\forall k} \delta_k^c e^{-iE_k\tau/\hbar} \left[\sum_{\forall l} \frac{1}{l!} \left(-\frac{iE_k\theta}{2\hbar} \right)^l \right] \phi_k(x) - \sum_{\forall k} a_k e^{-iE_k\tau/\hbar} \left[\sum_{\forall l} \frac{1}{l!} \left(\frac{iE_k\theta}{2\hbar} \right)^l \right] \phi_k(x) \right\} \frac{1}{\theta} \right\} . \tag{127}$$

Since $\theta \to 0$:

$$\begin{aligned} \left[\frac{1}{\theta} \sum_{\forall l} \frac{1}{l!} \left(-\frac{iE_k\theta}{2\hbar} \right)^l \right] & = \ \sum_{\forall l} \frac{1}{l!} \left(-\frac{iE_k}{2\hbar} \right)^l \theta^{l-1} = \left[\frac{1}{\theta} + \left(-\frac{iE_k}{2\hbar} \right) \right] + \sum_{\forall l \geq 2} \frac{1}{l!} \left(-\frac{iE_k}{2\hbar} \right)^l \theta^{l-1} \\ & \overset{!}{\cong} \ \frac{1}{\theta} - \frac{iE_k}{2\hbar} ; \end{aligned} \tag{128}$$

$$\left[\frac{1}{\theta}\sum_{\forall l}\frac{1}{l!}\left(\frac{iE_k\theta}{2\hbar}\right)^l\right] = \sum_{\forall l}\frac{1}{l!}\left(\frac{iE_k}{2\hbar}\right)^l\theta^{l-1} = \left[\frac{1}{\theta}+\left(\frac{iE_k}{2\hbar}\right)\right]+\sum_{\forall l\geq 2}\frac{1}{l!}\left(\frac{iE_k}{2\hbar}\right)^l\theta^{l-1}$$
$$\overset{!}{\cong} \frac{1}{\theta}+\frac{iE_k}{2\hbar}. \tag{129}$$

Substituting the results from the Eqs. (128) and (129) within the Eq. (127), one reaches:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)\Big|_{t=\tau} \overset{!}{\cong} i\hbar\sum_{\forall k}\delta_k^c\left(\frac{1}{\theta}-\frac{iE_k}{2\hbar}\right)e^{-iE_k\tau/\hbar}\phi_k(x) - i\hbar\sum_{\forall k}a_k\left(\frac{1}{\theta}+\frac{iE_k}{2\hbar}\right)e^{-iE_k\tau/\hbar}\phi_k(x)$$
$$= i\hbar\sum_{\forall k}\left(\frac{\delta_k^c-a_k}{\theta}\right)e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}\frac{1}{2}E_k\left(\delta_k^c+a_k\right)e^{-iE_k\tau/\hbar}\phi_k(x)$$
$$= i\hbar\sum_{\forall k}\left(\frac{\delta_k^c-a_k}{\theta}\right)e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}E_k\left(\delta_k^c-\frac{1}{2}\delta_k^c+\frac{1}{2}a_k\right)e^{-iE_k\tau/\hbar}\phi_k(x)$$
$$= \sum_{\forall k}E_k\delta_k^c e^{-iE_k\tau/\hbar}\phi_k(x)+\frac{1}{2}\sum_{\forall k}\left(a_k-\delta_k^c\right)E_k e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}i\hbar\left(\frac{\delta_k^c-a_k}{\theta}\right)e^{-iE_k\tau/\hbar}\phi_k(x)$$
$$= \sum_{\forall k}E_k\delta_k^c e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}i\hbar\left[\frac{(\delta_k^c-a_k)}{\theta}-\frac{(\delta_k^c-a_k)}{2}\frac{E_k}{i\hbar}\right]e^{-iE_k\tau/\hbar}\phi_k(x)$$
$$= \sum_{\forall k}E_k\delta_k^c e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}i\hbar\frac{(\delta_k^c-a_k)}{\theta}\left(1+i\frac{E_k}{2\hbar}\theta\right)e^{-iE_k\tau/\hbar}\phi_k(x) \tag{130}$$
$$= \sum_{\forall k}E_k\delta_k^c e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}i\hbar\frac{d\beta_k}{dt}\Big|_{t=\tau}z(\theta)\,e^{-iE_k\tau/\hbar}\phi_k(x), \tag{131}$$

where:

$$z(\theta)\equiv 1+i\frac{E_k}{2\hbar}\theta\overset{!}{\cong}1; \tag{132}$$

and:

$$\beta_k|_{t=\tau^+} = \delta_k^c, \tag{133}$$
$$\beta_k|_{t=\tau^-} = a_k; \tag{134}$$

$$\tau^+ = \tau+\frac{\theta}{2},\ \tau^-=\tau-\frac{\theta}{2}, \tag{135}$$

in virtue of the Eqs. (13), (14) and (114), from which:

$$\frac{d\beta_k}{dt}\Big|_{t=\tau}\overset{!}{\cong}\frac{(\delta_k^c-a_k)}{\theta}. \tag{136}$$

Hence, the Eq. (131) turns out to read, as $\theta\to 0$:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)\Big|_{t=\tau}=\sum_{\forall k}E_k\delta_k^c e^{-iE_k\tau/\hbar}\phi_k(x)+\sum_{\forall k}i\hbar\frac{d\beta_k}{dt}\Big|_{t=\tau}e^{-iE_k\tau/\hbar}\phi_k(x). \tag{137}$$

In fact, in virtue of the Eq. (113):

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi\left(x,t\right) &= i\hbar\frac{\partial}{\partial t}\sum_{\forall k}\beta_k e^{-iE_k t/\hbar}\phi_k\left(x\right) \overset{\text{Eq. (114)}}{=} i\hbar\frac{\partial}{\partial t}\sum_{\forall k}\left[\left(1-\delta_{t\bar{t}}\right)a_k+\delta_{t\bar{t}}\delta_k^c\right]e^{-iE_k t/\hbar}\phi_k\left(x\right) \\
&= i\hbar\sum_{\forall k}\left\{\left[\left(1-\delta_{t\bar{t}}\right)a_k+\delta_{t\bar{t}}\delta_k^c\right]\frac{\partial}{\partial t}\left[e^{-iE_k t/\hbar}\phi_k\left(x\right)\right]+e^{-iE_k t/\hbar}\phi_k\left(x\right)\frac{\partial}{\partial t}\left[\left(1-\delta_{t\bar{t}}\right)a_k+\delta_{t\bar{t}}\delta_k^c\right]\right\} \\
&= i\hbar\sum_{\forall k}\left\{\left[\left(1-\delta_{t\bar{t}}\right)a_k+\delta_{t\bar{t}}\delta_k^c\right]\left[\phi_k\left(x\right)\frac{\partial}{\partial t}\left(e^{-iE_k t/\hbar}\right)\right]+e^{-iE_k t/\hbar}\phi_k\left(x\right)\left[a_k\frac{\partial}{\partial t}\left(1-\delta_{t\bar{t}}\right)+\delta_k^c\frac{\partial}{\partial t}\delta_{t\bar{t}}\right]\right\} \\
&= i\hbar\sum_{\forall k}\left\{\left[\left(1-\delta_{t\bar{t}}\right)a_k+\delta_{t\bar{t}}\delta_k^c\right]\left[\left(-\frac{iE_k}{\hbar}\right)e^{-iE_k t/\hbar}\phi_k\left(x\right)\right]+e^{-iE_k t/\hbar}\phi_k\left(x\right)\left[-a_k\frac{\partial}{\partial t}\delta_{t\bar{t}}+\delta_k^c\frac{\partial}{\partial t}\delta_{t\bar{t}}\right]\right\} \\
&= \sum_{\forall k}\left[\left(1-\delta_{t\bar{t}}\right)a_k+\delta_{t\bar{t}}\delta_k^c\right]E_k e^{-iE_k t/\hbar}\phi_k\left(x\right)+\sum_{\forall k}i\hbar\left[\left(\delta_k^c-a_k\right)\frac{\partial}{\partial t}\delta_{t\bar{t}}\right]e^{-iE_k t/\hbar}\phi_k\left(x\right)\ \therefore \\
i\hbar\left.\frac{\partial}{\partial t}\Psi\left(x,t\right)\right|_{t=\tau} &= \sum_{\forall k}\left[\left(1-\delta_{\tau\bar{t}}\right)a_k+\delta_{\tau\bar{t}}\delta_k^c\right]E_k e^{-iE_k\tau/\hbar}\phi_k\left(x\right)+\sum_{\forall k}i\hbar\left(\delta_k^c-a_k\right)\left.\frac{\partial}{\partial t}\delta_{t\bar{t}}\right|_{t=\tau}e^{-iE_k\tau/\hbar}\phi_k\left(x\right) \\
&= \sum_{\forall k}E_k\delta_k^c e^{-iE_k\tau/\hbar}\phi_k\left(x\right)+\sum_{\forall k}i\hbar\left(\delta_k^c-a_k\right)\left.\frac{\partial}{\partial t}\delta_{t\bar{t}}\right|_{t=\tau}e^{-iE_k\tau/\hbar}\phi_k\left(x\right),
\end{aligned}
\tag{138}
$$

since:

$$\delta_{\tau\bar{t}}=1, \tag{139}$$

in virtue of the Eq. (14). Furthermore:

$$\frac{\partial}{\partial t}\delta_{t\bar{t}}=\frac{d}{dt}\delta_{t\bar{t}}=\delta\left(t-\tau\right), \tag{140}$$

where $\delta\left(t-\tau\right)$ is the Dirac delta function, to be discussed below in terms of delta sequences, but, for now, we will simply write down:

$$\delta\left(x\right) = \infty,\ \text{for}\ x\in\mathbb{R}\wedge x=0; \tag{141}$$
$$\delta\left(x\right) = 0\ ,\ \text{for}\ x\in\mathbb{R}\wedge x\neq 0, \tag{142}$$

which raises the high peak sifting property of transition between dialectical descriptions for the physical system, which would be referred as a discontinuous transition under the usual definition for continuous functions, but, once understood as a distribution, it is the highly concentrated behaviour that naturally occurs, which is quite common in Nature. Hence, from the Eqs. (137), (138) and (140), one obtains:

$$\left.\frac{d}{dt}\beta_k\right|_{t=\tau}=\left(\delta_k^c-a_k\right)\left.\frac{d}{dt}\delta_{t\bar{t}}\right|_{t=\tau}=\left(\delta_k^c-a_k\right)\left.\delta\left(t-\tau\right)\right|_{t=\tau}, \tag{143}$$

where:

$$
\begin{aligned}
\left.\frac{\partial}{\partial t}\delta_{t\bar{t}}\right|_{t=\tau} &= \left.\frac{d}{dt}\delta_{t\bar{t}}\right|_{t=\tau}=\left.\delta\left(t-\tau\right)\right|_{t=\tau}=\delta\left(0s\right) \\
&= s^{-1}\delta\left(0\right),
\end{aligned}
\tag{144}
$$

i.e., a highly concentrated distribution over the chronological set $\left(-\infty,\tau\right)\cup\left[\tau,+\infty\right)$.

Within the ordinary physics, one may construct the Dirac delta function via delta sequences to study its highly peaked sifting property, but one is usually not so quite interested in the details that occur at, say, in an analogy with our case being discussed in this paper, τ, i.e., e.g., an instantaneous localized force, i.e., an impulsive force that frequently occur in classical mechanics, may be directly modelled by the use of the delta function, once the physical relevance turns out to be the linear momentum variation, the impulse obtained by the integration of the impulsive force modelled by the delta function which neglects the details around the instant τ at which the impulsive force is applied. This is a mechanically idealized scenario, and, so far one is not interested in the details within the impulsive process, the delta function turns out to be a quite good mathematical object to mathematically describe this localized phenomenon.

