Indefinite Summation

Jörg Siegler*

February 1, 2013

Abstract

This paper about indefinite summation describes a natural approach to discrete calculus. Two natural operators for discrete difference and summation are defined. They preserve symmetry and show a duality in contrast to the classical operators. Several summation and differentiation algorithms will be presented.

1 Introduction

After a short summary of the well known classical discrete calculus in this introduction, we derive the natural form of discrete calculus in chapter 2 and show some of it's remarkable properties. In chapter 3 we present several summation algorithms which were used to obtain the formulae listed in chapter 4.

1.1 Notation

Throughout this paper we focus on intuitive readability. So we try to use non-letter symbols as operators and avoid indices where possible. Therefore all operators like increment $\uparrow$, summation Σ, integration $\int$, discrete difference Δ and derivative ∂ are with respect to the variable x. The summation step size δ is always assumed to be one because $\delta \neq 1$ can be achieved by $\Sigma f(\delta x)$.

Consequently all operators (especially the evaluation $\big|_c$) are used as prefix operators. They will be evaluated from left to right. The precedence of all operators $\uparrow, \Sigma, \int, \Delta$ and ∂ shall be the same as for the addition operation. Hence there is $\Sigma f + g = (\Sigma f) + g$ and $\Sigma f \cdot g = \Sigma(f \cdot g)$.

*Address: D–73061 Roßwälden, sk69202@web.de *Keywords*: indefinite summation, discrete difference, discrete calculus. *Classification MSC 2000*: 39A99, 65B15. This paper was supported by nobody.

$\uparrow f(x) = f(x+1)$	increment of x by 1
$\Big\vert_{c} f(x) = \Big\vert_{0} \uparrow^{c} f(x) = f(c)$	evaluation at $x = c$
$\Big\vert_{b}^{a} f(x) = \Big\vert_{b} f(x) - \Big\vert_{a} f(x) = f(b) - f(a)$	interval evaluation
$\Big\vert_{cx} f(x) = f(cx)$	substitution of x by cx
$\mathbf{1} f(x) = f(x)$	identity operator
$\Big\vert_{a}^{b} \Sigma_0 = \sum_{x=a}^{b-1}$	classical definite sum over x
$\Sigma_0 = \Delta_0^{-1} = (\uparrow - \mathbf{1})^{-1}$	classical indefinite summation operator over x
$\Sigma = \Delta^{-1} = \frac{1}{2} \cdot \frac{\uparrow + \mathbf{1}}{\uparrow - \mathbf{1}}$	natural indefinite summation operator over x
$\partial = \frac{\mathrm{d}}{\mathrm{d}x}$	indefinite derivative operator with respect to x
$\int = \partial^{-1} = \int \cdots \frac{\mathrm{d}}{\mathrm{d}x}$	indefinite integration operator with respect to x

1.2 Increment Operator

In the beginning there was the increment operator $\uparrow$. It allows us to count and defines the natural numbers given a first number usually called „one“. The increment is the basis for summation and discrete calculus.

$$\boxed{\uparrow f(x) := f(x+1)} \qquad \text{Increment Operator } \uparrow \tag{1.1}$$

$$\begin{aligned} \uparrow f(x) &:= f(x+1) & \qquad \uparrow^{a} f(x) &= f(x+a) \\ \downarrow f(x) &:= f(x-1) & \qquad \downarrow^{a} f(x) &= f(x-a) \\ \uparrow^{a}\uparrow^{b} &= \uparrow^{a+b} & & \\ \downarrow &= \uparrow^{-1} & \qquad \uparrow\downarrow = \downarrow\uparrow &= \mathbf{1} \end{aligned}$$

1.3 Classical Discrete Calculus

The classical discrete difference operator Δ_0 is defined as an increment minus the identity. The subscript 0 indicates here the classical operator because Δ will later be used for the natural difference operator. Classical literature about discrete calculus is e.g. [Jo65]. A modern description can be found in [GKP95] and some classical summation results in [WP1]. Elementary formulae can be found in [NI10, Ab72].

$$\boxed{\Delta_0 := \uparrow - \mathbf{1}} \qquad \Delta_0 f(x) = f(x+1) - f(x) \tag{1.2}$$

The classical summation operator is simply defined as the inverse discrete difference.

$$\boxed{\Sigma_0 := \Delta_0^{-1}} \qquad \text{classical summation operator} \tag{1.3}$$

Like in the case of geometric progressions we use the telescoping method to obtain a relation between definite and indefinite sums.

$$\Sigma_0^{-1} \sum_{k=a}^{b-1} \uparrow^k = (\uparrow - \mathbf{1}) \cdot \sum_{k=a}^{b-1} \uparrow^k = \sum_{k=a+1}^{b} \uparrow^k - \sum_{k=a}^{b-1} \uparrow^k = \uparrow^b - \uparrow^a$$
$$\sum_{k=a}^{b-1} \uparrow^k = (\uparrow^b - \uparrow^a) \Sigma_0 = \Big|_a^b \Sigma_0$$

$$\boxed{\sum_{x=a}^{b-1} f(x) = \Big|_a^b \Sigma_0 f(x)} \tag{1.4}$$

1.4 General Operators

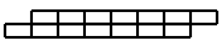
By introducing the weighted summation operator $\Sigma_w = \Sigma_0 + w \cdot \mathbf{1}$ we can control the borders of the summation interval $\left(\uparrow^b - \uparrow^a\right) \Sigma_w = (1-w)\uparrow^a + \sum_{k=a+1}^{b-1} \uparrow^k + w\uparrow^b$.

$$\Sigma_w = \Sigma_0 + w \cdot \mathbf{1} \qquad \Delta_w = \Sigma_w^{-1} = \frac{1}{w-1} + \sum_{k=1}^{\infty} \frac{w^{k-1}}{(w-1)^{k+1}} \uparrow^k \tag{1.5}$$

The classical operators Σ_0 and $\Delta_0 = \mathbf{\uparrow} - \mathbf{1}$ with weight $w = 0$ represent the lower border summation $\sum_a^{b-1}$. The weight $w = 1$ results in the upper border summation $\sum_{a+1}^{b}$ with operators $\Sigma_1 = \Sigma_0 + \mathbf{1}$ and $\Delta_1 = \mathbf{\downarrow}\Delta_0 = \mathbf{1} - \mathbf{\downarrow}$.

The symmetric half border summation with weight $w = \frac{1}{2}$ is the approach to the natural discrete calculus presented in the next chapter.

2 Natural Discrete Calculus

The natural convention is to count the interval borders of a sum only with their half values. So this natural sum Σ is simply the mean of two classical sums Σ_0 where one of them is shifted $\Sigma =$  $= \frac{1}{2}\Sigma_0\mathbf{\uparrow} + \frac{1}{2}\Sigma_0$ or after applying $\Sigma_0^{-1} = \Delta_0 = \mathbf{\uparrow} - \mathbf{1}$

$$\boxed{(\mathbf{\uparrow} - \mathbf{1})\,\Sigma = \frac{1}{2}(\mathbf{\uparrow} + \mathbf{1})} \tag{2.1}$$

This directly leads to the symmetrical functional equation:

$$\boxed{F(x+1) - F(x) = \frac{1}{2}f(x+1) + \frac{1}{2}f(x)} \tag{2.2}$$

$$\boxed{\Delta F(x) = f(x)} \qquad \boxed{F(x) = \sum f(x)} \tag{2.3}$$

Instead of arguments x and $x+1$ the functional equation (2.2) may also be written with the symmetric arguments $x \pm \frac{1}{2}$ as $F(x+\frac{1}{2}) - F(x-\frac{1}{2}) = \frac{1}{2}f(x+\frac{1}{2}) + \frac{1}{2}f(x-\frac{1}{2})$.

Both functions $f(x)$ and $F(x)$ are evaluated only at points x with integer distance, although x itself needs not to be an integer. This explains the attribute „discrete". So both functions may also be finite or infinite series $f_x = f(x)$, general number sequences or numerical data in array or vector form.

2.1 Linearity

The functional equation (2.2) is linear with respect to the argument function.

$$\Delta c \cdot f(x) = c \cdot \Delta f(x) \qquad \sum c \cdot f(x) = c \cdot \sum f(x)$$

$$\Delta\,(f(x) + g(x)) = \Delta f(x) + \Delta g(x) \qquad \sum\,(f(x) + g(x)) = \sum f(x) + \sum g(x)$$

2.2 Translation Invariance

The functional equation (2.2) is invariant under the translation $x \to x+a$ and negative under argument negation $x \to -x$.

$$\begin{aligned} \Delta \Big|_{x+a} &= \Big|_{x+a} \Delta & \qquad \Sigma \Big|_{x+a} &= \Big|_{x+a} \Sigma \\ \Delta \Big|_{-x} &= -\Big|_{-x} \Delta & \qquad \Sigma \Big|_{-x} &= -\Big|_{-x} \Sigma \end{aligned} \tag{2.4}$$

2.3 Symmetry

The functional equation (2.2) preserves the symmetry of the argument function by changing the sign of the symmetry. This can easily be shown by inserting $-x$ into (2.2) and utilize the translation invariance (2.4) to increment x. For even $f_e(-x) = f_e(x)$ we get $f_e(-x+1) \pm f_e(-x) = f_e(x-1) \pm f_e(x) = \pm f_e(x'+1) + f_e(x')$ and for odd $f_o(-x) = -f_o(x)$ we get $f_o(-x+1) \pm f_o(-x) = -f_o(x-1) \mp f_o(x) = \mp f_o(x'+1) - f_e(x')$. So an even $f_e(x)$ maps to an odd $F_o(x)$ in the functional equation and vice versa. Note that all functions $f(x) = f_e(x) + f_o(x)$ can be split into even and odd parts.

$$\boxed{\begin{aligned} \Delta\text{even} &= \text{odd} & \qquad \sum\text{even} &= \text{odd} \\ \Delta\text{odd} &= \text{even} & \qquad \sum\text{odd} &= \text{even} \end{aligned}} \tag{2.5}$$

Later the expansion (3.4) into odd powers of the derivative operator ∂ shows, that this symmetry feature has it's origin is infinitesimal calculus. So the sum and difference operator inherits all symmetry problems from calculus. E.g. the odd function $1/x$ has the integral $\ln|x|$ and the sum $\Psi(x) + (2x)^{-1}$ without evident symmetry.

Half-Symmetry

In addition to the even/odd symmetry around $x = 0$ we also have to pay attention to the half-even/half-odd symmetry around $\frac{1}{2}$. The function $f_E\left(\frac{1}{2} - x\right) = f_E\left(\frac{1}{2} + x\right)$ is half-even and the function $f_O\left(\frac{1}{2} - x\right) = -f_O\left(\frac{1}{2} + x\right)$ is half-odd. So there is $f_E(x+1) = f_E(-x)$ and $f_O(x+1) = -f_O(-x)$.

Combined symmetry around zero and one half

Now we discuss functions which show both symmetries around $x = 0$ and $x = \frac{1}{2}$. We denote function to be even and half-odd as $f_{eO}(x)$. So there is $f_{eE}(x+1) = f_{eE}(x)$ and $f_{oO}(x+1) = f_{oO}(x)$ but $f_{eO}(x+1) = -f_{eO}(x)$ and $f_{oE}(x+1) = -f_{oE}(x)$.

$f(x)$ Symmetry	$f_{eE}(x)$ even, 1/2-even	$f_{eO}(x)$ even, 1/2-odd	$f_{oE}(x)$ odd, 1/2-even	$f_{oO}(x)$ odd, 1/2-odd
$\uparrow$	$f_{eE}(x)$	$-f_{eO}(x)$	$-f_{oE}(x)$	$f_{oO}(x)$
Δ	0	$4x \cdot f_{eO}(x)$	$4x \cdot f_{oE}(x)$	0
Σ	$x \cdot f_{eE}(x)$	0	0	$x \cdot f_{oO}(x)$

Even/half-even functions $f_{eE}(x)$ and odd/half-odd functions $f_{oO}(x)$ are constant functions also described in chapter 2.10. Even/half-odd functions $f_{eO}(x)$ and odd/half-even functions $f_{oE}(x)$ are anti-constant functions also described in chapter 2.11.