But in our case being discussed in this paper we will be interested in the details around τ, which will be discussed in the next section.

ON THE COLLAPSE OF THE WAVE FUNCTION

Let $\Psi\left(x,t\right)$ be the wave function of a quantum object. What is the *utility* for an observer i, concerning this wave function as a representation of a quantum object? The level of representation must be the mathematical one, since this is the spirit of the mathematics in physics. By the way, the answer is within the question. Any observer would agree: the utility of $\Psi\left(x,t\right)$ is the representation of a quantum object. In a mathematical representation of a quantum object, a basis must be choosen. In any case, choosen or given/prescribed, the quantum object is absolute in existence, i.e., remains the same quantum object, remains independent of the basis choosen or given by anything external to it. Since the quantum object to be mathematically represented is absolute, a choosen basis must be equivalent to a given/prescribed one regarding this mathematical representation of $\Psi(x,t)$, $\forall$ observers i. Hence, the utility of $\Psi(x,t)$'s mathemathical representation must be exactly the same as the utility of $\Psi(x,t)$'s mathematical representation given a basis [5]. Now, we reformulate, equivalently, our question: Given a basis, what is the utility of the mathematical representation of $\Psi(x,t)$ for an observer? Any observer would agree: The utility is on the knowledge of the set of coefficients of the representation in the given basis, since one would not be able to mathematically construct a quantum object without these coefficients ([1], in Sect. 6).

This said, once the dialectics remains unchanged:

$$\frac{d}{dt}\beta_k = 0, \tag{145}$$

for an object $\Psi(x,t)$ mathematically represented by:

$$\Psi(x,t) = \sum_{\forall k} \beta_k \Psi_k(x,t). \tag{146}$$

Now, to change the dialectics, lets suppose an influence $f(t)$ acts on the system under scrutinity within the interval $(\tau-\theta/2, \tau+\theta/2)$, $\theta \in (0,+\infty)$, so that:

$$\frac{d}{dt}\beta_k = f(t), \ \tau - \frac{\theta}{2} < t < \tau + \frac{\theta}{2}. \tag{147}$$

Hence, one has, for the physical system under scrutinity:

$$\begin{aligned}
\frac{d}{dt}\beta_k &= 0, \ t < \tau - \frac{\theta}{2}; & (148)\\
\frac{d}{dt}\beta_k &= f(t), \ \tau - \frac{\theta}{2} < t < \tau + \frac{\theta}{2}; & (149)\\
\frac{d}{dt}\beta_k &= 0, \ t > \tau + \frac{\theta}{2}, & (150)
\end{aligned}$$

once well defined dialectics hold within $(-\infty,+\infty) - (\tau-\theta/2, \tau+\theta/2)$, but not necessarily the same ones. The Eqs. (148) and (150) have got the following solutions:

$$\begin{aligned}
\beta_k &= a_k, \ a_k \in \mathbb{C}, \ t < \tau - \frac{\theta}{2}; & (151)\\
\beta_k &= c_k, \ c_k \in \mathbb{C}, \ t > \tau + \frac{\theta}{2}, & (152)
\end{aligned}$$

where a_k and c_k are constants providing their respective constant dialectics within their respective dialectical domains. Since there is dialectical change due to $f(t)$ within the interval $(\tau-\theta/2, \tau+\theta/2)$, one cannot refer to an unique dialectical characterization within this interval, and it seems inteligibility lacks. Integrating the Eq. (149), one has got:

$$\begin{aligned}
\int_{\tau-\theta/2}^{\tau+\theta/2} d\beta_k &= \int_{\tau-\theta/2}^{\tau+\theta/2} f(t)\,dt \Rightarrow\\
\beta_k(\tau+\theta/2) &= \beta_k(\tau-\theta/2) + \int_{\tau-\theta/2}^{\tau+\theta/2} f(t)\,dt
\end{aligned} \tag{153}$$

Once one expects spacetime as being homogeneous, specially in relation to the chronological domain, since there is no reason to suppose a measure that is accomplished at a given instant τ turns out to be different from any other measure at any other instant within $(-\infty,+\infty)$, mantained the conditions for reprodutibility, viz., mantained the same characteristics of any of the identical designed apparata that eventually is used to scrutinize the physical system under consideration. In fact, under chronological homogeneity, the assertion that for a given instant at t for a measure that will be accomplished at a future instant τ in relation to t one has got its future symmetrical instant $\tau + (\tau - t) = 2\tau - t$ in relation to τ such that:

- the [preterit, in relation to τ] chronologically probabilistical weight:

$$\pi(t \to \tau) = \frac{dt}{\tau - t}, \tag{154}$$

and:

- the [future, in relation to τ] chronologically probabilistical weight:

$$\pi[\tau \to (2\tau - t)] = \frac{dt}{(2\tau - t) - \tau} = \frac{dt}{\tau - t}, \tag{155}$$

as being the same ones, as required by an assumption of chronological homogeneity, viz., having got the same chronological partition $dt > 0$ [since one takes $|d(\tau - t)| = |-dt| = dt$, and t increases, and, also, it is the norm of an elementary piece of chronological interval $\tau - t$ that matters to calculate the chronological probability within each of the above mentioned intervals] over identically symmetrical [in relation to τ] intervals:

$$\tau - t = (2\tau - t) - \tau. \tag{156}$$

Hence, once considering t will symmetrical and futurely vary through τ from its presently instantaneous distance $\theta(t)/2 = \tau - t$ from τ, at $t < \tau$, being τ arbitrary, the symmetrically averaged value of $d\beta_k/dt$ over $(t, 2\tau - t)$ turns out to read:

$$\begin{aligned}
\left\langle \frac{d\beta_k}{dt} \right\rangle_{(t,2\tau-t)} &= \left(\int_t^\tau + \int_\tau^{2\tau-t} \right) \frac{d\beta_k}{d\xi} \frac{d\xi}{\tau - \xi}\\
&= \left(\int_t^\tau + \int_\tau^{2\tau-t} \right) \dot{\beta}_k(\xi) \frac{d\xi}{\tau - \xi},
\end{aligned} \tag{157}$$

where the superscripted dot represents the total differentiation in relation to time $[\xi]$. $\dot{\beta}_k(\xi)$ is being considered as a function of ξ. With a change of variable:

$$u(\xi) \equiv 2\tau - \xi, \tag{158}$$

the second integration at the right-hand side of the Eq. (157) turns out to read:

$$\begin{aligned}\int_{\tau}^{2\tau-t} \dot{\beta}_k(\xi)\frac{d\xi}{\tau-\xi} &= \int_{u(\tau)}^{u(2\tau-t)} \dot{\beta}_k(\xi(u))\frac{d\xi(u)}{\tau-\xi(u)}\\ &= \int_{\tau}^{t} \dot{\beta}_k(2\tau-u)\frac{(-du)}{u-\tau}\\ &= \int_{t}^{\tau} \dot{\beta}_k(2\tau-u)\frac{du}{u-\tau}\\ &= \int_{t}^{\tau} -\dot{\beta}_k(2\tau-u)\frac{du}{\tau-u}.\end{aligned} \tag{159}$$

Now, the variable u, in virtue of the Eq. (159), varies exactly as the variable ξ in the first integral at the right-hand side of the Eq. (157), viz., from t to τ, the reason by which one may relable $u \to \xi$ within the right-hand side of the Eq. (159), rewriting:

$$\int_{\tau}^{2\tau-t} \dot{\beta}_k(\xi)\frac{d\xi}{\tau-\xi} = \int_{t}^{\tau} -\dot{\beta}_k(2\tau-\xi)\frac{d\xi}{\tau-\xi}. \tag{160}$$

Henceforth, from the Eq. (160), the Eq. (157) reads:

$$\left\langle\frac{d\beta_k}{dt}\right\rangle_{(t,2\tau-t)} = \int_{t}^{\tau}\left[\dot{\beta}_k(\xi) - \dot{\beta}_k(2\tau-\xi)\right]\frac{d\xi}{\tau-\xi}. \tag{161}$$

Now, one turns out to be interested in an existence condition $\mathcal{E}$ such that:

$$\mathcal{E} \Leftrightarrow \exists \left\langle\frac{d\beta_k}{dt}\right\rangle_{(t,2\tau-t)}, \tag{162}$$

under the chronological homogeneity we are considering, which, physically, consubstantiates the existence of wavelike objects [mathematically represented wavelike objects as the ones in the Eq. (146), for which the superposition coefficients matter]. For this purpose, we define a new variable:

$$\zeta \equiv \tau - \xi \Rightarrow d\xi = -d\zeta, \tag{163}$$

[τ is a constant with time dimension] allowing to rewrite the Eq. (161):

$$\begin{aligned}\left\langle\frac{d\beta_k}{dt}\right\rangle_{(t,2\tau-t)} &= \int_{\tau-t}^{0}\left[\dot{\beta}_k(\tau-\zeta) - \dot{\beta}_k(\tau+\zeta)\right]\frac{(-d\zeta)}{\zeta}\\ &= \int_{0}^{\tau-t}\left[\dot{\beta}_k(\tau-\zeta) - \dot{\beta}_k(\tau+\zeta)\right]\frac{d\zeta}{\zeta}\\ &= \int_{0}^{\tau-t}\left[\dot{\beta}_k(\tau-\zeta) - \dot{\beta}_k(\tau+\zeta)\right]d\ln|\zeta|.\end{aligned} \tag{164}$$

Using the first mean value theorem, deliberately, for the integration at the right-hand side of the Eq. (164), one reaches:

$$\left\langle\frac{d\beta_k}{dt}\right\rangle_{(t,2\tau-t)} = \left[\dot{\beta}_k(\tau-\bar{\zeta}) - \dot{\beta}_k(\tau+\bar{\zeta})\right]\ln|\zeta|\Big|_{0}^{\tau-t}, \tag{165}$$

where $\left(\bar{\zeta} > 0\right) \in (0, \tau - t)$, which suggests a weaker condition for $\mathcal{E}$ [cf. Eqs. (161), (162) and (165) and the inherent discussion]:

$$\dot{\beta}_k(\tau+\zeta) = \dot{\beta}_k(\tau-\zeta), \tag{166}$$

with $(\zeta > 0) \in (0, \tau - t)$, i.e., being the function, $\dot{\beta}_k(\xi)$, even, symmetric, in relation to τ. A stronger condition would read:

$$\ddot{\beta}_k(\xi) = \frac{d^2}{d\xi^2}\beta_k(\xi) = 0,\ \forall\ \xi \in (t, 2\tau - t), \tag{167}$$

for which, $\dot{\beta}_k(\xi)$, is, also, an even function. If an instant τ turns out to be special in any sense, i.e., if τ turns out to separate two chronologically homogeneous regions, over τ's left and over τ's right, so that τ would be being a chronologically local boundary for different dialectics, one over (t, τ) and the other over $(\tau, 2\tau - t)$, $\forall\ t$, it would become a case for the condition stated by the Eq. (166); otherwise, the condition given by the Eq. (166) would apply $\forall\ \tau$, since there would not be any special τ, or, which is the same, with every τ identically special, a case in which a constant function $\dot{\beta}_k(\xi)$ would be required, the content of the Eq. (167) to satisfy the condition established by the Eq. (166), $\forall\ \tau$, and not just for an unique one. Now, we will study, more carefully, the Eq. (157) from a more general condition.