$f(x)$ Symmetry	$\cos(2n\pi x)$ even, 1/2-even	$\cos((2n+1)\pi x)$ even, 1/2-odd	$\sin((2n+1)\pi x)$ odd, 1/2-even	$\sin(2n\pi x)$ odd, 1/2-odd
Δ	0	$4x \cdot \cos((2n+1)\pi x)$	$4x \cdot \sin((2n+1)\pi x)$	0
Σ	$x \cdot \cos(2n\pi x)$	0	0	$x \cdot \sin(2n\pi x)$

2.4 Duality

The most remarkable feature of the functional equation (2.2) is a duality between difference Δ and summation operator $\Sigma = \Delta^{-1}$. Simply multiply the functional equation by $(-1)^x$ to get $(-1)^{x+1}F(x+1) + (-1)^x F(x) = \frac{1}{2}(-1)^{x+1} f(x+1) - \frac{1}{2}(-1)^x f(x)$.

$$\boxed{\begin{aligned} \sum f(x) &= \frac{1}{4}(-1)^x \,\Delta (-1)^x \cdot f(x) \\ \Delta f(x) &= 4\,(-1)^x \sum (-1)^x \cdot f(x) \end{aligned}} \tag{2.6}$$

$(-1)^x f(x)$ is called the alternate function of $f(x)$.

2.5 Additive Intervals

The functional equation (2.2) is additive in the summation intervals.

$$\Big|_a^b \Sigma f(x) \;+\; \Big|_b^c \Sigma f(x) \;=\; \Big|_a^c \Sigma f(x) \tag{2.7}$$

2.6 Comparison with Classical Discrete Calculus

By inserting (1.2) $\Sigma_0 = (\mathbf{\uparrow} - \mathbf{1})^{-1}$ into the functional equation (2.1) we get $\Sigma = \frac{1}{2}\Sigma_0\,(\mathbf{\uparrow} + \mathbf{1})$. Now we replace $\mathbf{\uparrow}$ by $\Delta_0 + \mathbf{1}$ and find the simple relation between the classical and natural summation.

$$\boxed{\Sigma \;=\; \Sigma_0 + \frac{1}{2}\mathbf{1}} \tag{2.8}$$

This means that we can use our classical summation formulae and simply add one half of the original function to get the natural indefinite summation formula. As in the definition of natural summation, the interval borders are counted only with their half value $\big|_a^b \Sigma f(x) = \sum_a^{b-1} f(x) + \big|_a^b \frac{1}{2} f(x) = \frac{1}{2} f(a) + \sum_{a+1}^{b-1} f(x) + \frac{1}{2} f(b)$.

$$\boxed{\Big|_a^b \Sigma f(x) \;=\; \sum_{x=a}^{b-1} f(x) \;+\; \frac{1}{2}\Big|_a^b f(x)} \tag{2.9}$$

Note that on the left side of this equation we have a natural definite sum, whereas on the right side we have the classical definite sum.

2.7 Series Expansion of Operators

By using the relation $\Delta_0\Delta \;=\; \Delta\Delta_0 \;=\; 2\,(\Delta_0 - \Delta)$ between natural and classical discrete difference, it is possible to expand the natural operators Δ in terms of the classical operators Δ_0. These are only formal infinite equations. We have to assure a terminating expansion either by $\Delta_0^n f(x) \;=\; 0$ (like a geometric progression) or in case of a periodic sequence by $\Delta_0^n f(x) = f(x)$.

$$\Sigma = -\frac{1}{2}\sum_{k=1}^{\infty}(\Delta_0\Sigma)^k = -\frac{1}{2} - \sum_{k=1}^{\infty}\uparrow^k \tag{2.10}$$

$$\Delta = \Delta_0\sum_{k=0}^{\infty}\left(-\frac{\Delta_0}{2}\right)^k = -2 - 4\sum_{k=1}^{\infty}(-1)^k\uparrow^k \tag{2.11}$$

2.8 Definite Summation

The following equation (2.12) can either be derived from (1.4) and (2.1) or by factorizing $\uparrow^b - \uparrow^a = \uparrow^a(\uparrow - \mathbf{1})\sum_{k=0}^{b-a-1}\uparrow^k$. The factor $\uparrow - \mathbf{1}$ in this geometric progression cancels the denominator of $\Sigma = \frac{1}{2}(\uparrow + \mathbf{1})/(\uparrow - \mathbf{1})$ (3.2). So we see that the definite natural summation consists of five parts.

$$\Big|_a^b\Sigma = \Big|_0\left(\uparrow^b - \uparrow^a\right)\Sigma = \overbrace{\Big|_0}^{5)}\ \overbrace{(\uparrow^b - \uparrow^a)}^{4)\ \ 3)}\ \overbrace{\tfrac{1}{2}(\uparrow + \mathbf{1})}^{2)}\ \overbrace{(\uparrow - \mathbf{1})^{-1}}^{1)} \tag{2.12}$$

From right to left there are:

1. Classical indefinite summation $\Sigma_0 = (\uparrow - \mathbf{1})^{-1} = -\mathbf{1} - \uparrow - \uparrow^2 - \uparrow^3 - \uparrow^4 - \cdots$
2. Symmetrization $\frac{1}{2}(\uparrow + \mathbf{1})$ to the natural form
3. Border shifts $\uparrow^b$ and $\uparrow^a$
4. Interval forming by subtracting the borders
5. Evaluation $|_0$ at zero

2.9 Definite Difference

Accordingly we get expansion (2.13) of the definite difference by further factorizing $\uparrow^b - \uparrow^a = \uparrow^a(\uparrow - \mathbf{1})(\uparrow + \mathbf{1})\sum_{k=0}^{n/2-1}\uparrow^{2k}$ for even integer $n = b - a$. The additional factor $\uparrow + \mathbf{1}$ now cancels the denominator of $\Delta = 2(\uparrow - \mathbf{1})/(\uparrow + \mathbf{1})$ (3.3).

$$\begin{aligned}\Big|_a^b\Delta &= \Big|_0\left(\uparrow^b - \uparrow^a\right)\Delta = 2\uparrow^a\left(\uparrow - \mathbf{1}\right)^2\sum_{k=0}^{\frac{b-a}{2}-1}\uparrow^{2k} = \qquad \text{even } b-a \\ &= 2\uparrow^a + 4\sum_{k=a+1}^{b-1}(-1)^{k-a}\uparrow^k + 2\uparrow^b\end{aligned} \tag{2.13}$$

$$\Big|_a^b \Delta f(x) = 2f(a) + 4\sum_{k=a+1}^{b-1}(-1)^{k-a}f(k) + 2f(b) \qquad \text{even } b-a \tag{2.14}$$

There is no such factorization and series expansion for odd $n = b - a$.

2.10 Constants

To all sums we can add functions $c(x)$ which are Δ-constant in the sense $\Delta c(x) = 0$ without changing the derivative of the sum. From (2.11) follows that $\Delta_0 f(x) = 0 \implies \Delta f(x) = 0$. In other words: All Δ_0-constant functions are also Δ-constant. Δ_0-constant functions fulfill the condition $\Delta_0 f(x) = f(x+1) - f(x) = 0$ or $f(x+1) = f(x)$ and are either the constant real value c or any periodic function $p(x)$ with period one. Such Δ_0- and Δ-constant periodic functions are e.g. $\cos(2\pi kx)$, $\sin(2\pi kx)$, $\exp(2\pi \mathtt{i} kx)$ or $\{x\}$ (3.33). See also chapter 2.3 about symmetry.

2.11 Anti-Constants

Analogously we can define anti-constant functions $a(x)$ which fulfill $\sum a(x) = 0 \impliedby a(x+1) = -a(x)$. Anti-constant functions have a period of two, which will be be relevant later in chapter 3.5 when we discuss duplication. Such anti-constant functions are e.g. $(-1)^x$ or $\cos(\pi x)$.

By using the product rules (3.39) and (3.40), we get an expression for the sum of constant function and the difference of anti-constant functions.

$$\begin{aligned} \Delta c(x) &= 0 & \qquad \textstyle\sum c(x) &= x \cdot c(x) \\ \textstyle\sum a(x) &= 0 & \qquad \Delta a(x) &= 4x \cdot a(x) \end{aligned}$$

2.12 Equivalence Relation

The relation $\equiv$ is defined to skip all constants and anti-constants.

$$\begin{aligned} \Delta F(x) &= f(x) \equiv f(x) + a(x) & \quad \text{with} \quad \textstyle\sum a(x) &= 0 \\ \textstyle\sum f(x) &= F(x) \equiv F(x) + c(x) & \quad \text{with} \quad \Delta c(x) &= 0 \end{aligned}$$

3 Summation and Differentiation Algorithms

In this chapter we provide several algorithms to calculate sums and discrete differences with the natural operators.

3.1 Functional Equation

Like all functional equations also (2.2) $(\boldsymbol{\uparrow}-\mathbf{1})\Sigma = \frac{1}{2}(\boldsymbol{\uparrow}+\mathbf{1})$ is not really suitable for directly obtaining an indefinite summation formula. The functional equation may be used for guessing a solution but the main field of application is the verification of a solution found by one of the other algorithms.

Example 1: With $(\boldsymbol{\uparrow}-\mathbf{1})e^x = (e-1)e^x$ and $(\boldsymbol{\uparrow}+\mathbf{1})e^x = (e+1)e^x$ we find $\Sigma e^x = \frac{1}{2}\frac{e+1}{e-1}e^x = \frac{1}{2}\coth\frac{1}{2}\cdot e^x$.

3.2 Classical Summation

With relation (2.8) $\Sigma = \Sigma_0 + \frac{1}{2}\mathbf{1}$ we are able to obtain natural sums from all existing classical sums simply by adding one half of the original function. Especially the powerful Gosper and Wilf-Zeilberger algorithms [Go78, Wi90, Ze91, Ko94] can be used to generate natural summation formulae.

Example 2: From the classical sum $\sum_{k=1}^{n}\frac{1}{k} = \Psi(n+1)+\gamma$ and $\Psi(1) = -\gamma$ we find $\Sigma\frac{1}{x} = \Psi(x)+\frac{1}{2x} = \ln|x| - \sum_{k=1}^{\infty}\frac{B_{2k}}{2k}x^{-2k}$.

3.3 Infinitesimal Calculus

The discrete calculus operators Σ, Σ_0 and Δ commute with the integration $\int$ and differentiation ∂ operators except for some constant or anti-constant terms. Therefore if we know e.g. the sum $\Sigma\partial f(x)$ of the derivative, we can calculate the sum of $f(x)$ by $\Sigma f(x) \equiv \int\Sigma\partial f(x)$. The result must be checked in any case with the functional equation (2.2) to get the correct constant and anti-constant terms or for a double integration (i.e. $\int\Sigma$ or $\Sigma\int$) also the linear term.

$$\Sigma f(x) \equiv \int \Sigma\partial f(x) \qquad \Sigma f(x) \equiv \partial\Sigma\int f(x)$$
$$\Delta f(x) \equiv \int \Delta\partial f(x) \qquad \Delta f(x) \equiv \partial\Delta\int f(x)$$

Example 3: We already know $\Sigma\frac{1}{x} = \Psi(x) + \frac{1}{2x}$ with $\Psi(x) = \partial\ln\Gamma(x)$ and $\partial\ln|x| = \frac{1}{x}$. So we get $\Sigma\ln x \equiv \ln\Gamma(x) + \frac{1}{2}\ln x$.

3.4 Operator Expansion

Starting with the Taylor expansion $f(x+\delta) = \sum_{k=0}^{\infty} \frac{\delta^k}{k!}\partial^k f(x)$ at distance δ around x we get for $\delta = 1$ the equation $\uparrow f(x) = f(x+1) = \sum_{k=0}^{\infty} \frac{1}{k!}\partial^k f(x)$ which formally equals the exponential function of the derivative operator $\uparrow = \sum_{k=0}^{\infty} \frac{1}{k!}\partial^k = e^{\partial}$. Analogously we get for the decrement $\downarrow \ = \ e^{-\partial}$.

$$\boxed{\uparrow \ = \ e^{\partial}} \qquad = \sum_{k=0}^{\infty} \frac{\partial^k}{k!} \ \approx \ 1 + \partial + \frac{1}{2}\partial^2 + \frac{1}{6}\partial^3 + \frac{1}{24}\partial^4 + \cdots \tag{3.1}$$

It is allowed to handle the derivative operator ∂ as argument of a power series, because ∂ obeys all necessary rules like linearity, commutativity and e.g. $\partial^n \partial^m = \partial^n \cdot \partial^m = \partial^{n+m}$. But keep in mind that instead of a multiplication, the operator concatenation $\circ$ is used in power series of operators. Therefore the function evaluation $g(\partial)$ on an operator is itself an operator. When we sloppy write $g(\partial)f(x)$ we actually mean $g(\partial) \circ f(x)$ the application of the operator $g(\partial)$ on $f(x)$. So $g(\partial)$ shall only be applied in the form of it's expansion into a power series. These arguments are in the same sense valid for all other linear operators which commute with addition like $\uparrow$, $\downarrow$, Δ_0, Σ_0, Δ, Σ and $\int$.