To begin with, the Eq. (157) may, in fact, be understood as being generated by the following differential equation for $d\beta_k/dt$ [we are back to the original notation for time, t, instead of ξ, since we are to work out the infinitesimal version of the Eq. (157) around τ (being τ arbitrary), as, below, we are to see and to define]:

$$\frac{d}{dt}\dot{\beta}_k = \frac{\dot{\beta}_k}{\tau - t} = -\frac{\dot{\beta}_k}{t-\tau}, \tag{168}$$

where the interpretation for $\left\langle\dot{\beta}_k\right\rangle$, left-hand side of the Eq. (157), emerges under the construction we discussed, leading to the integration of the Eq. (168), which is the infinitesimal version of that former Eq. (157), around τ, where:

$$\frac{d}{dt}\beta_k \equiv \dot{\beta}_k, \tag{169}$$

viz., as before, the superscripted dot represents the total differentiation in relation to time. The Eq. (157) is elementary, singular, due to the following aspects:

- (i): Is singular, since the integral related to it is critical at τ [cf. the previous discussion leading to the Eqs. (165), (166) and (167), as well as the right-hand side of the Eq. (168) which turns out to originate that discussion on criticality];

- (ii): Is singular, once the probability distribution given by the Eq. (154) [and (155)] is unique, viz., does not depend on which [specific] τ is choosen to accomplish a measure [one infers it symmetrically depends on the chronological distance to τ], once the chronological domain is taken as homogeneous, from which the solution of the Eq. (168) will not depend on specific initial conditions, viz., it must remain valid under any specific situaton [if one want to make a reference to Born's rule here, for instructive purposes, one turns out to infer this rule establishes the probabilistical meaning for measures, given a solution for the Schrodinger equation, with no reference to specific instants; the homogeneity, here, is related to the fact that, under Born's rule, there is not any chronological bias. But, in fact, there is a chronological bias in the sense the wave function turns out to change its dialectical description in virtue of a potential energy operator related to this change of description at τ: it seems some sort of paradoxical objectivity raises, which will be discussed and solved within this section];

- (iii): Is singular, once the Eq. (168) will turn out to be compatible with the characteristic of a distribution, as we will see, even with its [apparent] incompatibility with continuity in the usual mathematical sense.

We start with the verification of the assertion we have pointed out by the item (iii) above. To verify that $\dot{\beta}_k\left(t\right)$ is compatible with a distribution characteristic, we, firstly, will solve the Eq. (168). Its solution reads:

$$\dot{\beta}_k\left(t\right)=\frac{\gamma_k}{\left|t-\tau\right|}=\frac{\gamma_k}{\sqrt{\left(t-\tau\right)^2}}, \tag{170}$$

with $\gamma_k\in\mathbb{C}$, as one may verify by substitution. In fact, the left-hand side of the Eq. (168) reads:

$$\begin{aligned}\frac{d}{dt}\dot{\beta}_k &\overset{\text{Eq. (170)}}{=} \frac{d}{dt}\left(\frac{\gamma_k}{\left|t-\tau\right|}\right)=\frac{d}{dt}\left[\frac{\gamma_k}{\sqrt{\left(t-\tau\right)^2}}\right]=\gamma_k\frac{d}{dt}\left\{\left[\left(t-\tau\right)^2\right]^{-1/2}\right\}=\gamma_k\left(-\frac{1}{2}\right)\left[\left(t-\tau\right)^2\right]^{-3/2}\left(2\right)\left(t-\tau\right)\\ &= -\gamma_k\frac{t-\tau}{\left[\left(t-\tau\right)^2\right]^{3/2}}=-\gamma_k\frac{t-\tau}{\left[\sqrt{\left(t-\tau\right)^2}\right]^3}=-\frac{\gamma_k\left(t-\tau\right)}{\left|t-\tau\right|^3},\end{aligned} \tag{171}$$

and the right-hand side of the Eq. (168) also reads:

$$-\frac{\dot{\beta}_k}{t-\tau}\overset{\text{Eq. (170)}}{=}-\frac{\gamma_k}{\left(t-\tau\right)\left|t-\tau\right|}=-\frac{\gamma_k}{\left(t-\tau\right)\left|t-\tau\right|}\frac{t-\tau}{t-\tau}=-\frac{\gamma_k\left(t-\tau\right)}{\left(t-\tau\right)^2\left|t-\tau\right|}=-\frac{\gamma_k\left(t-\tau\right)}{\left|t-\tau\right|^2\left|t-\tau\right|}=-\frac{\gamma_k\left(t-\tau\right)}{\left|t-\tau\right|^3}. \tag{172}$$

The Eqs. (171) and (172) lead to the Eq. (168), as required. We may verify the condition required for $\beta_k\left(\xi\right)$ as being an even function in relation to τ, Eq. (166), is satisfied by the solution given by the Eq. (170):

$$\begin{aligned}\dot{\beta}_k\left(\tau+\zeta\right) &= \left.\frac{\gamma_k}{\left|\xi-\tau\right|}\right|_{\xi=\tau+\zeta}=\frac{\gamma_k}{\left|\left(\tau+\zeta\right)-\tau\right|}=\frac{\gamma_k}{\left|\zeta\right|},\\ \dot{\beta}_k\left(\tau-\zeta\right) &= \left.\frac{\gamma_k}{\left|\xi-\tau\right|}\right|_{\xi=\tau-\zeta}=\frac{\gamma_k}{\left|\left(\tau-\zeta\right)-\tau\right|}=\frac{\gamma_k}{\left|-\zeta\right|}.\end{aligned}$$

Hence:

$$\dot{\beta}_k\left(\tau+\zeta\right)=\dot{\beta}_k\left(\tau-\zeta\right)=\frac{1}{\zeta}>0, \tag{173}$$

since $\zeta>0$ [cf. the discussion inherent to the march that led to the the Eq. (166), from the Eq. (154)]. The distribution characteristic related to the Eq. (168) arises from the singular characteristic we have pointed out by the items (i), (ii) and (iii) above, but one must analyze this issue through a deeper mathematical property: the singular characteristic per se. The fundamental equation to obtain this distribution behaviour is not the solution given by the Eq. (170) for the Eq. (168), i.e., it is the Eq. (168) per se that generates, as we will see. In fact, an inherent paradox turs out to arise, at a first glance, and we will explain why. Here is the paradox: one deliberately rewrites the Eq. (168):

$$(\xi-\tau)\,d\dot{\beta}_k(\xi)=-\dot{\beta}_k(\xi)\,d\xi=-\frac{d\beta_k(\xi)}{d\xi}d\xi, \tag{174}$$

which, by an integration over $\xi\in(t,2\tau-t)$, leads to:

$$\int_t^{2\tau-t}(\xi-\tau)\,d\dot{\beta}_k(\xi)=\int_t^{2\tau-t}-\dot{\beta}_k(\xi)\,d\xi, \tag{175}$$

which, after integrating by parts the left-hand side of the Eq. (175):

$$\int \mu d\nu=\mu\nu-\int\nu d\mu, \tag{176}$$

with:

$$\mu \equiv \xi-\tau, \tag{177}$$
$$d\nu \equiv d\dot{\beta}_k(\xi), \tag{178}$$

yields:

$$\begin{aligned}\int_t^{2\tau-t}-\dot{\beta}_k(\xi)\,d\xi &= (\xi-\tau)\,\dot{\beta}_k(\xi)\Big|_t^{2\tau-t}+\\ &-\int_t^{2\tau-t}\dot{\beta}_k(\xi)\,d(\xi-\tau),\\ &=(\xi-\tau)\,\dot{\beta}_k(\xi)\Big|_t^{2\tau-t}+\\ &-\int_t^{2\tau-t}\dot{\beta}_k(\xi)\,d\xi.\end{aligned} \tag{179}$$

Hence:

$$(\xi-\tau)\,\dot{\beta}_k(\xi)\Big|_t^{2\tau-t}=0, \tag{180}$$

which is an absurd, since, with the Eq. (170) [also an absurd by consideration on parity of $\dot{\beta}_k(\xi)$, by the Eqs. (166) and (173)]:

$$\begin{aligned}(\xi-\tau)\,\dot{\beta}_k(\xi)\Big|_t^{2\tau-t} &= (\xi-\tau)\frac{\gamma_k}{|\xi-\tau|}\Big|_t^{2\tau-t}\\ &=\gamma_k\frac{[(2\tau-t)-\tau]}{|(2\tau-t)-\tau|}-\gamma_k\frac{t-\tau}{|t-\tau|}\\ &=\gamma_k\frac{\tau-t}{|\tau-t|}-\gamma_k\frac{t-\tau}{|t-\tau|}\\ &=\gamma_k\frac{\tau-t}{|\tau-t|}+\gamma_k\frac{\tau-t}{|\tau-t|}\\ &=2\gamma_k\frac{\tau-t}{|\tau-t|}=2\gamma_k,\end{aligned} \tag{181}$$

since, as discussed, $\tau>t$ [remember we have been considering τ as an future event for t at t; the reason by which we must relable the chronological variable, ξ instead of t, when needed]. By the Eqs. (180) and (181), one faces a paradox. One would argue $\gamma_k=0$ circumvents the paradox, but, unfortunatelly, this argument is to assert an unique solution for the Eq. (168), in our context, just the trivial one. There is a richer argument for the circumvention, as we are to also develop within the next lines of this section.

The word is singularity, as it very seems. One may ask if the solution for the Eq. (168), given by the Eq. (170), is obeying this former Eq. (168) in any sense, once the Eq. (170) really obeys the Eq. (168) and avoids the establishment of a particular constant of integration: since τ is arbitrary and appears within the Eq. (168), viz., appears in this generating differential equation, which is not a behaviour for a constant of integration [of course, the constant of integration seems to be γ_k, but the situation is not typical, and for the sake of digression, we are considering all the arbitrary, say, parameters]; since γ_k, which may, clearly, be arbitrary within the march that led to the verification that the Eq. (170) is solution for the Eq. (168), via the obtained Eqs. (171) and (172), $\forall$ $\gamma_k\in\mathbb{C}$, avoids its establishment:

- Unfortunatelly, it turned out to lead to that, say, paradoxical restriction for γ_k: $\gamma_k=0$.

The source of this paradox resides in the use of the Eq. (176), which borns from the rule for differentiation:

$$\frac{d}{d\xi}[\mu(\xi)\,\nu(\xi)]=\mu(\xi)\frac{d}{d\xi}\nu(\xi)+\nu(\xi)\frac{d}{d\xi}\mu(\xi), \tag{182}$$

which is legitimate under the assumption the function being differentiated, $\mu(\xi)\,\nu(\xi)$, is continuous within the domain for which this function is being differentiated. In other words, the left-hand side of the Eq. (182) must exist, once one would be mathematically considering a rule of existence for an object that lacks over the entire domain for differentiation, leading to a paradox, as it led. Again, from the elementary calculus, one knows the necessary condition for the existence of the derivative of a function at a given point of its domain is the continuity of the function at the point, viz., mathematically:

$$\exists\ \frac{d}{d\xi}f(\xi)\Big|_{\xi=\tau} \Rightarrow \lim_{\xi\to\tau}f(\xi)=f\left(\lim_{\xi\to\tau}\xi\right) \tag{183}$$
$$\Rightarrow \lim_{\xi\to\tau^-}f(\xi)=\lim_{\xi\to\tau^+}f(\xi), \tag{184}$$

where $\tau\in\mathcal{D}\left(\xi\xrightarrow{f}f(\xi)\right)$, being $\xi\in\mathcal{D}\left(\xi\xrightarrow{f}f(\xi)\right)$ the domain points of the funtion $f(\xi)$. The right-hand side of the Eq. (184) is just one of the necessary conditions for the right-hand side of the Eq. (183) [$\exists\ f(\tau)$ and $\lim_{\xi\to\tau}f(\xi)=f(\tau)$ are the remaining ones, and the three are encapsulated within the right-hand side of the Eq. (183)]. Thus, from the Eq. (184):

$$\lim_{\xi\to\tau^-} f(\xi) \neq \lim_{\xi\to\tau^+} f(\xi) \Rightarrow \nexists \left.\frac{d}{d\xi} f(\xi)\right|_{\xi=\tau}. \tag{185}$$

Now, with:

$$f(\xi) = \mu(\xi)\,\nu(\xi) = \overbrace{(\xi-\tau)}^{\mu(\xi)} \underbrace{\frac{\gamma_k}{|\xi-\tau|}}_{\nu(\xi)}, \tag{186}$$

one has got:

$$\begin{aligned}
\lim_{\xi\to\tau^-} f(\xi) &= \lim_{\xi\to\tau^-} \gamma_k \frac{\xi-\tau}{|\xi-\tau|} = \gamma_k \lim_{h\to0} \frac{(\tau-h)-\tau}{|(\tau-h)-\tau|} \\
&= \gamma_k \lim_{h\to0} \frac{(-h)}{|-h|} = \gamma_k \lim_{h\to0} -\frac{h}{|h|} \\
&= \gamma_k \lim_{h\to0} -\frac{h}{h} = \gamma_k \lim_{h\to0} -1 = \gamma_k(-1) \\
&= -\gamma_k, \qquad (187)
\end{aligned}$$

where $h \in (0, +\infty)$ [with dimension of time], and, analogously, one also has got:

$$\begin{aligned}
\lim_{\xi\to\tau^+} f(\xi) &= \lim_{\xi\to\tau^+} \gamma_k \frac{\xi-\tau}{|\xi-\tau|} = \gamma_k \lim_{h\to0} \frac{(\tau+h)-\tau}{|(\tau+h)-\tau|} \\
&= \gamma_k \lim_{h\to0} \frac{h}{|h|} = \gamma_k \lim_{h\to0} \frac{h}{h} \\
&= \gamma_k \lim_{h\to0} 1 \\
&= \gamma_k, \qquad (188)
\end{aligned}$$

where, again, $h \in (0, +\infty)$ [with dimension of time]. Once again, with $\gamma_k \neq 0$, one concludes from the Eqs. (185), (187) and (188) that:

$$\nexists \left.\frac{d}{d\xi}\left[\mu(\xi)\,\nu(\xi)\right]\right|_{\xi=\tau}. \tag{189}$$

It very seems the constant γ_k is an intrinsical property of a family of solutions related to the Eq. (170) [relating Eq. (170) to a family of solutions, being the Eq. (170) a singular solution that avoids establishment from such family], that cannot be established, given a particular point, namely $\xi = \tau$, from the Eq. (168), hence, pointing to the possibility of a singular solution for the Eq. (168). One observes that the Eq. (170) will also obey the following differential equation, which is the very same as the Eq. (168):

$$\eta(\xi)\left[\frac{d}{d\xi}\dot{\beta}_k(\xi) + \frac{\dot{\beta}_k(\xi)}{\xi-\tau}\right] = 0, \tag{190}$$

provided $\eta(\xi) \neq 0$. To raise the singularity, one observes the highest order for a pole at $\xi = \tau$ follows from the Eq. (171) [and, equivalently, from the Eq. (172)], from which $\xi = \tau$ turns out to be a pole of second order. Hence, the imposition:

$$\eta(\xi)(\xi-\tau)^2 = \eta(\xi)|\xi-\tau|^2 \propto 1, \tag{191}$$

is sufficient to remove the singularity due to a second order pole, being the proporcionality achieved by a constant, say κ [which must be a non vanishing one, since $\eta(\xi) \neq 0$], that will be absorbed below, since the Eq. (190) may be equivalently rewritten as:

$$\begin{aligned}
\frac{d}{dt}\dot{\beta}_k(\xi) &= -\frac{\eta(\xi)}{\eta(\xi)}\frac{\dot{\beta}_k(\xi)}{\xi-\tau} \\
&= \left[-\eta(\xi)\right]\left[\frac{1}{\eta(\xi)}\right]\frac{\dot{\beta}_k(\xi)}{\xi-\tau} \\
&= \left[-\eta(\xi)\right]\left[\frac{(\xi-\tau)^2}{\kappa}\right]\frac{\dot{\beta}_k(\xi)}{\xi-\tau} \\
&= \left[-\frac{\eta(\xi)}{\kappa}\right](\xi-\tau)\,\dot{\beta}_k(\xi),
\end{aligned}$$

from which, henceforth, the Eq. (190) turns out to read:

$$\frac{d}{dt}\dot{\beta}_k(\xi) = \tilde{\eta}(\xi)(\xi-\tau)\,\dot{\beta}_k(\xi). \tag{192}$$

Of course, the Eq. (192) is identical to the Eq. (168), but in a context one tries to determine emergent solutions for the Eq. (192) with these solutions being particular cases of a general solution $G(\xi)$, provided the Eq. (170) is also a solution, albeit unique, singular, for the Eq. (192) [and, which is the same, for the Eq. (168)]. These facts are to be putted within a sufficiently rigorous framework within the subsequent lines of this section. The general solution for the Eq. (192) may be putted under the form:

$$G\left(\dot{\beta}_k, \xi, C\right) = 0, \tag{193}$$

where C is a constant of integration, so that: particularizing a value for C gives a particular solution for the Eq. (192), albeit the Eq. (170) could not be obtained from any particular C. In principle, one asserts, the Eq. (193) is such that it may be inverted for C, viz., that it may be rewritten as:

$$C = C(\xi) = C\left(\dot{\beta}_k(\xi), \xi\right). \tag{194}$$

For a given particular solution for the Eq. (192), the Eq. (194) would establish an unique, a fixed constant for the considered particular solution, provided some specific point, say S, with coordinates $\left(\xi_S, \dot{\beta}_k(\xi_S)\right)$, so that:

$$C_S = C(\xi_S) = C\left(\dot{\beta}_k(\xi_S), \xi_S\right), \tag{195}$$

i.e., the constant of integration has turned out to be fixed, generating the particular solution:

$$G_S\left(\dot{\beta}_k, \xi, C_S\right) = 0, \tag{196}$$

via the Eq. (193). This process of specification for the obtention of a particular solution would generate all the possible solutions for the Eq. (192), obeying their specific initial conditions via Eq. (195), except for an unique, a singular solution, which, in our case, very seems to be given by the Eq. (170), which needs to be verified. Specification would lack for singular solution. Furthermore, once the singular solution also obeys its generating differential equation, the points pertaining to the singular solution also pertain to particular *solutions*, so that $C(\xi)$ would not be a constant over the singular solution, albeit constant at each non-singular particular solution. Given these considerations, the Eq. (192):

$$\begin{aligned} \frac{d}{dt}\dot{\beta}_k(\xi) &= \tilde{\eta}(\xi(C))(\xi-\tau)\dot{\beta}_k(\xi) \\ &= \tilde{\eta}(C)(\xi-\tau)\dot{\beta}_k(\xi), \end{aligned} \tag{197}$$

turns out to have got two distinct but complementary contexts:

- The singular context, for which $C(\xi)$ varies, and $\tilde{\eta}(\xi)$ also varies, but it is not felt [at least under an explicit sense] by the singular solution, once $\eta(\xi)$ cancels out within the Eq. (190), generating the Eq. (168) which is obeyed by the singular solution, namely, in our case, by the Eq. (170);

- The general solution context, which encapsulates all the particular solutions, but not the singular one, a context for which $C(\xi)$ varies only over distinct particular solutions, being a fixed parameter for a particular solution for the Eq. (192); thus, each particular solution turns out to obey the Eq. (192) with a fixed $\tilde{\eta}(\xi)$.

Here, one needs to put these characteristics under a concrete mathematical formulation.

Firstly, one denotes the singular solution for the Eq. (197) [which is the very same Eq. (192), remembering $C = C(\xi)$, under the singular context] by:

$$\left[\dot{\beta}_k(\xi)\right]_{\text{Singular}} \equiv s(\xi). \tag{198}$$

Now, since a point S, with coordinates $(\xi_S, s(\xi_S))$, that pertain to the singular solution also pertain to some particular solution for which the constant of integration reads $C_S = C(s(\xi_S), \xi_S)$, due to the Eq. (194), the set of coordinated points $\{(\xi, s(\xi))\}$ which gives the entire singular solution determines a miriad of constants of integration $C(s(\xi), \xi)$, viz., the points pertaining to the singular solution pertain, each, to a given particular solution, the reason why one cannot specify the singular solution by a particularization of C, once a miriad of particular solutions is necessary to geometrically give the entire singular solution. Hence, the equation for the singular solution turns out to be geometrically given by its miriad of points $(\xi, s(\xi))$ obeying:

$$G(s(\xi), \xi, C(s(\xi), \xi)) = 0, \tag{199}$$

since these are the points pertaining, one by one, to some particular solution which, particularly, is given by the Eq. (193). Furthermore, considering $C(s, \xi)$ $[s = s(\xi)]$ as being differentiable, non entirely constant nor by pieces [once the singular solution would, entirely or by pieces, degenerate into a particular solution], one differentiates the equation for the singular solution, the Eq. (199), in relation to ξ, i.e.:

$$\begin{aligned} \frac{d}{d\xi}G(s, \xi, C) &= \frac{\partial G}{\partial s}\frac{ds}{d\xi} + \frac{\partial G}{\partial \xi}\frac{d\xi}{d\xi} + \frac{\partial G}{\partial C}\frac{dC}{d\xi} \\ &= \frac{\partial G}{\partial s}\frac{ds}{d\xi} + \frac{\partial G}{\partial \xi} + \frac{\partial G}{\partial C}\frac{dC}{d\xi} \\ &= 0. \end{aligned} \tag{200}$$

But, for the singular solution, which is the case:

$$\begin{aligned} \frac{dC}{d\xi} = \frac{d}{d\xi}C(s, \xi) &= \frac{\partial C}{\partial s}\frac{ds}{d\xi} + \frac{\partial C}{\partial \xi}\frac{d\xi}{d\xi} \\ &= \frac{\partial C}{\partial s}\frac{ds}{d\xi} + \frac{\partial C}{\partial \xi}. \end{aligned} \tag{201}$$

Thus, from the Eqs. (200) and (201):

$$\begin{aligned} \frac{d}{d\xi}G(s, \xi, C) &= \frac{\partial G}{\partial s}\frac{ds}{d\xi} + \frac{\partial G}{\partial \xi} + \\ &+ \frac{\partial G}{\partial C}\left(\frac{\partial C}{\partial s}\frac{ds}{d\xi} + \frac{\partial C}{\partial \xi}\right) \\ &= 0. \end{aligned} \tag{202}$$

For a particular solution, implying a particular C, that shares a same point with the singular solution, the Eq. (202) reads:

$$\frac{\partial G}{\partial s}\frac{ds}{d\xi} + \frac{\partial G}{\partial \xi} = 0. \tag{203}$$

The Eq. (203) is valid for any particular solution, since it also follows from the Eq. (193), once C remains constant along any particular solution. At a sharing point, the Eq. (203) is valid for the singular solution, once both the solutions must satisfy the Eq. (197), implying they share the same derivative, the same coordinates [hence, the same C], instantaneously at the sharing point. This reasoning for each point of the singular solution leads to the conclusion the Eq. (203) is valid for all the points of the singular solution, hence for the singular solution itself. Henceforth, the Eq. (202) turns out to read:

$$\frac{\partial G}{\partial C}\left(\frac{\partial C}{\partial s}\frac{ds}{d\xi} + \frac{\partial C}{\partial \xi}\right) = 0. \tag{204}$$