We rewrite the functional equation (2.2) as rational function of the increment operator and transform it with $\uparrow = e^{\partial}$ (3.1) into a function of the derivative operator.

$$\boxed{\Sigma \ = \ \frac{1}{2}\frac{\uparrow + \mathbf{1}}{\uparrow - \mathbf{1}} \ = \ \frac{1}{2}\coth\frac{\partial}{2}} \qquad \text{Summation Operator} \tag{3.2}$$

$$\boxed{\Delta \ = \ 2\frac{\uparrow - \mathbf{1}}{\uparrow + \mathbf{1}} \ = \ 2\tanh\frac{\partial}{2}} \qquad \text{Difference Operator} \tag{3.3}$$

Expanding the rational functions in (3.2) and (3.3) into a series of increments will give (2.10) and (2.11) which were already described in chapter 2.7. The series expansions of $\frac{1}{2}\coth\frac{\partial}{2}$ and $2\tanh\frac{\partial}{2}$ in terms of the derivative operator ∂ are as follows.

$$\begin{aligned} \Sigma &= \frac{1}{2}\coth\frac{\partial}{2} \approx \partial^{-1} + \frac{1}{12}\partial - \frac{1}{720}\partial^3 + \frac{1}{30240}\partial^5 - \frac{1}{1209600}\partial^7 + \cdots \\ \Delta &= 2\tanh\frac{\partial}{2} \approx \partial - \frac{1}{12}\partial^3 + \frac{1}{120}\partial^5 - \frac{17}{20160}\partial^7 + \cdots \end{aligned} \tag{3.4}$$

It is remarkable that the series expansions of $\frac{1}{2}\coth\frac{\partial}{2}$ and $2\tanh\frac{\partial}{2}$ consist only of odd derivative powers. So applying Σ or Δ will swap the even/odd symmetry of the argument. Also the series expansions of Σ contains an integral $\int = \partial^{-1}$ whereas the series expansions of Δ contains only derivatives.

It seems that all numerators in these series expansions are one. But especially because of $B_{12} = -\frac{691}{2730}$ this is *not* the case.

$$\begin{aligned} \frac{1}{2}\coth\frac{x}{2} &= \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k-1} \approx && (3.5) \\ &\approx \frac{1}{x} + \frac{x}{12} - \frac{x^3}{720} + \frac{x^5}{30240} - \frac{x^7}{1209600} + \frac{x^9}{47900160} - \frac{691\,x^{11}}{1307674368000} + \cdots \\ 2\tanh\frac{x}{2} &= 4\sum_{k=1}^{\infty} \frac{B_{2k}\left(2^{2k}-1\right)}{(2k)!} x^{2k-1} \approx && (3.6) \\ &\approx x - \frac{x^3}{12} + \frac{x^5}{120} - \frac{17\,x^7}{20160} + \frac{31\,x^9}{362880} - \frac{691\,x^{11}}{79833600} + \cdots \end{aligned}$$

[GKP95] describes that the series expansion of $\frac{x}{2}\coth\frac{x}{2}$ and $\frac{x}{e^x-1}$ differ by the term $\frac{x}{2}$. This appears as difference $\frac{1}{2}$ between the natural and classical sum $\Sigma - \Sigma_0$ in equation (2.8). So omitting the single non-zero odd Bernoulli term $B_1 = -\frac{1}{2}$ is responsible for the symmetry of our natural summation.

$$\begin{aligned} \frac{x}{2}\coth\frac{x}{2} = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} x^{2k} &= \frac{x}{2}\,\frac{e^x+1}{e^x-1} = \frac{x}{e^x-1} + \frac{x}{2} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k + \frac{x}{2} \\ \text{formally} \qquad \sum B_{\text{even}} &= \sum B_{\text{all}} - B_1 \end{aligned}$$

To get rid of the factors two and one half, we tentatively introduced new and much simpler operators for Σ and Δ without these factors. But it turned out to be more confusing to handle new operators than to handle constant factors. So we continue to use our operators Σ and Δ.

3.5 Duplication

The factor $2^{2k} - 1$ in the hyperbolic tangent expansion (3.6) leads to the duplication formula (3.7) for the hyperbolic cotangent

$$2\coth x = \coth\frac{x}{2} + \tanh\frac{x}{2} \tag{3.7}$$

and thus to the duplication formula (3.8) for the discrete calculus.

Even Duplication

$$\boxed{2\sum_{2x} = \sum + \frac{1}{4}\Delta} \qquad \text{Duplication Formula} \tag{3.8}$$

Where $\sum_{2x}$ is the even sum i.e. the sum over all even indices.

$$\boxed{\sum_{2x} = \Big|_{x/2} \sum \Big|_{2x} = \frac{1}{2}\frac{\mathbf{\uparrow}^2 + \mathbf{1}}{\mathbf{\uparrow}^2 - \mathbf{1}} = \frac{1}{2}\coth\partial} \qquad \text{even duplication} \tag{3.9}$$

$$\begin{aligned} \sum_{2x} &= \frac{1}{2}\coth\partial = \sum_{k=0}^{\infty} \frac{B_{2k}\, 2^{2k-1}}{(2k)!}\partial^{2k-1} \approx \\ &\approx \frac{1}{2}\partial^{-1} + \frac{1}{6}\partial - \frac{1}{90}\partial^3 + \frac{1}{945}\partial^5 - \frac{1}{9450}\partial^7 + \cdots \end{aligned}$$

Odd Duplication

The odd sum $\sum_{2x+1}$ over all odd indices and the even sum $\sum_{2x}$ together must be equal to the sum over all indices.

$$\boxed{\sum = \sum_{2x} + \sum_{2x+1}} \qquad \boxed{2\sum_{2x+1} = \sum - \frac{1}{4}\Delta} \tag{3.10}$$

$$\boxed{\sum_{2x+1} = \frac{\mathbf{\uparrow}}{\mathbf{\uparrow}^2 - \mathbf{1}} = \frac{1}{2\sinh\partial}} \qquad \text{odd duplication} \tag{3.11}$$

$$\begin{aligned} \sum_{2x+1} &= \frac{1}{2\sinh\partial} = \sum_{k=0}^{\infty} \frac{B_{2k}\,(1-2^{2k-1})}{(2k)!}\partial^{2k-1} \approx \\ &\approx \frac{1}{2}\partial^{-1} - \frac{1}{12}\partial + \frac{7}{720}\partial^3 - \frac{31}{30240}\partial^5 + \frac{127}{1209600}\partial^7 - \cdots \end{aligned}$$

Inverse Duplication

We obtain inverse duplication operators simply by inverting the fractions in (3.9) and (3.11).

$$\boxed{\underset{2x}{\Delta} = 2\frac{\mathbf{\uparrow}^2 - \mathbf{1}}{\mathbf{\uparrow}^2 + \mathbf{1}} = 2\tanh\partial} \qquad \text{even duplication} \tag{3.12}$$

$$\begin{aligned}\underset{2x}{\Delta} &= 2\tanh\partial = 2\sum_{k=1}^{\infty}\frac{B_{2k}\,2^{2k}\left(2^{2k}-1\right)}{(2k)!}\partial^{2k-1} \approx \\ &\approx 2\partial - \frac{2}{3}\partial^3 + \frac{4}{15}\partial^5 - \frac{34}{315}\partial^7 + \frac{124}{2835}\partial^9 - \frac{2764}{155925}\partial^{11} + \cdots\end{aligned}$$

In contrast to $\underset{2x}{\Delta}$ the odd duplicate difference $\underset{2x+1}{\Delta} = \frac{\mathbf{\uparrow}^2 - \mathbf{1}}{\mathbf{\uparrow}} = \mathbf{\uparrow} - \mathbf{\downarrow}$ has a very simple representation $\underset{2x+1}{\Delta} f(x) = f(x+1) - f(x-1)$. In chapter 3.15 we take advantage of this fact when handling numerical data.

$$\boxed{\underset{2x+1}{\Delta} = \mathbf{\uparrow} - \mathbf{\downarrow} = 2\sinh\partial} \qquad \text{odd duplication} \tag{3.13}$$

$$\begin{aligned}\underset{2x+1}{\Delta} &= 2\sinh\partial = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\partial^{2k+1} \approx \\ &\approx 2\partial + \frac{1}{3}\partial^3 + \frac{1}{60}\partial^5 + \frac{1}{2520}\partial^7 + \frac{1}{181440}\partial^9 + \frac{1}{19958400}\partial^{11} + \cdots\end{aligned}$$

Duplication in detail

The connection between even/odd and total/alternating sum can intuitively be seen with the following picture of summation coefficients (+ represents +1, − represents −1 and a space represents 0):

$$\begin{array}{lclcll}
\sum & = & + + + + + + + + + + + + \quad \cdots & = & \underset{2x}{\sum} + \underset{2x+1}{\sum} & \text{total} \\
\frac{1}{4}\Delta & = & + - + - + - + - + - + - \quad \cdots & = & \underset{2x}{\sum} - \underset{2x+1}{\sum} & \text{alternating} \\
\underset{2x}{\sum} & = & + \quad + \quad + \quad + \quad + \quad + \quad\; \cdots & = & \frac{1}{2}\sum + \frac{1}{2}\frac{1}{4}\Delta & \text{even} \\
\underset{2x+1}{\sum} & = & \quad + \quad + \quad + \quad + \quad + \quad + \; \cdots & = & \frac{1}{2}\sum - \frac{1}{2}\frac{1}{4}\Delta & \text{odd}
\end{array}$$

There are special duplication formulae for constant $\Delta c(x) = 0$ and anti-constant functions $\Sigma a(x) = 0$.

$$\begin{array}{rclcl} \sum\limits_{2x} c(x) & = & \sum\limits_{2x+1} c(x) = \frac{1}{2}\Sigma c(x) & \Delta c(x) = 0 & \text{constant } c(x) \\ \sum\limits_{2x} a(x) & = - & \sum\limits_{2x+1} a(x) = \frac{1}{8}\Delta a(x) & \Sigma a(x) = 0 & \text{anti-constant } a(x) \end{array}$$

It shall be emphasized that *all* sums have a duplication formula. You will find them for trigonometric and hyperbolic functions, for $\ln\Gamma(x)$, $\Psi^{(n)}(x)$ and $\zeta(x)$.

We also observed that even $\sum\limits_{2x}$ and $\Delta\limits_{2x}$ show the half border behavior of natural discrete calculus whereas $\sum\limits_{2x+1}$ and $\Delta\limits_{2x+1}$ are equal to their corresponding classical sum or difference. This fact will be generalized in the next chapter 3.6 about scaling with arbitrary factors $n \neq 2$.

The duplication formula (3.8) together with the duality relation (2.6) results in following formula for the summation of alternating functions:

$$\Sigma(-1)^x f(x) = \frac{1}{4}(-1)^x \Delta f(x) = (-1)^x \left(2\sum_{2x} - \Sigma\right) f(x) \tag{3.14}$$

Example 4:

$$\begin{array}{rclcl} \Sigma x^5 & = & \frac{1}{2}\coth\frac{\partial}{2}\, x^5 & = & \frac{1}{6}x^6 + \frac{5}{12}x^4 - \frac{1}{12}x^2 + \frac{1}{252} \\ \sum\limits_{2x} x^5 & = & \frac{1}{2}\coth\partial\, x^5 & = & \frac{1}{12}x^6 + \frac{5}{6}x^4 - \frac{2}{3}x^2 + \frac{8}{63} \\ \sum\limits_{2x+1} x^5 & = & \frac{1}{2\sinh\partial}\, x^5 & = & \frac{1}{12}x^6 - \frac{5}{12}x^4 + \frac{7}{12}x^2 - \frac{31}{252} \\ \Delta x^5 & = & 2\tanh\frac{\partial}{2}\, x^5 & = & 5x^4 - 5x^2 + 1 \\ \Delta\limits_{2x} x^5 & = & 2\tanh\partial\, x^5 & = & 10x^4 - 40x^2 + 32 \\ \Delta\limits_{2x+1} x^5 & = & 2\sinh\partial\, x^5 & = & 10x^4 + 20x^2 + 2 = (x+1)^5 - (x-1)^5 \end{array}$$

3.6 Scaling

A difficult problem in discrete calculus are scaled functions $\Sigma f(ax)$ or sums $\sum\limits_{nx} f(x)$ with a step size n different from one. In the previous chapter 3.5 we discussed the case $n = 2$ of duplication which already showed a lot of remarkable features.