Since:

$$C(s, \xi) = C(s(\xi), \xi) \neq \text{constant}, \tag{205}$$

for the singular solution, viz., along the singular solution, one has got:

$$\frac{\partial C}{\partial s}\frac{ds}{d\xi}+\frac{\partial C}{\partial \xi}\neq 0. \tag{206}$$

Hence, the singular solution must also satisfy:

$$\frac{\partial}{\partial C}G\left(s,\xi,C\right)=0. \tag{207}$$

Collecting these results, we have got:

- A differential equation for which the singularity was raised, Eq. (197):

  $$\frac{d}{d\xi}\dot{\beta}_k\left(\xi\right)=\tilde{\eta}\left(C\right)\left(\xi-\tau\right)\dot{\beta}_k\left(\xi\right), \tag{208}$$

  which is to be solved to give its general solution, its family of particular solutions, Eq. (193):

  $$G\left(\dot{\beta}_k,\xi,C\right)=0; \tag{209}$$

- A singular solution $s\left(\xi\right)$ for the Eq. (208) which is obtained from both the impositions, Eqs. (199) and (207):

  $$\begin{aligned} G\left(s,\xi,C\right) &= 0, & (210)\\ \frac{\partial}{\partial C}G\left(s,\xi,C\right) &= 0. & (211)\end{aligned}$$

To solve the Eq. (208), one rewrites:

$$\frac{d\dot{\beta}_k}{\dot{\beta}_k}=\tilde{\eta}\left(C\right)\left(\xi-\tau\right)d\xi, \tag{212}$$

integrates:

$$\int\frac{d\dot{\beta}_k}{\dot{\beta}_k}=\int\tilde{\eta}\left(C\right)\left(\xi-\tau\right)d\xi=\int\tilde{\eta}\left(C\right)\left(\xi-\tau\right)d\left(\xi-\tau\right), \tag{213}$$

leading to:

$$\ln\dot{\beta}_k=\frac{1}{2}\tilde{\eta}\left(C\right)\left(\xi-\tau\right)^2+\text{constant}. \tag{214}$$

Writing:

$$\text{constant}\equiv\ln C, \tag{215}$$

where $C\neq 0$ is a constant of integration, one reaches, for the Eq. (214):

$$\dot{\beta}_k\left(\xi\right)=Ce^{\tilde{\eta}(C)(\xi-\tau)^2/2}, \tag{216}$$

from which, one may establish:

$$G\left(\dot{\beta}_k,\xi,C\right)=\dot{\beta}_k-Ce^{\tilde{\eta}(C)(\xi-\tau)^2/2}=0, \tag{217}$$

as demanded by the Eq. (209) for the general solution of the Eq. (208). One may equally well establish, from the Eq. (214), instead of from the Eq. (216):

$$G\left(\dot{\beta}_k,\xi,C\right)=\ln\dot{\beta}_k-\frac{1}{2}\tilde{\eta}\left(C\right)\left(\xi-\tau\right)^2-\ln C=0, \tag{218}$$

since both the Eqs., (217) and (218), are general solutions for the Eq. (208). We will use the Eq. (218) to find the singular solution for the Eq. (208) from the Eqs. (210) and (211), albeit one may equally well use the Eq. (217) if prefers. Hence, from the Eq. (218), the Eq. (210) reads:

$$G\left(s,\xi,C\right)=\ln s-\frac{1}{2}\tilde{\eta}\left(C\right)\left(\xi-\tau\right)^2-\ln C=0. \tag{219}$$

Now, applying the Eq. (211) to the Eq. (219), one reaches:

$$\begin{aligned}\frac{\partial}{\partial C}G\left(s,\xi,C\right) &= \frac{\partial}{\partial C}\left[\ln s-\frac{1}{2}\tilde{\eta}\left(C\right)\left(\xi-\tau\right)^2-\ln C\right]\\ &= -\frac{1}{2}\left(\xi-\tau\right)^2\frac{d\tilde{\eta}\left(C\right)}{dC}-\frac{1}{C}\\ &= 0. \end{aligned} \tag{220}$$

Remembering that within the march that led from the Eq. (191) to (192) we had defined:

$$\tilde{\eta}\left(\xi\right)\equiv-\frac{\eta\left(\xi\right)}{\kappa}\overset{\text{Eq. (191)}}{=}-\frac{1}{\kappa}\frac{\kappa}{\left(\xi-\tau\right)^2}=-\frac{1}{\left(\xi-\tau\right)^2}, \tag{221}$$

where κ is the constant of proporcionality used in the Eq. (191), one also reaches:

$$\left(\xi-\tau\right)^2=-\frac{1}{\tilde{\eta}\left(\xi\right)}=-\frac{1}{\tilde{\eta}\left(\xi\left(C\right)\right)}=-\frac{1}{\tilde{\eta}\left(C\right)}. \tag{222}$$

Henceforth, by the Eqs. (220) and (222) one has got:

$$-\frac{1}{2}\left[-\frac{1}{\tilde{\eta}\left(C\right)}\right]\frac{d\tilde{\eta}\left(C\right)}{dC}-\frac{1}{C}=0, \tag{223}$$

which leads to the following differential equation relating $\tilde{\eta}\left(C\right)$ and C:

$$\frac{1}{\tilde{\eta}\left(C\right)}\frac{d\tilde{\eta}\left(C\right)}{dC}=\frac{2}{C}. \tag{224}$$

To the integration of the Eq. (224), one rewrites:

$$\frac{d\tilde{\eta}\left(C\right)}{\tilde{\eta}\left(C\right)}=2\frac{dC}{C}, \tag{225}$$

and integrates:

$$\int\frac{d\tilde{\eta}\left(C\right)}{\tilde{\eta}\left(C\right)}=2\int\frac{dC}{C}. \tag{226}$$

From the Eq. (222) one concludes that $\tilde{\eta}(C) \in \mathbb{R}$ [with dimension of time^{-2}]. Hence, the Eq. (226) reads:

$$\ln|\tilde{\eta}(C)| = \ln\left(C^2\right) - \ln\lambda = \ln\left(\frac{C^2}{\lambda}\right), \tag{227}$$

where $\lambda \in \mathbb{C}$. Thus, the Eq. (227) leads to:

$$\ln\left[\lambda\frac{|\tilde{\eta}(C)|}{C^2}\right] = 0, \tag{228}$$

from which:

$$C^2 = \lambda|\tilde{\eta}(C)| \overset{\text{Eq. (221)}}{=} \frac{\lambda}{(\xi-\tau)^2}, \tag{229}$$

and, also:

$$|\tilde{\eta}(C)| = \frac{C^2}{\lambda}, \tag{230}$$

which is a positive real number, from which one concludes that:

$$\tilde{\eta}(C) = -\frac{C^2}{\lambda}, \tag{231}$$

since $\tilde{\eta}(C) < 0$ by the Eq. (221) and the Eq. (230) must be satisfied. Hence, in virtue of the Eq. (231), the Eq. (212) reads:

$$\frac{d}{d\xi}\dot{\beta}_k(\xi) = -\frac{C^2}{\lambda}(\xi-\tau)\dot{\beta}_k(\xi), \tag{232}$$

satisfied the Eq. (230):

$$\frac{C^2}{\lambda} \in (0, +\infty). \tag{233}$$

Hence the general solution for the Eq. (232), by the Eqs. (216) and (231), is given by:

$$\dot{\beta}_k(\xi) = Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}. \tag{234}$$

To obtain the singular solution $s(\xi)$ for the Eq. (232), one writes:

$$s(\xi) = C(\xi)\, e^{-\left\{[C(\xi)]^2/\lambda\right\}(\xi-\tau)^2/2}, \tag{235}$$

in virtue of the the Eq. (234), which turns out to read:

$$\begin{aligned} s(\xi) &= \frac{\sqrt{\lambda}}{|\xi-\tau|}e^{-\left\{1/(\xi-\tau)^2\right\}(\xi-\tau)^2/2} \\ &= \frac{\sqrt{\lambda}}{|\xi-\tau|}e^{-1/2} \\ &= \frac{\sqrt{\lambda/e}}{|\xi-\tau|}, \end{aligned} \tag{236}$$

by the Eq. (229) for $C(\xi)$:

$$[C(\xi)]^2 = \frac{\lambda}{(\xi-\tau)^2} \Rightarrow C(\xi) = \frac{\sqrt{\lambda}}{\sqrt{(\xi-\tau)^2}} = \frac{\sqrt{\lambda}}{|\xi-\tau|}, \tag{237}$$

remembering $\lambda \in \mathbb{C}$. Now, one is ready to obtain the differential equation that has got its general solution given by the Eq. (234), and, also, having got the Eq. (236) as its singular solution. To accomplish this, one uses the Eq. (232) and its general solution given by the Eq. (234) to write the resultant differential equation satisfying both these equations with no explicit constant of integration, since a given differential equation must be the same in spite of a particular value for the constant of integration, viz., one has got a family of solutions, not a family of differential equations [the Eq. (232) is explicitizing the constant of integration, from which, hence, it is not, purely, the differential equation we are looking for, albeit the Eq. (234) satisfies it]. From the Eq. (231), we have got:

$$\begin{aligned} \ln[\tilde{\eta}(C)] &= \ln\left(-\frac{C^2}{\lambda}\right) = \ln\left(-\frac{1}{\lambda}\right) + \ln\left(C^2\right) \\ &= \ln\left(-\frac{1}{\lambda}\right) + 2\ln C, \end{aligned} \tag{238}$$

from which:

$$\begin{aligned} \ln[\tilde{\eta}(C)] - \ln\left(-\frac{1}{\lambda}\right) &= 2\ln C \Rightarrow \\ \ln\left[\tilde{\eta}(C) \div \left(-\frac{1}{\lambda}\right)\right] &= 2\ln C \Rightarrow \\ \ln[-\lambda\tilde{\eta}(C)] &= 2\ln C \Rightarrow \\ \ln C &= \frac{1}{2}\ln[-\lambda\tilde{\eta}(C)]. \end{aligned} \tag{239}$$

From the Eqs. (214), (215) and (239), one reaches:

$$\ln\dot{\beta}_k = \frac{1}{2}\tilde{\eta}(C)(\xi-\tau)^2 + \frac{1}{2}\ln[-\lambda\tilde{\eta}(C)]. \tag{240}$$

From the Eq. (208), one catches up:

$$\tilde{\eta}(C) = \frac{d\dot{\beta}_k}{d\xi}\frac{1}{(\xi-\tau)\dot{\beta}_k}. \tag{241}$$