Quadruplication

Before we discuss the scaling $\sum\limits_{nx}$ with arbitrary factors n, let us have a closer look to the interesting case $n = 4$. To split $\sum\limits_{4x} = \big|_{x/4} \sum \big|_{4x} = \frac{1}{2}(\uparrow^4 + \mathbf{1}) / (\uparrow^4 - \mathbf{1})$ into terms we use the partial fraction decomposition. $\mathtt{i} := \sqrt{-1}$ shall denote the imaginary unit.

$$\begin{aligned}\sum_{4x} = \frac{1}{2}\frac{\uparrow^4 + \mathbf{1}}{\uparrow^4 - \mathbf{1}} &= \frac{1}{2} + \frac{1}{4}\frac{1}{\uparrow - 1} + \frac{\mathtt{i}}{4}\frac{1}{\uparrow - \mathtt{i}} - \frac{1}{4}\frac{1}{\uparrow + 1} - \frac{\mathtt{i}}{4}\frac{1}{\uparrow + \mathtt{i}} = \\ &= \left(\frac{1}{8} + \frac{1}{4}\frac{1}{\uparrow - 1}\right) + \left(\frac{1}{8} + \frac{\mathtt{i}}{4}\frac{1}{\uparrow - \mathtt{i}}\right) + \left(\frac{1}{8} - \frac{1}{4}\frac{1}{\uparrow + 1}\right) + \left(\frac{1}{8} - \frac{\mathtt{i}}{4}\frac{1}{\uparrow + \mathtt{i}}\right) = \\ &= \frac{1}{8}\frac{\uparrow + 1}{\uparrow - 1} + \frac{1}{8}\frac{\uparrow + \mathtt{i}}{\uparrow - \mathtt{i}} + \frac{1}{8}\frac{\uparrow - 1}{\uparrow + 1} + \frac{1}{8}\frac{\uparrow - \mathtt{i}}{\uparrow + \mathtt{i}}\end{aligned}$$

According to the integer scaling $\sum\limits_{4x}$ we derive the fractional scaling $\sum\limits_{4x+k}$ for $k = 1, 2, 3$.

$$\begin{aligned}\sum_{4x} &= \frac{1}{2}\frac{\uparrow^4 + \mathbf{1}}{\uparrow^4 - \mathbf{1}} = \frac{1}{8}\frac{\uparrow + 1}{\uparrow - 1} + \frac{1}{8}\frac{\uparrow + \mathtt{i}}{\uparrow - \mathtt{i}} + \frac{1}{8}\frac{\uparrow - 1}{\uparrow + 1} + \frac{1}{8}\frac{\uparrow - \mathtt{i}}{\uparrow + \mathtt{i}} \\ \sum_{4x+1} &= \frac{\uparrow}{\uparrow^4 - \mathbf{1}} = \frac{1}{8}\frac{\uparrow + 1}{\uparrow - 1} + \frac{\mathtt{i}}{8}\frac{\uparrow + \mathtt{i}}{\uparrow - \mathtt{i}} - \frac{1}{8}\frac{\uparrow - 1}{\uparrow + 1} - \frac{\mathtt{i}}{8}\frac{\uparrow - \mathtt{i}}{\uparrow + \mathtt{i}} \\ \sum_{4x+2} &= \frac{\uparrow^2}{\uparrow^4 - \mathbf{1}} = \frac{1}{8}\frac{\uparrow + 1}{\uparrow - 1} - \frac{1}{8}\frac{\uparrow + \mathtt{i}}{\uparrow - \mathtt{i}} + \frac{1}{8}\frac{\uparrow - 1}{\uparrow + 1} - \frac{1}{8}\frac{\uparrow - \mathtt{i}}{\uparrow + \mathtt{i}} \\ \sum_{4x+3} &= \frac{\uparrow^3}{\uparrow^4 - \mathbf{1}} = \frac{1}{8}\frac{\uparrow + 1}{\uparrow - 1} - \frac{\mathtt{i}}{8}\frac{\uparrow + \mathtt{i}}{\uparrow - \mathtt{i}} - \frac{1}{8}\frac{\uparrow - 1}{\uparrow + 1} + \frac{\mathtt{i}}{8}\frac{\uparrow - \mathtt{i}}{\uparrow + \mathtt{i}}\end{aligned}$$

These results for $n = 4$ are rather disturbing but show an interesting general pattern.

Scaling with arbitrary factors

With the duplication in chapter 3.5 and the quadruplication in chapter 3.6 we are now prepared to formulate the scaling $\sum\limits_{nx+k}$ with arbitrary factors n i.e. the sum over all indices $i \equiv k \pmod{n}$.

$$\boxed{\sum_{nx+k} = \frac{\uparrow^k}{\uparrow^n - \mathbf{1}} + \frac{1}{2}\mathbf{1}\delta_{k=0}} \qquad \sum_{nx} = \frac{1}{2} \cdot \frac{\uparrow^n + \mathbf{1}}{\uparrow^n - \mathbf{1}} = \Big|_{x/n} \sum \Big|_{nx} \tag{3.15}$$

$$\boxed{\underset{nx+k}{\Delta} = \begin{cases} 2\,\dfrac{\uparrow^n - \mathbf{1}}{\uparrow^n + \mathbf{1}} & \text{for} \quad k = 0 \\ \uparrow^{n-k} - \downarrow^{k} & \text{for} \quad k \neq 0 \end{cases}} \tag{3.16}$$

$$\underset{nx+k}{\Sigma} = \begin{cases} \dfrac{1}{2}\coth\dfrac{n\partial}{2} \\ \dfrac{e^{k\partial}}{2\,\sinh\frac{n\partial}{2}} \end{cases} \qquad \underset{nx+k}{\Delta} = \begin{cases} 2\tanh\dfrac{n\partial}{2} & \text{for} \quad k = 0 \\ 2\,e^{-k\partial}\sinh\dfrac{n\partial}{2} & \text{for} \quad k \neq 0 \end{cases} \tag{3.17}$$

This reflects the fact that $\underset{nx}{\Sigma}$ and $\underset{nx}{\Delta}$ with $k = 0$ show the half border behavior of natural discrete calculus whereas $\underset{nx+k}{\Sigma}$ and $\underset{nx+k}{\Delta}$ with $k \neq 0$ are equal to their corresponding classical sum or difference. Also for $k \neq 0$ there is $\underset{nx+k}{\Delta} f(x) = f(x+n-k) - f(x-k)$. Now we define the circular sums:

$$\boxed{\underset{k/n}{\Sigma} := \frac{1}{2}\cdot\frac{\uparrow + \mathbf{1}e^{2\pi \mathtt{i} k/n}}{\uparrow - \mathbf{1}e^{2\pi \mathtt{i} k/n}} = \frac{1}{2}\coth\left(\frac{\partial}{2} - \frac{\pi \mathtt{i} k}{n}\right)} \qquad \text{circular sum} \tag{3.18}$$

So the discrete Fourier transform of the circular sum operators equals the scaled sums $\underset{nx+k}{\Sigma}$, ideally suitable for applying the FFT *(Fast Fourier Transform)*.

$$\boxed{\underset{nx+k}{\Sigma} = \frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi \mathtt{i} jk/n}\cdot\underset{j/n}{\Sigma}} \qquad \underset{nx}{\Sigma} = \frac{1}{n}\sum_{j=0}^{n-1}\underset{j/n}{\Sigma} \tag{3.19}$$

3.7 Operators in Product Representation

In chapter 4.36 of [NI10] we find product representations for $\cosh x$ and $\frac{\sinh x}{x}$.

$$\ln\frac{\sinh\frac{\partial}{2}}{\frac{\partial}{2}} = \ln\prod_{k=1}^{\infty}\left(1 + \frac{\partial^2}{(2k)^2\pi^2}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!\,k}\partial^{2k} \tag{3.20}$$

$$\ln\cosh\frac{\partial}{2} = \ln\prod_{k=1}^{\infty}\left(1 + \frac{\partial^2}{(2k-1)^2\pi^2}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{2k}\,(2^{2k}-1)}{(2k)!\,k}\partial^{2k} \tag{3.21}$$

Together with $\ln\frac{\sinh\partial}{\partial} = \ln\frac{\sinh(\partial/2)}{\partial/2} + \ln\cosh\frac{\partial}{2}$ we get a product representation of the summation and discrete difference operator which may be useful for number theoretic applications.

$$\mathcal{P} = \ln\cosh\frac{\partial}{2} - \ln\frac{\sinh\frac{\partial}{2}}{\frac{\partial}{2}} = \sum_{k=1}^{\infty}\frac{B_{2k}\,(2^{2k-1}-1)}{(2k)!\,k}\partial^{2k} \tag{3.22}$$

$$\Sigma = e^{\mathcal{P}}\int \qquad \Delta = e^{-\mathcal{P}}\partial$$

3.8 Laplace Transformation

The Laplace transformation is defined by:

$$\mathcal{L}f(x) := \int_0^{\infty} f(t)\,e^{-xt}\,\mathrm{d}t \tag{3.23}$$

$$\mathcal{L}f(ax) = \frac{1}{a}\Big|_{x/a}(\mathcal{L}f) \tag{3.24}$$

$$\mathcal{L}x^n = \frac{n!}{x^{n+1}} \tag{3.25}$$

$$\mathcal{L}(\mathrm{step}_a(x)\cdot f(x-a)) = e^{-ax}\mathcal{L}f(x) \qquad a>0 \tag{3.26}$$

$$\text{with}\quad \mathrm{step}_a(x) := \begin{cases} 1 & \text{for} \quad x>a \\ 0 & \text{for} \quad x<a \end{cases}$$

The Laplace transformation is used to solve the functional equation (2.2) $F(x) - F(x-1) = \frac{1}{2}f(x) + \frac{1}{2}f(x-1)$ by application of (3.26). This results in $\mathcal{L}F(x) - e^{-x}\mathcal{L}F(x) = \frac{1}{2}\mathcal{L}f(x) + \frac{1}{2}e^{-x}\mathcal{L}f(x)$ which can be rewritten in terms of the hyperbolic tangent function.

$$\mathcal{L}F(x) = \frac{1}{2}\coth\frac{x}{2}\cdot\mathcal{L}f(x) \qquad \text{or} \qquad \mathcal{L}\Sigma = \frac{1}{2}\coth\frac{x}{2}\cdot\mathcal{L} \tag{3.27}$$

$$\mathcal{L}f(x) = 2\tanh\frac{x}{2}\cdot\mathcal{L}F(x) \qquad \text{or} \qquad \mathcal{L}\Delta = 2\tanh\frac{x}{2}\cdot\mathcal{L} \tag{3.28}$$

Example 5: For the calculation of Δx^3 by (3.28) we multiply $\mathcal{L}x^3 = 6x^{-4}$ with $2\tanh\frac{x}{2}$ (3.6) which gives $6x^{-3} - \frac{1}{2}x^{-1}$ plus terms $x^{n\geq 0}$ which can be ignored. We then get the final result $\Delta x^3 = 3x^2 - \frac{1}{2}$ by an inverse Laplace transform. The ignored constant term would transform to a Dirac delta peak and the terms $x^{n>0}$ would transform to Dirac derivatives. Analogously we calculate $\Sigma x^3 = \frac{1}{4}x^4 + \frac{1}{4}x^2 - \frac{1}{120}$ by (3.27) from the Laplace transform $6x^{-5} + \frac{1}{2}x^{-3} - \frac{1}{120}x^{-1}$.

3.9 Euler-Maclaurin Summation

As described in [Ap99], it is possible to express sums by integrals plus some correction terms. This is called the Euler-Maclaurin summation.

$$\begin{aligned}\sum_{k=a}^{b} f(k) &= \int_a^b f(x)\,\mathrm{d}x + f(a) + \int_a^b (x - \lfloor x \rfloor)\, f'(x)\,\mathrm{d}x = \\ &= \int_a^b f(x)\,\mathrm{d}x + \frac{1}{2}(f(a) + f(b)) + \int_a^b \left(x - \lfloor x \rfloor - \frac{1}{2}\right) f'(x)\,\mathrm{d}x\end{aligned} \tag{3.29}$$

$$\boxed{\sum f(x) = \int f(x)\ \mathrm{d}x + \int \{x\}\, f'(x)\ \mathrm{d}x} \tag{3.30}$$

Where the half fraction $\{x\} = x - \lfloor x \rfloor - \frac{1}{2}$ is defined by (3.33). The Euler-Maclaurin summation is only valid for integer interval borders a and b because the Bernoulli numbers B_{2k} are the values of the Bernoulli function at integer borders 0 and 1.

$$\boxed{\Big|_{k=a}^{b} \sum f(k) = \int_a^b f(x)\,\mathrm{d}x + \Big|_a^b \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x) + R_{2n}} \tag{3.31}$$

$$R_{2n} = \frac{1}{(2n+1)!} \int_a^b \overline{B}_{2n+1}(x) f^{(2n+1)}(x)\,\mathrm{d}x$$

With the Fourier expansion (3.34) of the half fraction x and integration by parts we see the connection between the Euler-Maclaurin summation and the Poisson summation.

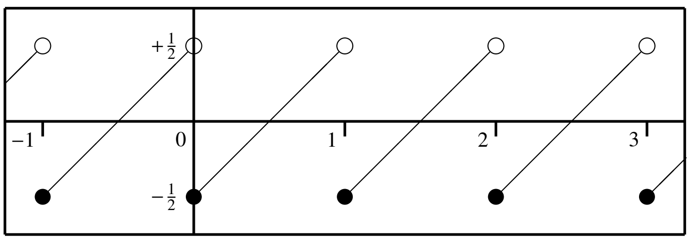
$$\sum f(x) = \int f(x)\,\mathrm{d}x + 2 \sum_{k=1}^{\infty} \int f(x)\ \cos(2\pi k x)\,\mathrm{d}x \tag{3.32}$$

3.10 Half Fraction

For the Euler-Maclaurin summation we have to introduce the half fraction. This is simply the non-integer part of the argument minus one half.

$$\boxed{\{x\} = x - \lfloor x \rfloor - \frac{1}{2}} \tag{3.33}$$

So these half fraction values are symmetrical around zero.



$$\{\lfloor x \rfloor\} \;=\; -\frac{1}{2}$$

$$\lfloor x \rfloor \leq \quad x \quad < \lfloor x \rfloor + 1$$

The half fraction has a simple Fourier series:

$$\{x\} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k} \tag{3.34}$$

3.11 Products

We obtain a product rule for the natural sum and difference simply by inserting the product into the functional equation (2.2).

$$\boxed{\Delta f \cdot g \;=\; f \cdot \Delta g \;+\; g \cdot \Delta f \;-\; \frac{1}{4}\Delta\,(\Delta f \cdot \Delta g)} \qquad \text{product difference} \tag{3.35}$$

The product difference 3.35 is symmetric with respect to the functions and may also be written as $f \cdot g = \Sigma(f \cdot \Delta g + g \cdot \Delta f) - \frac{1}{4}\Delta f \cdot \Delta g$ or as $\Sigma f \cdot \Sigma g = \Sigma(f \cdot \Sigma g + g \cdot \Sigma f) - \frac{1}{4} f \cdot g$.

Additionally we get the sum of a product (3.36) in the form of summation by parts. It shall be noted, that the summation by parts is *not* symmetric with respect to the functions.

$$\boxed{\Sigma f \cdot g \;=\; g \cdot \Sigma f \;+\; \frac{1}{4} f \cdot \Delta g \;-\; \Sigma\,(\Delta g \cdot \Sigma f)} \qquad \text{summation by parts} \tag{3.36}$$

The difference of a product can be expanded into a series of differences.

$$\Delta f \cdot g \;=\; \sum_{k=0}^{n-1} \left(-\tfrac{1}{4}\right)^k \left(\Delta^k f \cdot \Delta^{k+1} g + \Delta^{k+1} f \cdot \Delta^k g\right) \;+\; \left(-\tfrac{1}{4}\right)^n \Delta\left(\Delta^n f \cdot \Delta^n g\right) \tag{3.37}$$

$$\Delta f^2 \;=\; 2f\Delta f \;-\; \tfrac{1}{4}\Delta\left(\Delta f\right)^2 \;=\; 2\sum_{k=0}^{n-1} \left(-\tfrac{1}{4}\right)^k \Delta^k f \cdot \Delta^{k+1} f \;+\; \left(-\tfrac{1}{4}\right)^n \Delta\left(\Delta^n f\right)^2 \tag{3.38}$$

The difference of a product has a finite number of terms when a) there is a zero nth difference $\Delta^n f = 0$ or when b) the nth difference equals the identity operator $\Delta^n f = f$ and thus produces a periodic sequence of terms.

The case a) with a zero nth difference leads to the integer powers which are discussed in chapter 4.4 and 4.5. For products of small integer powers and arbitrary functions $f(x)$ we find the equations:

$$\Delta x \cdot f \;=\; f \;+\; x \cdot \Delta f \;-\; \frac{1}{4}\Delta^2 f \tag{3.39}$$

$$\sum x \cdot f \;=\; x \cdot \sum f \;+\; \frac{1}{4} f \;-\; \sum\sum f \tag{3.40}$$

3.12 Product Pattern

By inserting the product pattern (3.41) into the functional equation (2.2) we get the relation (3.42) which allows us to calculate $f(x)$ and $F(x)$ for any given kernel $k(x)$ which has a product representation.

$$\sum k(x) \cdot f(x) \;=\; k(x) \cdot F(x) \tag{3.41}$$

$$\frac{k(x+1)}{k(x)} \;=\; \frac{2F(x) + f(x)}{2F(x+1) - f(x+1)} \tag{3.42}$$

Example 6:

$k(x)$	$\frac{k(x+1}{k(x)}$	$f(x)$	$F(x)$
$x!$	$x + 1$	x	$\frac{1}{2}x + 1$
$(2x)!$	$(2x + 1) \cdot (2x + 2)$	$\frac{4}{3}x^2 + 2x + \frac{1}{3}$	$\frac{2}{3}x^2 + x + \frac{1}{2}$
$\binom{2x}{x}4^{-x}$	$\frac{2x+1}{2x+2}$	1	$2x + \frac{1}{2}$

3.13 Multiple Summation and Differentiation

Multiple summation Σ^n and differentiation operators Δ^n can be obtained by raising (3.2) and (3.3) to the n'th power.

$$\Delta^n = 2^n \left(\frac{\boldsymbol{\uparrow} - \mathbf{1}}{\boldsymbol{\uparrow} + \mathbf{1}}\right)^n = 2^n \tanh^n \frac{\partial}{2} \qquad \Sigma^n = \frac{1}{2^n}\left(\frac{\boldsymbol{\uparrow} + \mathbf{1}}{\boldsymbol{\uparrow} - \mathbf{1}}\right)^n = \frac{1}{2^n} \coth^n \frac{\partial}{2} \tag{3.43}$$

The expansion of these operators with respect to ∂ show us that $\Sigma^n \approx \partial^{-n} = \int^n$ and $\Delta^n \approx \partial^n$.

$$\begin{aligned} \Delta^n &\approx \partial^n - \tfrac{n}{12}\partial^{n+2} + \tfrac{n(5n+7)}{60\cdot 4!}\partial^{n+4} - \tfrac{n(35n^2+147n+124)}{504\cdot 6!}\partial^{n+6} + \tfrac{n(175n^3+1470n^2+3509n+2286)}{2160\cdot 8!}\partial^{n+8} - \cdots \\ \Sigma^n &\approx \partial^{-n} + \tfrac{n}{12}\partial^{2-n} + \tfrac{n(5n-7)}{60\cdot 4!}\partial^{4-n} + \tfrac{n(35n^2-147n+124)}{504\cdot 6!}\partial^{6-n} + \tfrac{n(175n^3-1470n^2+3509n-2286)}{2160\cdot 8!}\partial^{8-n} + \cdots \end{aligned} \tag{3.44}$$

Here we should stop and think about commutativity. Multiple differentiation Δ^n is uncomplicated, because it depends only on derivative operators ∂. But multiple summation also contains integration operators $\int_c = \partial^{-1} + c$ which do not commute with the derivative operator $\left[\int_c, \partial\right] = \int_c \partial - \partial \int_c = c + a$ when the integration $\int_c$ adds an arbitrary constant c. The second constant residue a of the commutator is due to $\partial a = 0$. So even if we define $c = 0$, there is no commutativity between integration and the derivative operator. This commutativity situation also exists between summation and difference operator.

So each consecutive application of a summation operator as part of Σ^n can add a different constant c_k with $k = 1 \ldots n$. This results in additional terms $\Sigma^{n-k} c_k$ for Σ^n which are no longer constant. So we can have very different kinds of multiple summation operators Σ^n. Normally we define $c_k = 0$ for the iterative summation $\Sigma \circ \Sigma \circ \cdots \circ \Sigma$ but our natural discrete calculus uses different constants when Σ^n is represented by (3.43). A third representation of Σ^n is described in chapter 3.14. There the summation constants c_n are chosen to satisfy $e_n(0) := 0$ for the elementary Taylor polynomials $e_n(x) := \Sigma^n 1$.

Natural Unit Sums

Natural unit sums $s_n(x)$ are defined by $s_n(x) = \Sigma^n 1$ with the natural representation (3.43) and (3.44).

$$
\begin{aligned}
s_0(x) &= 1 \\
s_1(x) &= x \\
s_2(x) &= \tfrac{1}{2}x^2 + \tfrac{1}{6} \\
s_3(x) &= \tfrac{1}{6}x^3 + \tfrac{1}{4}x \\
s_4(x) &= \tfrac{1}{24}x^4 + \tfrac{1}{6}x^2 + \tfrac{13}{360} \\
s_5(x) &= \tfrac{1}{120}x^5 + \tfrac{5}{72}x^3 + \tfrac{1}{16}x \\
s_6(x) &= \tfrac{1}{720}x^6 + \tfrac{1}{48}x^4 + \tfrac{23}{480}x^2 + \tfrac{251}{30240} \\
s_7(x) &= \tfrac{1}{5040}x^7 + \tfrac{7}{1440}x^5 + \tfrac{49}{2160}x^3 + \tfrac{1}{64}x \\
s_8(x) &= \tfrac{1}{40320}x^8 + \tfrac{1}{1080}x^6 + \tfrac{11}{1440}x^4 + \tfrac{11}{840}x^2 + \tfrac{3551}{1814400} \\
s_9(x) &= \tfrac{1}{362880}x^9 + \tfrac{1}{6720}x^7 + \tfrac{19}{9600}x^5 + \tfrac{409}{60480}x^3 + \tfrac{1}{256}x
\end{aligned}
$$

We get an analytic formula for the natural unit sums by using the Laplace representation (3.27) $\Sigma f(x) = \mathcal{L}^{-1}\left(\frac{1}{2}\coth\frac{x}{2}\cdot\mathcal{L}f(x)\right)$ of the summation operator. Applying this Laplace transformation on the constant $f(x) = 1$ with $\mathcal{L}1 = \frac{1}{x}$ and on the Taylor expansion of $\coth^n\frac{x}{2}$ results in:

$$
\begin{aligned}
s_n(x) &= \Sigma^n 1 = \frac{1}{2^n}\mathcal{L}^{-1}\left(\coth^n\frac{x}{2}\cdot\mathcal{L}1\right) = \frac{1}{2^n}\mathcal{L}^{-1}\left(\frac{1}{x}\coth^n\frac{x}{2}\right) = \qquad (3.45) \\
&= \frac{1}{n!}\,x^{-n} + \frac{n}{12(n-2)!}\,x^{n-2} + \frac{n(5n-7)}{60\cdot 4!\cdot(n-4)!}\,x^{n-4} + \\
&+ \frac{n(35n^2-147n+124)}{504\cdot 6!\cdot(n-6)!}\,x^{n-6} + \frac{n(175n^3-1470n^2+3509n-2286)}{2160\cdot 8!\cdot(n-8)!}\,x^{n-8} + \cdots
\end{aligned}
$$

Zero Unit Sums

Zero unit sums $z_n(x)$ are defined by $z_n(x) := \Sigma z_{n-1}(x)$ and $z_0(x) := 1$ as iterative summation of 1. Note that even the additive summation constant $c = 0$ allows here constant terms (e.g. $\frac{1}{12}$ in $z_2(x)$).

$$\begin{aligned}
z_0(x) &= 1 \\
z_1(x) &= x \\
z_2(x) &= \tfrac{1}{2}x^2 + \tfrac{1}{12} \\
z_3(x) &= \tfrac{1}{6}x^3 + \tfrac{1}{6}x \\
z_4(x) &= \tfrac{1}{24}x^4 + \tfrac{1}{8}x^2 + \tfrac{1}{80} \\
z_5(x) &= \tfrac{1}{120}x^5 + \tfrac{1}{18}x^3 + \tfrac{23}{720}x \\
z_6(x) &= \tfrac{1}{720}x^6 + \tfrac{5}{288}x^4 + \tfrac{7}{240}x^2 + \tfrac{1}{448} \\
z_7(x) &= \tfrac{1}{5040}x^7 + \tfrac{1}{240}x^5 + \tfrac{11}{720}x^3 + \tfrac{11}{1680}x \\
z_8(x) &= \tfrac{1}{40320}x^8 + \tfrac{7}{8640}x^6 + \tfrac{19}{3456}x^4 + \tfrac{409}{60480}x^2 + \tfrac{1}{2304} \\
z_9(x) &= \tfrac{1}{362880}x^9 + \tfrac{1}{7560}x^7 + \tfrac{43}{28800}x^5 + \tfrac{359}{90720}x^3 + \tfrac{563}{403200}x
\end{aligned}$$