From the Eqs. (240) and (241), one also reaches:

$$
\begin{aligned}
\ln\dot{\beta}_k &= \frac{1}{2}(\xi-\tau)^2\frac{d\dot{\beta}_k}{d\xi}\frac{1}{(\xi-\tau)\dot{\beta}_k}+\frac{1}{2}\ln\left[-\lambda\frac{d\dot{\beta}_k}{d\xi}\frac{1}{(\xi-\tau)\dot{\beta}_k}\right] \\
&= \frac{1}{2}(\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k+\frac{1}{2}\ln\left[\frac{-\lambda}{(\xi-\tau)}\frac{d}{d\xi}\ln\dot{\beta}_k\right]\Rightarrow \\
\ln\left[\left(\dot{\beta}_k\right)^2\right] &= (\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k+\ln\left[\frac{-\lambda}{(\xi-\tau)}\frac{d}{d\xi}\ln\dot{\beta}_k\right]\Rightarrow \\
0 &= (\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k+\ln\left[\frac{-\lambda}{(\xi-\tau)}\left(\frac{d}{d\xi}\ln\dot{\beta}_k\right)\frac{1}{\left(\dot{\beta}_k\right)^2}\right]\Rightarrow \\
\ln\left[\frac{-\lambda}{(\xi-\tau)}\left(\frac{d}{d\xi}\ln\dot{\beta}_k\right)\frac{1}{\left(\dot{\beta}_k\right)^2}\right] &= -(\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k\Rightarrow \\
\frac{-\lambda}{(\xi-\tau)}\left(\frac{d}{d\xi}\ln\dot{\beta}_k\right)\frac{1}{\left(\dot{\beta}_k\right)^2} &= e^{-(\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k},
\end{aligned}
\tag{242}
$$

from which one reaches the differential equation for which the Eq. (234) is the general solution and the Eq. (236) is the singular solution:

$$
-\lambda\frac{d}{d\xi}\ln\dot{\beta}_k=\left(\dot{\beta}_k\right)^2(\xi-\tau)\exp\left[-(\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k\right].
\tag{243}
$$

Before going further, we will verify this assertion really holds, viz., that both the Eqs., (234) [as the general solution] and (236) [as the singular solution that does not depend on C, and connot be obtained from the general solution], are solutions for the Eq. (243). This is accomplished by substitution. Firstly, the left-hand side of the Eq. (243), using the singular solution given by the Eq. (236), reads:

$$
\begin{aligned}
-\lambda\frac{d}{d\xi}\ln[s(\xi)] &= -\lambda\frac{1}{s(\xi)}\frac{d}{d\xi}s(\xi)=-\lambda\frac{|\xi-\tau|}{\sqrt{\lambda/e}}\frac{d}{d\xi}\left[\frac{\sqrt{\lambda/e}}{|\xi-\tau|}\right]=-\lambda\frac{|\xi-\tau|}{\sqrt{\lambda/e}}\frac{d}{d\xi}\left\{\frac{\sqrt{\lambda/e}}{\sqrt{(\xi-\tau)^2}}\right\} \\
&= -\lambda\frac{|\xi-\tau|}{\sqrt{\lambda/e}}\sqrt{\lambda/e}\frac{d}{d\xi}\left[(\xi-\tau)^2\right]^{-1/2}=-\lambda|\xi-\tau|(-1/2)\left[(\xi-\tau)^2\right]^{-3/2}\frac{d}{d\xi}\left[(\xi-\tau)^2\right] \\
&= -\lambda|\xi-\tau|(-1/2)\left[(\xi-\tau)^2\right]^{-3/2}(2)(\xi-\tau)=\lambda\frac{|\xi-\tau|}{\left[(\xi-\tau)^2\right]^{3/2}}(\xi-\tau)=\lambda\frac{|\xi-\tau|(\xi-\tau)}{\left\{\left[(\xi-\tau)^2\right]^{1/2}\right\}^3} \\
&= \lambda\frac{|\xi-\tau|(\xi-\tau)}{|\xi-\tau|^3}=\lambda\frac{(\xi-\tau)}{|\xi-\tau|^2}.
\end{aligned}
\tag{244}
$$

Now, the right-hand side of the Eq. (243), also using the singular solution given by the Eq. (236), reads:

$$\begin{aligned}
s^2(\xi)(\xi-\tau)\exp\left\{-(\xi-\tau)\frac{d}{d\xi}\ln[s(\xi)]\right\} &= \frac{\lambda/e}{|\xi-\tau|^2}(\xi-\tau)\exp\left\{-(\xi-\tau)\left[-\frac{(\xi-\tau)}{|\xi-\tau|^2}\right]\right\} \\
&= \frac{\lambda/e}{|\xi-\tau|^2}(\xi-\tau)\exp\left[\frac{(\xi-\tau)^2}{|\xi-\tau|^2}\right] \\
&= \frac{\lambda/e}{|\xi-\tau|^2}(\xi-\tau)\exp\left[\frac{|\xi-\tau|^2}{|\xi-\tau|^2}\right] = \frac{\lambda/e}{|\xi-\tau|^2}(\xi-\tau)e^1 \\
&= \frac{\lambda/e}{|\xi-\tau|^2}(\xi-\tau)e = \lambda\frac{(\xi-\tau)}{|\xi-\tau|^2}. \qquad (245)
\end{aligned}$$

Hence, from the Eqs. (244) and (245), we conclude the Eq. (236) is [singular] solution for the Eq. (243). Now, the left-hand side of the Eq. (243), using the general solution given by the Eq. (234), reads:

$$\begin{aligned}
-\lambda\frac{d}{d\xi}\ln\dot{\beta}_k &= -\lambda\frac{d}{d\xi}\ln\left[Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\right] = -\lambda\frac{d}{d\xi}\left\{\ln C+\ln\left[e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\right]\right\} \\
&= -\lambda\frac{d}{d\xi}\left[\ln C-\frac{C^2}{\lambda}\frac{(\xi-\tau)^2}{2}\right] = \frac{C^2}{2}\frac{d}{d\xi}\left[(\xi-\tau)^2\right] = \frac{C^2}{2}(2)(\xi-\tau) \\
&= C^2(\xi-\tau). \qquad (246)
\end{aligned}$$

For the right-hand side of the Eq. (243), also using the general solution given by the Eq. (234), one has got:

$$\begin{aligned}
\left(\dot{\beta}_k\right)^2(\xi-\tau)\exp\left[-(\xi-\tau)\frac{d}{d\xi}\ln\dot{\beta}_k\right] &= C^2e^{-\left(C^2/\lambda\right)(\xi-\tau)^2}(\xi-\tau)e^{-(\xi-\tau)\left(-C^2/\lambda\right)(\xi-\tau)} \\
&= C^2(\xi-\tau)e^{-\left(C^2/\lambda\right)(\xi-\tau)^2+\left(C^2/\lambda\right)(\xi-\tau)^2} = C^2(\xi-\tau)e^0 \\
&= C^2(\xi-\tau). \qquad (247)
\end{aligned}$$

Hence, from the Eqs. (246) and (247), we conclude the Eq. (234) is the [general] solution for the Eq. (243). Now we will discuss physical interpretation and implications.

From the Eq. (229), within the context of the general solution, Eq. (234), for the Eq. (243), a given constant of integration C turns out to be defined by some ξ coordinate that uniquely characterizes a specific particular solution pertaining to the family [Eq. (234)]. Since the Eq. (234) is a family of gaussian curves and an inflection point is obtained from the condition:

$$\frac{d^2}{d\xi^2}\dot{\beta}_k(\xi) = 0, \qquad (248)$$

one has got for the family, Eq. (234):

$$\frac{d^2}{d\xi^2}\left[Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\right] = 0, \qquad (249)$$

from which:

$$
\begin{aligned}
\frac{d}{d\xi}\left\{\frac{d}{d\xi}\left[Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\right]\right\} &= \frac{d}{d\xi}\left\{Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\frac{d}{d\xi}\left[-\left(\frac{C^2}{\lambda}\right)\frac{(\xi-\tau)^2}{2}\right]\right\} \\
&= \frac{d}{d\xi}\left\{Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\left[\left(-\frac{C^2}{\lambda}\right)\right]\left(\frac{1}{2}\right)\frac{d}{d\xi}(\xi-\tau)^2\right\} \\
&= \frac{d}{d\xi}\left\{Ce^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\left[\left(-\frac{C^2}{\lambda}\right)\right]\left(\frac{1}{2}\right)(2)(\xi-\tau)\right\} \\
&= -\frac{C^3}{\lambda}\frac{d}{d\xi}\left[(\xi-\tau)e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\right] \\
&= -\frac{C^3}{\lambda}\left\{e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}+(\xi-\tau)e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\frac{d}{d\xi}\left[-\left(\frac{C^2}{\lambda}\right)\frac{(\xi-\tau)^2}{2}\right]\right\} \\
&= -\frac{C^3}{\lambda}\left\{e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}+(\xi-\tau)e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\left[-\left(\frac{C^2}{\lambda}\right)(2)\frac{(\xi-\tau)}{2}\right]\right\} \\
&= -\frac{C^3}{\lambda}e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\left\{1+(\xi-\tau)\left[-\left(\frac{C^2}{\lambda}\right)(2)\frac{(\xi-\tau)}{2}\right]\right\} \\
&= -\frac{C^3}{\lambda}e^{-\left(C^2/\lambda\right)(\xi-\tau)^2/2}\left[1-\frac{C^2}{\lambda}(\xi-\tau)^2\right] \\
&= 0
\end{aligned}
\tag{250}
$$

Hence, from the Eq. (250), one concludes that the coordinate ξ_c, critical one, at which a gaussian of the family given by the Eq. (234) has got an inflection point follows from:

$$1-\frac{C^2}{\lambda}(\xi_c-\tau)^2=0, \tag{251}$$

from which, henceforth:

$$\frac{C^2}{\lambda}=\frac{1}{(\xi_c-\tau)^2}, \tag{252}$$

i.e., the constant of integration for a given particular gaussian of the family given by the Eq. (234) is determined by inflection point coordinate ξ_c. From the Eq. (252), one concludes there are two inflection points, being the ξ_c coordinate of each obtained from the Eq. (252):

$$\xi_c=\tau\pm\sqrt{\frac{\lambda}{C^2}}. \tag{253}$$

Hence, the deviation, σ, of the ξ_c coordinate of the inflection point in relation to τ turns out to be:

$$\sigma\equiv|\xi_c-\tau|=\left|\pm\sqrt{\frac{\lambda}{C^2}}\right|=\sqrt{\frac{\lambda}{C^2}}\in\mathbb{R}_+^*, \tag{254}$$

in virtue of the Eq. (230). Henceforth, from the Eq. (254):

$$\frac{1}{\sigma^2}=\frac{C^2}{\lambda}, \tag{255}$$

and the Eq. (234) turns out to read:

$$\dot{\beta}_k(\xi)=\sqrt{\frac{\lambda}{\sigma^2}}e^{-(\xi-\tau)^2/\left(2\sigma^2\right)}, \tag{256}$$

remembering $\lambda\in\mathbb{C}$. Now, one infers the singular solution given by the Eq. (236) and the family of gaussians given by the Eq. (234) [also, equivalently, by the Eq. (256)] share the inflection points of the gaussians. If one plots just the inflection points of each gaussian of the family [Eq. (256)], the whole set of these inflection points turns out to be, exactly, the singular solution [curve]. In fact, for $\xi=\xi_c$, the Eq. (256) gives:

$$
\begin{aligned}
\dot{\beta}_k(\xi_c)=\dot{\beta}_k(\tau\pm\sigma) &= \sqrt{\frac{\lambda}{\sigma^2}}e^{-(\tau\pm\sigma-\tau)^2/\left(2\sigma^2\right)} \\
&= \sqrt{\frac{\lambda}{\sigma^2}}e^{-1/2}=\frac{\sqrt{\lambda/e}}{\sqrt{\sigma^2}} \\
&= \frac{\sqrt{\lambda/e}}{|\xi_c-\tau|},
\end{aligned}
\tag{257}
$$

where we have used the Eq. (254). And, also, by the Eq. (236):

$$s(\xi_c)=\frac{\sqrt{\lambda/e}}{|\xi_c-\tau|}, \tag{258}$$

which confirms the Eq. (257). Henceforth, the singular solution [Eq. (236)] and the members of the family of gaussians [Eq. (256)] share the points S:

$$S = \left(\xi_c, \frac{\sqrt{\lambda/e}}{|\xi_c - \tau|} \right), \tag{259}$$

which is also the singular solution curve given by the Eq. (236), as said, and asseverated:

$$\{S\} = \{(\xi_c, s(\xi_c))\}. \tag{260}$$

Now there seems to exist an important consequence for that Eq. (168) that came from the considerations on the Eqs. (154) and (155): its solution, being singular, turns out to require a family of gaussians [as discussed before, a miriad of gaussians is necessary to establish the solution, since the solution, being singular, cannot be established by an unique member of the general solution] which turns out to be a family of particular gaussians at different instants, which seems to require a dynamics for the collapse of the wave function. To better visualize, consider the graphs plotted below: Fig. 1 starts with $\sigma \to \infty$, with the sequence of remaining Figs., from Fig. 2 to Fig. 12, representing a decreasing σ, which turns out to lead to the collapse as $\sigma \to 0$, at which a measuring instant τ turns out to be a well defined present instant τ.