3.14 Taylor Expansion and Elementary Polynomials

The Taylor expansion is used in infinitesimal calculus to split a function $f(x)$ into a series $\sum f_k\, e_k(x)$ of elementary polynomials $e_k(x) = \frac{1}{k!}x^k$ with coefficients $f_k = \Big|_{x=0} \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x)$. Terms with order $< k$ vanish because of the repeated derivatives and terms with order $> k$ vanish because of $e_k(0) = \delta_{k=0}$. In discrete calculus we are also able to construct such elementary polynomials by repeated summation using the following definition:

$$\boxed{e_n(x) := \sum e_{n-1}(x)} \qquad e_0 := 1 \qquad e_n(0) := \delta_{n=0} \tag{3.46}$$

The undefined constant summation offsets are chosen to satisfy $e_n(0) = \delta_{n=0}$. All elementary polynomials evaluate to zero at the expansion point except $e_0 = 1$. So we get following representation of the elementary polynomials which are even/odd functions for even/odd n.

$$
\begin{aligned}
e_0(x) &= 1 \\
e_1(x) &= x \\
e_2(x) &= \tfrac{1}{2}x^2 \\
e_3(x) &= \tfrac{1}{6}x^3 + \tfrac{1}{12}x \\
e_4(x) &= \tfrac{1}{24}x^4 + \tfrac{1}{12}x^2 \\
e_5(x) &= \tfrac{1}{120}x^5 + \tfrac{1}{24}x^3 + \tfrac{1}{80}x \\
e_6(x) &= \tfrac{1}{720}x^6 + \tfrac{1}{72}x^4 + \tfrac{23}{1440}x^2 \\
e_7(x) &= \tfrac{1}{5040}x^7 + \tfrac{1}{288}x^5 + \tfrac{7}{720}x^3 + \tfrac{1}{448}x \\
e_8(x) &= \tfrac{1}{40320}x^8 + \tfrac{1}{1440}x^6 + \tfrac{11}{2880}x^4 + \tfrac{11}{3360}x^2 \\
e_9(x) &= \tfrac{1}{362880}x^9 + \tfrac{1}{8640}x^7 + \tfrac{19}{17280}x^5 + \tfrac{409}{181440}x^3 + \tfrac{1}{2304}x
\end{aligned}
$$

From equation (3.45) in chapter 3.13 we find following analytic formula without any proof. The prefix $x/n\partial$ cancels the constant term.

$$
\begin{aligned}
e_n(x) &= \frac{x}{n}\partial\, s_n(x) = \frac{x}{n}\partial \sum^n 1 = \text{conjecture} \qquad (3.47)\\
&= \frac{1}{n!}\,x^{-n} + \frac{1}{12(n-3)!}\,x^{n-2} + \frac{5n-7}{60\cdot 4!\cdot(n-5)!}\,x^{n-4} + \\
&+ \frac{35n^2-147n+124}{504\cdot 6!\cdot(n-6)!}\,x^{n-7} + \frac{175n^3-1470n^2+3509n-2286}{2160\cdot 8!\cdot(n-9)!}\,x^{n-8} + \cdots
\end{aligned}
$$

Example 7: The Taylor expansion of $f(x) = 11x^5 + 13x^4 + 17x^3 + 19x^2 + 23x + 43$ is $f_{0...5} = [43, \frac{51}{2}, -14, -228, 312, 1320]$. Because of the even/odd feature of the elementary polynomials only the first $c_0 = 43$, the last $c_5 = \frac{1320}{5!} = 11$ and the last but one $c_4 = \frac{312}{4!} = 13$ classical polynomial coefficients are directly visible. The great benefit of this Taylor expansion into elementary polynomials is the calculation of the discrete difference and especially the sum simply by shifting the coefficients. So the discrete difference is $f_{0...4} = [\frac{51}{2}, -14, -228, 312, 1320]$ or $\Delta f(x) = 55x^4 + 52x^3 - 4x^2 + 12x + \frac{51}{2}$. The sum is $f_{1...6} = [43, \frac{51}{2}, -14, -228, 312, 1320]$ or $\Sigma f(x) = \frac{11}{6}x^6 + \frac{13}{5}x^5 + \frac{53}{6}x^4 + \frac{32}{3}x^3 + \frac{89}{6}x^2 + \frac{686}{15}x$.

3.15 Local Approximation

Numerical data in form of a series or a vector $f_x = f(x)$ is only accessible at integer indices $x \in \mathbb{N}$. So a local approximation of order n around x shall depend only on values f_k with k near x or $|k - x| \le n$, i.e. only small powers $-n \cdots n$ of the increment operator $\uparrow^{k-x}$ can be used.

We are interested in a local approximation of the summation and discrete difference operator. So we have to express them in terms of the increment operator. This can be done by using the very simple and finite representation of $2\sinh(n\partial) = e^{n\partial} - e^{-n\partial} = \uparrow^n - \downarrow^n$ and $2\cosh(n\partial) = e^{n\partial} + e^{-n\partial} = \uparrow^n + \downarrow^n$ as increments.

$$\boxed{\begin{array}{rcl} 2\cosh(n\partial) &=& \uparrow^n + \downarrow^n \\ 2\sinh(n\partial) &=& \uparrow^n - \downarrow^n \end{array}} \qquad \begin{array}{l} \equiv \; f(x+n) + f(x-n) \\ \equiv \; f(x+n) - f(x-n) \end{array} \tag{3.48}$$

A short look on the series expansions (3.4) in terms of the derivative operator shows, that sinh, the sum and difference operator are odd functions whereas cosh is even. Furthermore only the sum operator contains a term ∂^{-1}. Therefore the sum can not be expanded into sinh and cosh. But it is possible to expand the difference operator in a series of $\sinh(n\partial)$ terms. We may use a coefficient comparison between the expansion (3.44) of Δ^n with respect to ∂ and $\sinh(n\partial)$ terms. But there is a simpler way to obtain an expansion by using half integer steps $\sinh\frac{n\partial}{2} = \uparrow^{n/2} - \downarrow^{n/2}$. In the following we define $2c := \cosh\frac{\partial}{2} = \uparrow^{1/2} + \downarrow^{1/2}$ and $2s := \sinh\frac{\partial}{2} = \uparrow^{1/2} - \downarrow^{1/2}$. The expansions of Δ^n in terms of $\sinh(k\partial)$ up to order $k \le m$ is denoted by $\overset{m}{\approx}$.

Odd Difference Powers

$$\begin{aligned} \Delta^n &= \left(2\tanh\frac{\partial}{2}\right)^n = 2^n\frac{s^n}{c^n} = 2^n c\frac{s^n}{(1+s^2)^{\frac{n+1}{2}}} \approx \qquad \boxed{\text{odd } n} \\ &\approx 2^n c s^n\left(1 - \frac{n+1}{2}s^2 + \frac{(n+1)(n+3)}{2\cdot 4}s^4 - \frac{(n+1)(n+3)(n+5)}{2\cdot 4\cdot 6}s^6 + \cdots\right) \\ &\overset{\frac{n+1}{2}}{\approx} 2^n c s^n = \frac{1}{2}\left(\uparrow^{\frac{1}{2}} + \downarrow^{\frac{1}{2}}\right)\left(\uparrow^{\frac{1}{2}} - \downarrow^{\frac{1}{2}}\right)^n = \frac{1}{2}\left(\uparrow - \downarrow\right)\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\uparrow^{k-\frac{n-1}{2}} \end{aligned}$$

$$
\begin{aligned}
\Delta &\overset{1}{\approx} \tfrac{1}{2}(\uparrow - \downarrow) \qquad \text{i.e.} \qquad \Delta f(x) \overset{1}{\approx} \tfrac{1}{2} f(x+1) - \tfrac{1}{2} f(x-1) \\
&\overset{2}{\approx} \tfrac{3}{4}(\uparrow - \downarrow) - \tfrac{1}{8}\left(\uparrow^2 - \downarrow^2\right) \\
&\overset{3}{\approx} \tfrac{29}{32}(\uparrow - \downarrow) - \tfrac{1}{4}\left(\uparrow^2 - \downarrow^2\right) + \tfrac{1}{32}\left(\uparrow^3 - \downarrow^3\right) \\
&\overset{4}{\approx} \tfrac{65}{64}(\uparrow - \downarrow) - \tfrac{23}{64}\left(\uparrow^2 - \downarrow^2\right) + \tfrac{5}{64}\left(\uparrow^3 - \downarrow^3\right) - \tfrac{1}{128}\left(\uparrow^4 - \downarrow^4\right)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^3 &\overset{2}{\approx} -(\uparrow - \downarrow) + \tfrac{1}{2}\left(\uparrow^2 - \downarrow^2\right) \\
&\overset{3}{\approx} -\tfrac{9}{4}(\uparrow - \downarrow) + \tfrac{3}{2}\left(\uparrow^2 - \downarrow^2\right) - \tfrac{1}{4}\left(\uparrow^3 - \downarrow^3\right)
\end{aligned}
$$

$$
\Delta^5 \overset{3}{\approx} \tfrac{5}{2}(\uparrow - \downarrow) - 2\left(\uparrow^2 - \downarrow^2\right) + \tfrac{1}{2}\left(\uparrow^3 - \downarrow^3\right)
$$

$$
\Delta^7 \overset{4}{\approx} -7(\uparrow - \downarrow) + 7\left(\uparrow^2 - \downarrow^2\right) - 3\left(\uparrow^3 - \downarrow^3\right) + \tfrac{1}{2}\left(\uparrow^4 - \downarrow^4\right)
$$

Even Difference Powers

$$
\begin{aligned}
\Delta^n &= \left(2 \tanh \frac{\partial}{2}\right)^n = 2^n \frac{s^n}{c^n} = 2^n \frac{s^n}{(1+s^2)^{\frac{n}{2}}} \approx \qquad \boxed{\text{even } n} \\
&\approx 2^n s^n \left(1 - \frac{n}{2} s^2 + \frac{n(n+2)}{2 \cdot 4} s^4 - \frac{n(n+2)(n+4)}{2 \cdot 4 \cdot 6} s^6 + \frac{n(n+2)(n+4)(n+6)}{2 \cdot 4 \cdot 6 \cdot 8} s^8 - \cdots\right) \\
&\overset{\frac{n}{2}}{\approx} 2^n s^n = \left(\uparrow^{\frac{1}{2}} - \downarrow^{\frac{1}{2}}\right)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \uparrow^{k-\frac{n}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2 &\overset{1}{\approx} -2 \cdot \mathbf{1} + (\uparrow + \downarrow) \qquad \text{i.e.} \qquad \Delta^2 f(x) \overset{1}{\approx} f(x-1) - 2f(x) + f(x+1) \\
&\overset{2}{\approx} -\tfrac{7}{2} \cdot \mathbf{1} + 2(\uparrow + \downarrow) - \tfrac{1}{4}\left(\uparrow^2 + \downarrow^2\right) \\
&\overset{3}{\approx} -\tfrac{19}{4} \cdot \mathbf{1} + \tfrac{47}{16}(\uparrow + \downarrow) - \tfrac{5}{8}\left(\uparrow^2 + \downarrow^2\right) + \tfrac{1}{16}\left(\uparrow^3 + \downarrow^3\right)
\end{aligned}
$$

$$
\Delta^4 \overset{2}{\approx} 6 \cdot \mathbf{1} - 4(\uparrow + \downarrow) + \left(\uparrow^2 + \downarrow^2\right)
$$

$$
\Delta^6 \overset{3}{\approx} -20 \cdot \mathbf{1} + 15(\uparrow + \downarrow) - 6\left(\uparrow^2 + \downarrow^2\right) + \left(\uparrow^3 + \downarrow^3\right)
$$

3.16 Periodic Numerical Data

The discrete difference and sum of periodic numerical data f_x with integer $0 \le x < n$ and period n or $f_{x+n} = f_x$ can be obtained using the Fourier transform $\tilde{f}_x = \mathcal{F} f_x$.

$$\begin{aligned} \tilde{f}_x &= \mathcal{F} f_x &&= \frac{1}{n}\sum_{k=0}^{n-1} f_k \cdot e^{-2\pi \mathrm{i} kx/n} \\ f_x &= \mathcal{F}^{-1}\tilde{f}_x &&= \sum_{k=0}^{n-1} \tilde{f}_k \cdot e^{2\pi \mathrm{i} kx/n} \end{aligned}$$

With the discrete difference and sum of the exponential function (4.1) as $\Delta^m e^{\mathrm{i}cx} = 2^m \mathtt{i}^m \tan^m \frac{c}{2} e^{\mathrm{i}cx}$ and $\Sigma e^{\mathrm{i}cx} = -\frac{1}{2}\mathtt{i} \cot \frac{c}{2} e^{\mathrm{i}cx}$ we get the equations:

$$\begin{aligned} \Delta^m f(x) &= 2^m \mathtt{i}^m \sum_{k=0}^{n-1} \tilde{f}_k \tan^m \tfrac{k\pi}{n} e^{2\pi \mathrm{i} kx/n} &&= 2^m \mathtt{i}^m \mathcal{F}^{-1}\left(\tan^m \tfrac{k\pi}{n} \cdot \tilde{f}_k\right) \\ \Sigma f(x) &= -\frac{1}{2}\mathtt{i} \sum_{k=0}^{n-1} \tilde{f}_k \cot \tfrac{k\pi}{n} e^{2\pi \mathrm{i} kx/n} &&= -\frac{1}{2}\mathtt{i}\mathcal{F}^{-1}\left(\cot \tfrac{k\pi}{n} \cdot \tilde{f}_k\right) \end{aligned}$$

There are some important remarks: Both equations are convolutions of f_k with $\mathcal{F}^{-1} \tan \frac{k\pi}{n}$ respectively $\mathcal{F}^{-1} \cot \frac{k\pi}{n}$. Further the difference of the constant is zero because $\tilde{f}_0$ is ignored due to $\tan 0 = 0$. The same holds for the zero sum of $\tilde{f}_{n/2}$. Finally the term $\tilde{f}_{n/2}$ in the difference formula has to be handled separately because of the pole $\tan \frac{\pi}{2}$. Simply set the frequency $\tilde{f}_{n/2} = 0$ when applying the difference formula and add $\tilde{f}_{n/2}\Delta e^{\mathrm{i}\pi x} = \tilde{f}_{n/2}\Delta(-1)^x = 4x(-1)^x \tilde{f}_{n/2}$ to the result. The same holds for the frequency $\tilde{f}_0$ in the sum formula with the additional term $4x\tilde{f}_0$.

4 Special Functions

In this chapter we provide sum and discrete difference formulae for elementary and some special functions. A few of these formulae were found by implementing an experimental symbolic relation generator under SYMPY [Sy] including subroutines for symbolic discrete differentiation and summation.

4.1 Exponential Function

The functional equation (2.2) with $x \pm \frac{1}{2}$ arguments shows that the exponential function e^{cx} is proportional to it's sum $\Sigma e^{cx} := f_c \cdot e^{cx}$. The functional equation can be simplified to $f_c \cdot (e^{c/2} - e^{-c/2}) = \frac{c}{2}(e^{1/2} + e^{c/2})$ or $f_c = \frac{1}{2}\coth\frac{c}{2}$.

$$\begin{aligned} \Delta e^{cx} &= 2\tanh\frac{c}{2}\cdot e^{cx} & \sum e^{cx} &= \frac{1}{2}\coth\frac{c}{2}\cdot e^{cx} \\ \Delta a^{x} &= 2\,\frac{a-1}{a+1}\cdot a^{x} & \sum a^{x} &= \frac{1}{2}\,\frac{a+1}{a-1}\cdot a^{x} \\ \Delta a^{bx+c} &= 2\,\frac{a^b-1}{a^b+1}\cdot a^{bx+c} & \sum a^{bx+c} &= \frac{1}{2}\,\frac{a^b+1}{a^b-1}\cdot a^{bx+c} \end{aligned} \tag{4.1}$$

We see that 3^x is equal to it's own sum because of $\coth\frac{\ln 3}{2} = 2$. So 3^x is the discrete version of the natural exponential function and 3 is the discrete $e \approx 2.71828$. 1^x is constant because of $\Delta 1^x = 0$ and $(-1)^x$ is anti-constant because of $\Sigma(-1)^x = 0$.

$$\begin{aligned} \Delta 1^{x} &= 0 & \sum 1^{x} &= x \\ \Delta(-1)^{x} &= 4x\cdot(-1)^{x} & \sum(-1)^{x} &= 0 \\ \Delta 3^{x} &= 3^{x} & \sum 3^{x} &= 3^{x} \\ \Delta 3^{-x} &= -3^{-x} & \sum 3^{-x} &= -3^{-x} \end{aligned} \tag{4.2}$$

The proportional factor will be the imaginary unit $\pm\mathtt{i}$ for $c = \ln\left(\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right)$ with $\left|\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right| = 1$ and $\left(\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right)^{-1} = \frac{3}{5} \mp \frac{4}{5}\mathtt{i}$. This is useful for trigonometric functions.

$$\Delta\left(\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right)^{x} = \pm\mathtt{i}\left(\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right)^{x} \qquad \sum\left(\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right)^{x} = \mp\mathtt{i}\left(\frac{3}{5} \pm \frac{4}{5}\mathtt{i}\right)^{x} \tag{4.3}$$

4.2 Hyperbolic Functions

With the exponential sum (4.1) we formulate the hyperbolic sums. The proportional factor will be one for $c = \ln 3 \approx 1.09861\,22886\,68109\,69140$.

$$\begin{aligned} \Delta\sinh(cx) &= 2\tanh\frac{c}{2}\cdot\cosh(cx) & \sum\sinh(cx) &= \frac{1}{2}\coth\frac{c}{2}\cdot\cosh(cx) \\ \Delta\cosh(cx) &= 2\tanh\frac{c}{2}\cdot\sinh(cx) & \sum\cosh(cx) &= \frac{1}{2}\coth\frac{c}{2}\cdot\sinh(cx) \end{aligned} \tag{4.4}$$

4.3 Trigonometric Functions

The imaginary argument in (4.3) introduces a sign change for trigonometric sums. It is remarkable that the trigonometric sums are proportional to the integrals. The sum for $c = 1$ has approximately 91.52% the size of the integral. The proportional factor will be one for $c = 2\arctan\frac{1}{2} \approx 0.92729\,52180\,01612\,23243$.

$$\begin{aligned} \Delta \sin(cx) &= 2\tan\frac{c}{2}\cdot\cos(cx) & \qquad \Sigma \sin(cx) &= -\frac{1}{2}\cot\frac{c}{2}\cdot\cos(cx) \\ \Delta \cos(cx) &= -2\tan\frac{c}{2}\cdot\sin(cx) & \qquad \Sigma \cos(cx) &= \frac{1}{2}\cot\frac{c}{2}\cdot\sin(cx) \end{aligned} \tag{4.5}$$

The degenerated cases for $c = n\pi$ represent constant and anti-constant functions which must be handled separately as described in chapters 2.3, 2.10 and 2.11. So we get $\Delta\cos((2n+1)\pi x) = 4x\cdot\cos((2n+1)\pi x)$, $\Delta\sin((2n+1)\pi x) = 4x\cdot\sin((2n+1)\pi x)$, $\Sigma\cos(2n\pi x) = x\cdot\cos(2n\pi x)$ and $\Sigma\sin(2n\pi x) = x\cdot\sin(2n\pi x)$.

4.4 Differences of Integer Powers

With the chapter 3.8 about Laplace transformations we are prepared to solve the general case Δx^n by using (3.28), (3.6) and $\mathcal{L}x^n = n!\,x^{-n-1}$. The Laplace transform is then $4n!\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(2^{2k}-1)x^{2k-n-2}$. For the inverse Laplace transform only the terms $k \le \frac{n+1}{2}$ are relevant. All other terms would transform to Dirac derivatives. The result is then $\Delta x^n = 4n!\sum_{k=1}^{k_{\max}} B_{2k}/(2k)!\,(2^{2k}-1)\,x^{n+1-2k}/(n+1-2k)!$.

$$\begin{aligned}
\Delta 1 = \Delta x^0 &= 0 \\
\Delta x = \Delta x^1 &= 1 \\
\Delta x^2 &= 2x \\
\Delta x^3 &= 3x^2 - \tfrac{1}{2} \\
\Delta x^4 &= 4x^3 - 2x \\
\Delta x^5 &= 5x^4 - 5x^2 + 1 \\
\Delta x^6 &= 6x^5 - 10x^3 + 6x \\
\Delta x^7 &= 7x^6 - \tfrac{35}{2}x^4 + 21x^2 - \tfrac{17}{4} \\
\Delta x^8 &= 8x^7 - 28x^5 + 56x^3 - 34x \\
\Delta x^9 &= 9x^8 - 42x^6 + 126x^4 - 153x^2 + 31
\end{aligned}$$

$$\Delta x^n = 4\frac{x^{n+1}}{n+1}\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor} B_{2k}\left(2^{2k}-1\right)\binom{n+1}{2k}x^{-2k} \tag{4.6}$$

4.5 Sums of Integer Powers

Analogously we calculate the general case Σx^n by using (3.27), (3.5) and $\mathcal{L}x^n = n!\,x^{-n-1}$.
The Laplace transform is $n!\sum_{k=0}^{\infty} B_{2k}/(2k)!x^{2k-n-2}$ with $k \le \frac{n+1}{2}$.
The result is $\Sigma x^n = n!\sum_{k=0}^{k_{\max}} B_{2k}/(2k)!x^{n+1-2k}/(n+1-2k)!$.

$$\begin{aligned}
\Sigma 1 = \Sigma x^0 &= x \\
\Sigma x = \Sigma x^1 &= \tfrac{1}{2}x^2 + \tfrac{1}{12} \\
\Sigma x^2 &= \tfrac{1}{3}x^3 + \tfrac{1}{6}x \\
\Sigma x^3 &= \tfrac{1}{4}x^4 + \tfrac{1}{4}x^2 - \tfrac{1}{120} \\
\Sigma x^4 &= \tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x \\
\Sigma x^5 &= \tfrac{1}{6}x^6 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2 + \tfrac{1}{252} \\
\Sigma x^6 &= \tfrac{1}{7}x^7 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x \\
\Sigma x^7 &= \tfrac{1}{8}x^8 + \tfrac{7}{12}x^6 - \tfrac{7}{24}x^4 + \tfrac{1}{12}x^2 - \tfrac{1}{240} \\
\Sigma x^8 &= \tfrac{1}{9}x^9 + \tfrac{2}{3}x^7 - \tfrac{7}{15}x^5 + \tfrac{2}{9}x^3 - \tfrac{1}{30}x
\end{aligned}$$

$$\Sigma x^n = \frac{x^{n+1}}{n+1}\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} B_{2k}\binom{n+1}{2k}x^{-2k} = \frac{B_{n+1}(x)}{n+1} + \frac{x^n}{2} \tag{4.7}$$

4.6 Rational Functions

The digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x) = \partial \ln\Gamma(x)$ is defined as the logarithmic derivative of the gamma function $\Gamma(x) = (x-1)!$ or factorial. The logarithmic derivative of the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ results in $\Psi(x+1) - \Psi(x) = \frac{1}{x}$. With the duplication formula (3.8) we get the discrete difference.

$$\Sigma \frac{1}{x} = \Psi(x) + \frac{1}{2x} = \ln|x| - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, x^{2k}} \tag{4.8}$$

$$\Delta \frac{1}{x} = 2\Psi\left(\frac{x}{2}\right) - 2\Psi\left(\frac{x+1}{2}\right) + \frac{2}{x} = 4\sum_{k=1}^{\infty} \frac{B_{2k}\left(1-2^{2k}\right)}{2k\, x^{2k}} \tag{4.9}$$

The Euler-Maclaurin sum of the digamma function $\Psi(x) = \ln x - \frac{1}{2x} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, x^{2k}}$ can be found as equation 5.11.2 in [NI10]. We recognize the sum (4.8) according to expansion (3.4) $\frac{1}{2} \coth \frac{\partial}{2} \approx \partial^{-1} + \frac{1}{12}\partial - \frac{1}{720}\partial^3 + \cdots$ as the integral $\int \frac{1}{x} = \ln x$ plus some Euler-Maclaurin correction terms. The sum of inverse numbers is sometimes called „harmonic number“ $H_n = \sum_{k=1}^{n} \frac{1}{k} = \gamma + \Psi(n+1)$ which can be expressed using the digamma function.