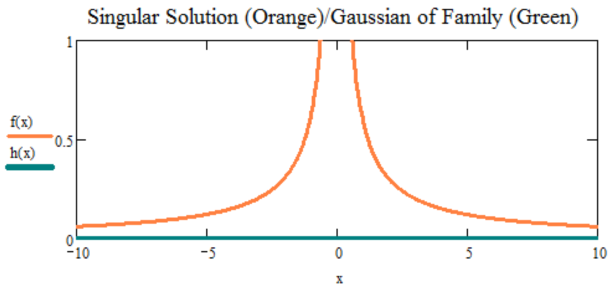

Fig. 1: Eq. (236) [Orange], were we have renamed: $\dot{\beta}_k \equiv f$, and $\xi \equiv x$. Eq. (256) [Green], were we have renamed: $s \equiv h$, and $\xi \equiv x$. We have taken $\tau = 0$. $\sigma \to \infty$.

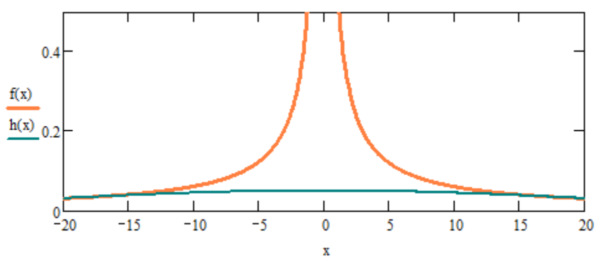

Fig. 2: The same as for the Fig. 1, but σ decreased.

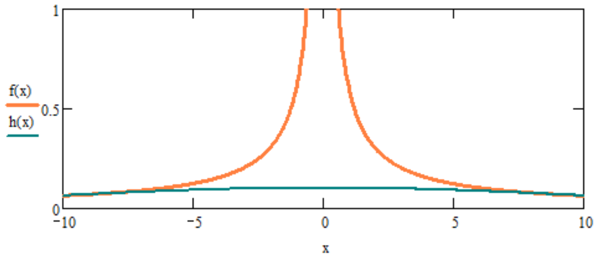

Fig. 3: The same as for the Fig. 2, but σ decreased.

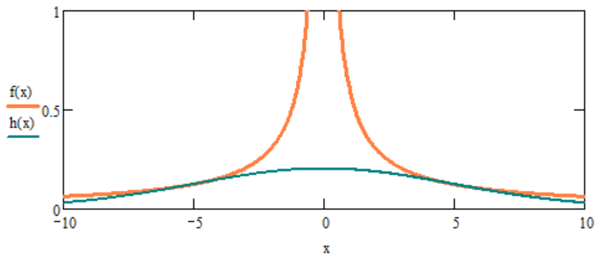

Fig. 4: The same as for the Fig. 3, but σ decreased.

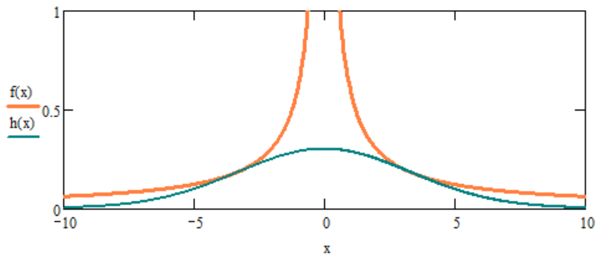

Fig. 5: The same as for the Fig. 4, but σ decreased.

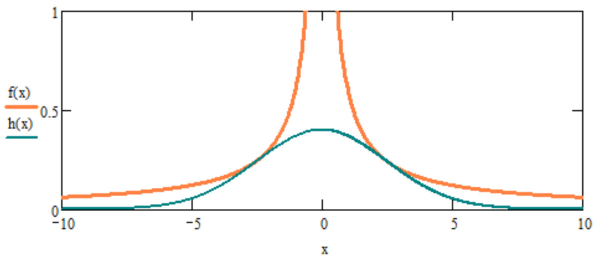

Fig. 6: The same as for the Fig. 5, but σ decreased.

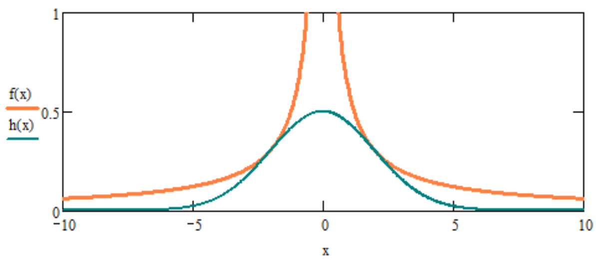

Fig. 7: The same as for the Fig. 6, but σ decreased.

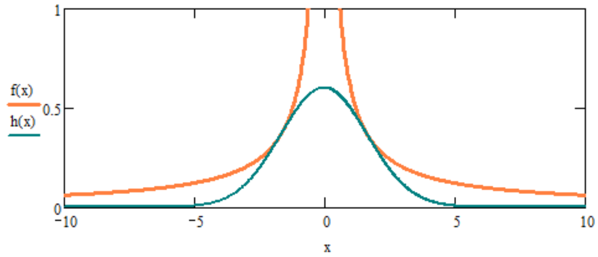

Fig. 8: The same as for the Fig. 7, but but σ decreased.

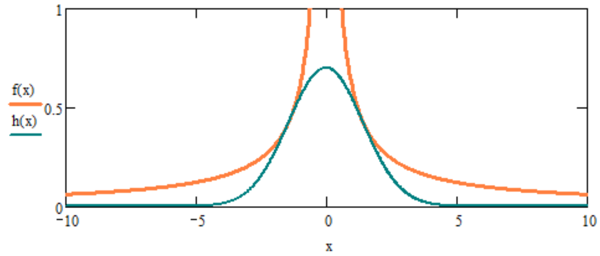

Fig. 9: The same as for the Fig. 8, but σ decreased.

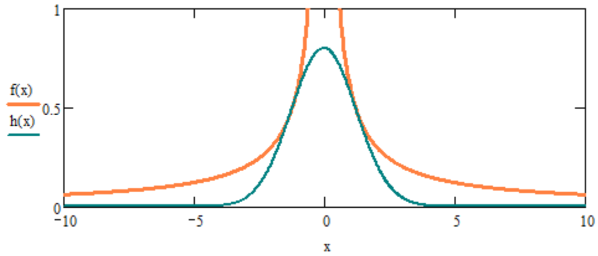

Fig. 10: The same as for the Fig. 9, but σ decreased.

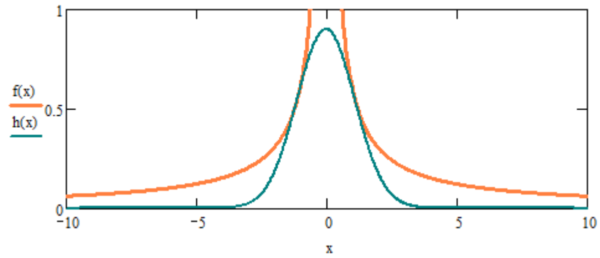

Fig. 11: The same as for the Fig. 10, but σ decreased.

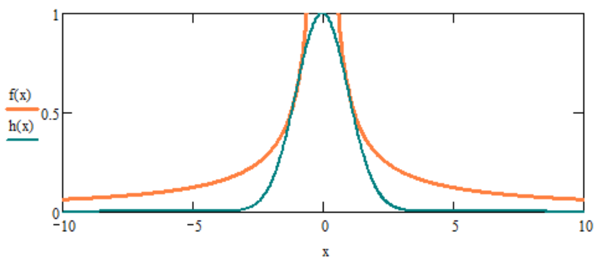

Fig. 12: The same as for the Fig. 11, but σ decreased.

Back to the Eq. (236), and, since the points of this curve are the critical points of the gaussians, one may write this Eq. (236) as a function of σ, i.e., by the Eqs. (236), (260) and (254):

$$s\left(\xi\right) \overset{\text{Eq. (260)}}{=} s\left(\xi_c\right) = s(\xi_c\left(\sigma\right)) \equiv \frac{\sqrt{\lambda/e}}{\sigma}. \tag{261}$$

For purposes of gaussian normalization, which will be discussed below, we will rewrite:

$$\sqrt{\lambda} \equiv \sqrt{r}\sqrt{\tilde{\lambda}}, \tag{262}$$

with:

$$\sqrt{r} \in \mathbb{R}_+^*, \tag{263}$$

remembering that $\lambda \in \mathbb{C}$, as discussed before. Hence, with the Eq. (262), the Eq. (261) turns out to read:

$$s(\xi_c\left(\sigma\right)) = \frac{\sqrt{\tilde{\lambda}}}{\sigma\sqrt{e/r}}. \tag{264}$$

Now, by the Eq. (136), which is the singular solution $s\left(\xi\right)$ [cf., also, the Eq. (170) and its entire context] for:

$$\theta \equiv \sigma\sqrt{e/r} \to 0, \tag{265}$$

[6] [since we are interested in the limit for the Eq. (136), the above definition, Eq. (265), for θ, is mathematically equivalent] one obtains:

$$\sqrt{\tilde{\lambda}} = \delta_k^c - a_k, \tag{266}$$

which is a complex number. Henceforth, from the Eqs. (262) and (266), the Eq. (256) reads:

$$\dot{\beta}_k\left(\xi\right) = \frac{d}{d\xi}\beta_k\left(\xi\right) = \left(\delta_k^c - a_k\right)\sqrt{\frac{r}{\sigma^2}}e^{-(\xi-\tau)^2/\left(2\sigma^2\right)}. \tag{267}$$

Now, one integrates the Eq. (267), writing:

$$\begin{aligned}\int_{-\infty}^{t} d\beta_k\left(\xi\right) &= \beta\left(t\right) - \beta\left(-\infty\right) \\ &= \left(\delta_k^c - a_k\right)\sqrt{\frac{r}{\sigma^2}}\int_{-\infty}^{t} e^{-(\xi-\tau)^2/\left(2\sigma^2\right)}d\xi\end{aligned} \tag{268}$$

Being the system to be putted under scrutinity a quantum mechanical one:

$$\beta_k\left(-\infty\right) = a_k, \tag{269}$$

the Eq. (268) reads:

$$\beta_k\left(t\right) = a_k + \left(\delta_k^c - a_k\right)\sqrt{\frac{r}{\sigma^2}}\int_{-\infty}^{t} e^{-(\xi-\tau)^2/\left(2\sigma^2\right)}d\xi \tag{270}$$

Once a dialectical change occurs, one certainly has got:

$$\beta_k(\infty) = \delta_k^c, \tag{271}$$

leading to the following normalization condition, with the Eq. (270):

$$(\delta_k^c - a_k)\left[1 - \sqrt{\frac{r}{\sigma^2}} \int_{-\infty}^{\infty} e^{-(\xi-\tau)^2/(2\sigma^2)} d\xi\right] = 0, \tag{272}$$

i.e.:

$$\sqrt{\frac{r}{\sigma^2}} \int_{-\infty}^{\infty} e^{-(\xi-\tau)^2/(2\sigma^2)} d\xi = 1. \tag{273}$$

To solve the Eq. (273), one may define:

$$x \equiv \frac{\xi - \tau}{\sigma\sqrt{2}}, \tag{274}$$

from which:

$$\sqrt{2}\sigma dx = d\xi, \tag{275}$$

leading to the Eq. (273), rewritten:

$$\begin{aligned} 1 &= \sqrt{\frac{r}{\sigma^2}} \int_{-\infty}^{\infty} e^{-x^2} \sqrt{2}\sigma dx \\ &= \sqrt{2r} \int_{-\infty}^{\infty} e^{-x^2} dx. \end{aligned} \tag{276}$$

Since the right-hand side of the Eq. (276) is a definite integration, the x variable is dummy, i.e., the Eq. (276) may be equally well written:

$$1 = \sqrt{2r} \int_{-\infty}^{\infty} e^{-y^2} dy. \tag{277}$$

Thus, from the Eqs. (276) and (277):

$$\begin{aligned} 1^2 &= 1 = \left(\sqrt{2r} \int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\sqrt{2r} \int_{-\infty}^{\infty} e^{-y^2} dy\right) \\ &= 2r \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dx dy, \end{aligned} \tag{278}$$

i.e., one turns out to have got, for the right-hand side of the Eq. (278), a double integration over the entire $\mathbb{R}^2$ plane, being $dS = dxdy$ the elementary element of area. Now, one uses polar coordinates:

$$x = \rho\cos\theta; \tag{279}$$

$$y = \rho\sin\theta, \tag{280}$$

leading to:

$$x^2 + y^2 = \rho^2. \tag{281}$$

The elementary element of area in polar coordinates reads:

$$(dS)_{\rho,\theta} = (\rho d\theta)(d\rho) = \rho\, d\rho\, d\theta. \tag{282}$$

Hence, the integration over the entire $\mathbb{R}^2$ plane, for the integrand:

$$e^{-\left(x^2+y^2\right)} \overset{\text{Eq. (281)}}{=} e^{-\rho^2}, \tag{283}$$

reads:

$$\int_{\mathbb{R}^2} e^{-\rho^2} (dS)_{\rho,\theta} = \int_0^{2\pi} \int_0^{\infty} e^{-\rho^2} \rho\, d\rho\, d\theta, \tag{284}$$

allowing:

$$\begin{aligned} 1 &= 2r \int_0^{2\pi} \int_0^{\infty} e^{-\rho^2} \rho\, d\rho\, d\theta \\ &= r \int_0^{2\pi} \int_0^{\infty} e^{-\rho^2} (2\rho\, d\rho)\, d\theta, \end{aligned} \tag{285}$$

the Eq. (278) equivalently rewritten. Now, one defines:

$$u \equiv -\rho^2, \tag{286}$$

from which:

$$du = -2\rho d\rho. \tag{287}$$

By the Eqs. (286) and (287), the Eq. (285) turns out to read:

$$\begin{aligned} 1 &= r \int_0^{2\pi} \int_0^{-\infty} e^u (-du)\, d\theta = r \int_0^{2\pi} \int_{-\infty}^{0} e^u du\, d\theta \\ &= r \left(\int_0^{2\pi} d\theta\right)\left(\int_{-\infty}^{0} e^u du\right) = r \left(\theta|_0^{2\pi}\right)\left(e^u|_{-\infty}^0\right) \\ &= r (2\pi - 0)\left(e^0 - e^{-\infty}\right) = r(2\pi)(1-0) = r(2\pi)(1) \\ &= 2\pi r, \end{aligned} \tag{288}$$

therefore:

$$r = \frac{1}{2\pi}. \tag{289}$$

With r obtained, given by the Eq. (289), one rewrites the Eq. (270):

$$\begin{aligned} \beta_k(t) &= a_k + (\delta_k^c - a_k)\sqrt{\frac{1}{2\pi\sigma^2}} \int_{-\infty}^{t} e^{-(\xi-\tau)^2/(2\sigma^2)} d\xi \\ &= a_k + (\delta_k^c - a_k)\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-(\xi-\tau)^2/(2\sigma^2)} d\xi \\ &= \left[1 - \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-(\xi-\tau)^2/(2\sigma^2)} d\xi\right] a_k \\ &+ \left[\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-(\xi-\tau)^2/(2\sigma^2)} d\xi\right] \delta_k^c. \end{aligned} \tag{290}$$

DISCUSSION

Since the information does not instantaneously propagate, once a system is putted under scrutinity, the emergent $\beta_k(t)$ is such that $t > \tau$. Also, once the scrutinity is chronologically well localized, viz., $\sigma \to 0$, one turns out to really infer, by the Eqs. (143), (267) and (289), in fact, as previously discussed and pointed out, the [and under] distribution context:

$$\delta(t-\tau) \overset{\mathcal{D}}{=} \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-\tau)^2/\left(2\sigma^2\right)}, \tag{291}$$

where the $\mathcal{D}$ is to asseverate the equality is to hold under a context of generalized functions [distributions].

Regarding the scrutinity, since $t = \tau^+ > \tau$, where t is the instant an emergent $\beta_k(t)$ is inferred from the system by some apparatus, once $\sigma \to 0$, the Eq. (290) states:

$$\begin{aligned} \beta_k\left(\tau^+\right) &= a_k + \left(\delta_k^c - a_k\right) \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\tau^+} e^{-(t-\tau)^2/\left(2\sigma^2\right)} dt \\ &\overset{\sigma \overset{!}{\to} 0}{=} a_k + \left(\delta_k^c - a_k\right) \underbrace{\lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty(!)} e^{-(t-\tau)^2/\left(2\sigma^2\right)}}_{=1} \\ &= \delta_k^c, \end{aligned} \tag{292}$$

being, hence, the collapse due to a chronologically well localized measure [$\sigma \to 0$], and due to the principle of causality, $\tau+ > \tau$ [once the process of mesurement is a non instantaneous process, not the contrary].

On the objectivity of a classical dialectical description, the fact that the causality principle holds does not restrict the description of a quantum mechanical system at τ is terms of an instrumental dialectic [classical], an one may refer to the collapse as being quite instantaneous. In fact, the Schrodinger equation, Eq. (34), with the Eqs. (14), (122) and (137), yields:

$$\sum_{\forall k} E_k \delta_k^c e^{-iE_k\tau/\hbar} \phi_k(x) + \sum_{\forall k} i\hbar \left.\frac{d\beta_k}{dt}\right|_{t=\tau} e^{-iE_k\tau/\hbar} \phi_k(x) = E \sum_{\forall k} \delta_k^c e^{-iE_k\tau/\hbar} \phi_k(x), \tag{293}$$

from which, at τ [classically, as discussed before, E manifests as a real constant]:

$$\delta_k^c E = \delta_k^c E_k + i\hbar \left.\frac{d\beta_k}{dt}\right|_{t=\tau}. \tag{294}$$

Under the instrumentally classical dialectics, once the quantum mechanical system turns out to be correlated to a measure apparatus at τ and, even with a small σ, for classical purposes, the [classical] dialectic manisfests as $\hbar \to 0$, and the Eq. (294) under classically dialectical manifestation/inteligibility [this is the dialectics the classical world manifests, and such manifestation is the inteligibility for classical interpreters] reads:

$$\delta_k^c E = \delta_k^c E_k + \left(i\hbar \left.\frac{d\beta_k}{dt}\right|_{t=\tau} \right)_{\hbar\to 0} \approx \delta_k^c E_k, \tag{295}$$

at τ. At τ^+, the equality is manifested, being, the collapse, a process that would not be fully accomplished at τ [cf. the discussion leading to the Eq. (153)]. One may write the Eq. (294) using the Eq. (267):

$$\begin{aligned} \delta_k^c E &= \delta_k^c E_k + i\hbar \left.\frac{1}{\sqrt{2\pi}\sigma} e^{-(t-\tau)^2/\left(2\sigma^2\right)}\right|_{t=\tau} \\ &= \delta_k^c E_k + i\frac{\hbar}{\sqrt{2\pi}\sigma} = \delta_k^c E_k + i\frac{\hbar}{\sqrt{2\pi}\sigma_t}, \end{aligned} \tag{296}$$

once $\sigma = \sigma_t$, since it is related to chronological measures. One should observe that, even with a gaussian distribution that turns out to be quite natural in classically random processes, the σ_t, here, turned out to be the *time interval spent by the system to substantially change its dialectical description*, which implies it is not, essentially, the standard deviation for a set of measures on the collapse instant. Such distinction is important, since, in classical quantum mechanics, the time is the independent variable, on which the dynamical quantities are functions. This said, σ_t has got the meaning of Δt within the *classical* Heisenberg indeterminacy principle. Hence, instrumentally, the Heinsenberg indeterminacy asserts:

$$\sigma_E \overbrace{\Delta t}^{= \text{our } \sigma_t} = \sigma_E \sigma_t \approx \hbar \Rightarrow \sigma_E \approx \frac{\hbar}{\sigma_t}, \tag{297}$$

and the Eq. (296) turns out to read:

$$\delta_k^c E \approx \delta_k^c E_k + \frac{i}{\sqrt{2\pi}} \sigma_E, \tag{298}$$

at τ. Under the classical inteligibility, intrumentally dialectical, at τ, σ_t may be small, since $\hbar$ is classically smaller, and one turns out to establish, classically, that, at τ:

$$\delta_k^c E \approx \delta_k^c E_k, \tag{299}$$

albeit the equality holds once the process of measure is indeed accomplished at $t = \tau^+ > \tau$, by causality.

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[1] A. Assis, Ann. Phys. (Berlin) **523**, 883 (2011).
[2] Classical, Galilean.
[3] The left-hand side of the Eq. (115) is 1, as discussed through the marches that led to the Eqs. (108) and (110), also, equivalently, in virtue of the Eqs. (117) and (118). The right-hand side of the Eq. (115) is $(1 - \delta_{t\bar{t}})^2 + 2\delta_{t\bar{t}} (1 - \delta_{t\bar{t}}) \mathbf{Re}[a_c] + (\delta_{t\bar{t}})^2$, and the Eq. (115) reads: $1 = 1 - 2\delta_{t\bar{t}} + 2(\delta_{t\bar{t}})^2 + 2\delta_{t\bar{t}} (1 - \delta_{t\bar{t}}) \mathbf{Re}[a_c]$, leading to the equation: $2\delta_{t\bar{t}} (1 - \delta_{t\bar{t}}) \{1 - \mathbf{Re}[a_c]\} = 0$, which is valid in spite of $\mathbf{Re}[a_c]$, in virtue of the Eqs. (13) and (14), from which $\mathbf{Re}[a_c]$ does not need to be 1, asseverating: to that raised fortuitous criticisms, respectfully.
[4] *Albert Einstein, Philosopher-Scientist*, edited by P.A. Schilpp (The Library of Living Philosophers, Inc., Evanston, Illinois, 1949).
[5] One may argue that $\Psi(x, t)$ could be treated in an abstract fashion, in an analogy with tensors in abstract forms, with no need of a basis. Such argument is void, since measures are to be performed by observers. As we argued, the utility would remain independent of any inserted basis.
[6] One may equally well impose $\theta \equiv 2\sigma\sqrt{e/r} \to 0$, since under the normalization condition for the r obtention such choice for $\theta \to 0$ turns out to be absorbed, leading to an identical equation, viz., equally leading to the Eq. (290), as one may confirm.