The n'th derivative $\Psi^{(n)}(x)$ of the digamma function is called polygamma function. So we get the generalized recurrence relation $\Psi^{(n)}(x+1) - \Psi^{(n)}(x) = (-1)^n\, n!\, x^{-n-1}$ and hence expressions for sum and difference of x^{-n} with $n > 1$.

$$\Sigma \frac{1}{x^n} = \frac{(-1)^{n-1}}{(n-1)!}\Psi^{(n-1)}(x) + \frac{1}{2x^n} \overset{n\geq 1}{=} \frac{x^{-n+1}}{-n+1} \sum_{k=0}^{\infty} B_{2k} \binom{2k+n-2}{2k} x^{-2k} \tag{4.10}$$

$$\Delta \frac{1}{x^n} = 4\frac{(-1)^{n-1}}{(n-1)!\, 2^n}\left(\Psi^{(n-1)}\left(\frac{x}{2}\right) - \Psi^{(n-1)}\left(\frac{x+1}{2}\right)\right) + \frac{2}{x^n} = \tag{4.11}$$

$$\overset{n\geq 1}{=} 4\frac{x^{-n+1}}{-n+1} \sum_{k=1}^{\infty} B_{2k} \binom{2k+n-2}{2k} \left(2^{2k}-1\right) x^{-2k}$$

With the equations above we are able to handle rational functions which have a partial fraction decomposition of the form $\sum \text{polynomial}/(ax+b)^n$. The recurrence relation $\Psi^{(n)}(x+b) = \Psi^{(n)}(x) + (-1)^n\, n! \sum_{k=0}^{b-1} (x+k)^{-n-1}$ of the polygamma function can used to expand the sums $\sum (ax+b)^{-n}$ for integer b.

$$\Sigma \frac{1}{(x-b)^n} = \frac{(-1)^{n-1}}{(n-1)!}\Psi^{(n-1)}(x) - \sum_{k=1}^{b-1} \frac{1}{(x-k)^n} - \frac{1}{2(x-b)^n} \qquad \text{for} \quad b \in \mathbb{N} \tag{4.12}$$

$$\Sigma \frac{1}{(x+b)^n} = \frac{(-1)^{n-1}}{(n-1)!}\Psi^{(n-1)}(x) + \sum_{k=0}^{b-1} \frac{1}{(x+k)^n} + \frac{1}{2(x+b)^n} \qquad \text{for} \quad b \in \mathbb{N} \tag{4.13}$$

Example 8:

$$\begin{aligned}
\sum \frac{1}{(x-3)^2} &= -\Psi_1(x) - \frac{1}{(x-1)^2} - \frac{1}{(x-2)^2} - \frac{1}{2(x-3)^2} \\
\sum \frac{1}{(x+3)^2} &= -\Psi_1(x) + \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{2(x+3)^2} \\
\sum \frac{1}{(x-2)^5} &= \frac{1}{24}\Psi_4(x) - \frac{1}{(x-1)^5} - \frac{1}{2(x-2)^5} \\
\sum \frac{1}{(x+2)^5} &= \frac{1}{24}\Psi_4(x) + \frac{1}{x^5} + \frac{1}{(x+1)^5} + \frac{1}{2(x+2)^5}
\end{aligned}$$

The difference between the shifted equations (4.12) and (4.13) above can be used to get rid of the polygamma term in the sum.

$$\sum \left(\frac{1}{(x+b)^n} - \frac{1}{(x+c)^n} \right) = \frac{1}{2(x+b)^n} + \sum_{k=b+1}^{c-1} \frac{1}{(x+k)^n} + \frac{1}{2(x+c)^n} \qquad \text{for} \quad b, c \in \mathbb{N} \tag{4.14}$$

Example 9:

$$\begin{aligned}
\sum \frac{1}{1-x^2} &= \sum \left(\frac{1/2}{x+1} - \frac{1/2}{x-1} \right) = \frac{1/4}{x-1} + \frac{1/2}{x} + \frac{1/4}{x+1} = \frac{2x^2-1}{2x\,(x^2-1)} \\
\sum \frac{1}{x(x\pm 1)} &= \sum \left(\pm\frac{1}{x} \mp \frac{1}{x\pm 1} \right) = \frac{-1/2}{x} + \frac{-1/2}{x\pm 1} = \frac{-2x \mp 1}{2x^2 \pm 2x} \\
\sum \frac{-2}{x^2+12x+35} &= \sum \left(\frac{1}{x+7} - \frac{1}{x+5} \right) = \frac{1/2}{x+5} + \frac{1}{x+6} + \frac{1/2}{x+7} = \frac{2x^2+24x+71}{x^3+18x^2+107x+210} \\
\sum \frac{-2x-1}{x^4+2x^3+x^2} &= \sum \left(\frac{1}{(x+1)^2} - \frac{1}{x^2} \right) = \frac{1/2}{x^2} + \frac{1/2}{(x+1)^2} = \frac{2x^2+2x+1}{2x^4+4x^3+2x^2}
\end{aligned}$$

Example 10: The Catalan number is $G := \sum_{k=0}^{\infty}(-1)^k\,(2k+1)^{-2}$. We use the duality relation (2.6) and (4.11) to get $\Sigma(-1)^k\,(2x+1)^{-2} = \frac{1}{4}(-1)^x\Delta(2x+1)^{-2} = \frac{1}{16}(-1)^x\Delta\left(x+\frac{1}{2}\right)^{-2} = \frac{-1}{16}(-1)^x\left(\Psi^{(1)}\left(\frac{2x+1}{4}\right) - \Psi^{(1)}\left(\frac{2x+3}{4}\right)\right) + \frac{1}{8}(-1)^x\left(x+\frac{1}{2}\right)^{-2}$. The sum vanishes for $x \to \infty$. With the half border value $\frac{1}{2}(2\cdot 0+1)^{-2} = \frac{1}{2}$ at $x = 0$ the Catalan number is $G = \frac{1}{2} - \Big|_0 \Sigma \cdots = \frac{1}{16}\left(\Psi^{(1)}\left(\frac{1}{4}\right) - \Psi^{(1)}\left(\frac{3}{4}\right)\right)$. By using $\Psi^{(1)}\left(\frac{1}{4}\right) + \Psi^{(1)}\left(\frac{3}{4}\right) = 2\pi^2$ we finally get $G = \frac{1}{8}\left(\Psi^{(1)}\left(\frac{1}{4}\right) - \pi^2\right) \approx 0.91596\,55941\,77219\,01505$.

4.7 Logarithms

The logarithmic sum can easily be deduced from the logarithm of the gamma function recurrence relation $\Gamma(x+1) = x\Gamma(x)$. According to the derivation of rational functions in chapter 4.6 we get $\Delta_0 \ln\Gamma(x) = \ln\Gamma(x+1) - \ln\Gamma(x) = \ln x$ or as a classical sum $\Sigma_0 \ln x = \ln\Gamma(x)$. The natural sum has due to (2.8) the additional term $\frac{1}{2}\ln x$ and thus corresponds to the Euler-Maclaurin sum $\ln\Gamma x = x\ln x - x - \frac{1}{2}\ln x + \frac{1}{2}\ln(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)\,x^{2k-1}}$ which can be found as equation 5.11.1 in [NI10].

$$\begin{aligned} \sum \ln x &= \ln\Gamma(x) + \frac{1}{2}\ln x = \overbrace{x\ln x - x}^{\int \ln x} + \frac{1}{2}\ln(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)\,x^{2k-1}} \quad (4.15) \\ \sum \ln(ax+b) &= \ln\left(a^x\Gamma\left(x+\frac{b}{a}\right)\right) + \frac{1}{2}\ln(ax+b) \end{aligned}$$

With the duplication formula (3.8) we get the discrete difference of the logarithm as:

$$\begin{aligned} \Delta \ln x &= 2\ln\frac{x}{2} + 4\ln\Gamma\left(\frac{x}{2}\right) - 4\ln\Gamma\left(\frac{x+1}{2}\right) \approx 4\sum_{k=1}^{\infty} \frac{B_{2k}\left(2^{2k}-1\right)}{2k(2k-1)\,x^{2k-1}} \quad (4.16) \\ \Delta \ln(ax) &= \Delta \ln a + \Delta \ln x = \Delta \ln x \end{aligned}$$

4.8 Polygamma Function

A recurrence relation for the polygamma sum $\Sigma\Psi^{(n)}(x)$ can be obtained from the product formula (3.40) $\Sigma x f = x\Sigma f + \frac{1}{4}f - \Sigma^2 f$ by a double summation of $f(x) = x^{n+1}$. The case $n = 0$ for the digamma function $\Psi(x)$ must be handled separately.

$$\sum \Psi(x) = \left(x - \frac{1}{2}\right)\cdot(\Psi(x) - 1) \quad (4.17)$$

$$\sum \Psi^{(n)}(x) = \left(x - \frac{1}{2}\right)\Psi^{(n)}(x) + n\Psi^{(n-1)}(x) \quad (4.18)$$

The difference is again calculated by the duplication formula (3.8).

$$\Delta \Psi(x) = \Psi\left(\frac{x+1}{2}\right) - \Psi\left(\frac{x}{2}\right) \quad (4.19)$$

$$\Delta \Psi^{(n)}(x) = \frac{1}{2^n}\left(\Psi^{(n)}\left(\frac{x+1}{2}\right) - \Psi^{(n)}\left(\frac{x}{2}\right)\right) \quad (4.20)$$

4.9 Factorials and Gamma Function

The factorial and the equivalent gamma function are not suited for discrete differentiation and summation, because they are defined as products. Hence their logarithm can be represented as a sum. We already saw this in chapters 4.6 and 4.7 about rational functions and logarithms.

$$\sum x \cdot x! = \left(\frac{x}{2} + 1\right) \cdot x! \tag{4.21}$$

$$\sum \frac{x-1}{x!} = -\frac{x+1}{2\,x!} \tag{4.22}$$

It is possible to obtain the summation formulae for rising and falling factorials described in [GKP95]. Also these equations are not really trivial.

$$\sum x^{\underline{n}} = \frac{1}{n+1} x^{\underline{n+1}} + \frac{1}{2} x^{\underline{n}} = \frac{2x - n + 1}{2(n+1)} x^{\underline{n}} \tag{4.23}$$

$$\sum x^{\overline{n}} = \frac{1}{n+1} (x-1)^{\overline{n+1}} + \frac{1}{2} x^{\overline{n}} = \frac{2x + n - 1}{2(n+1)} x^{\overline{n}} \tag{4.24}$$

The rather complicated formula for the gamma sum can be found in [WP1].

$$\sum \Gamma(x) = \left(\frac{(-1)^{x+1} \Re \Gamma(1-x, -1)}{e} + \frac{1}{2}\right) \Gamma(x) \tag{4.25}$$

4.10 Binomials

Like the factorials $x!$ before, the binomials $\binom{n}{k}$ are not really suited for discrete differentiation and summation. There are two different cases $\binom{n}{x}$ and $\binom{x}{k}$ with either $x = k$ or $x = n$ as free summation variable.

sum over k

$$\Delta \binom{n}{x} = \left(2 - \frac{4x}{n}\right) \binom{n}{x} \tag{4.26}$$

$$\sum \left(x - \frac{n}{2}\right) \binom{n}{x} = (x - n) \binom{n}{x} \tag{4.27}$$

sum over n

$$\sum\binom{x}{k} = \frac{2x+1-k}{2(k+1)}\binom{x}{k} \tag{4.28}$$

$$\sum\frac{1}{x}\binom{x}{k} = \frac{1}{k}\binom{x}{k} - \frac{1}{2x}\binom{x}{k} \tag{4.29}$$

$$\sum x\binom{x}{k} = \frac{-k^2x+2kx^2+kx+2k+2x^2}{2(k+1)(k+2)}\binom{x}{k} \tag{4.30}$$

Many other relations including binomials can be experimentally found. These relations are of minor interest.

$$\sum\binom{2x}{x}4^{-x} = 4^{-x}\binom{2x}{x}\frac{4x+1}{2} \tag{4.31}$$

$$\sum\frac{2x+1}{1}\binom{2x}{x}4^{-x} = 4^{-x}\binom{2x}{x}\frac{2x+1}{1}\cdot\frac{4x+3}{2\cdot 3} \tag{4.32}$$

$$\sum\frac{2x+1}{1}\cdot\frac{2x+3}{3}\binom{2x}{x}4^{-x} = 4^{-x}\binom{2x}{x}\frac{2x+1}{1}\cdot\frac{2x+3}{3}\cdot\frac{4x+5}{2\cdot 5} \tag{4.33}$$

4.11 Complex Powers of x

With (3.27) and $\mathcal{L}x^{-z} = \Gamma(1-z)x^{z-1}$ we get $\sum x^{-z} = x^{-z}\left(\frac{x}{1-z} - \frac{z}{12z} + \frac{z(z+1)(z+2)}{720x^3} - \frac{z(z+1)(z+2)(z+3)(z+4)}{30240x^5} + \cdots\right)$ which can be simplified by $\prod_{k=0}^{n-1}(z+k) = \Gamma(z+n)/\Gamma(z)$.

$$\sum x^{-z} = \frac{x^{1-z}}{1-z}\sum_{k=0}^{\infty}\frac{B_{2k}}{(2k)!}\frac{(z+2k-2)!}{(z-2)!}x^{-2k} \tag{4.34}$$

$$\Delta x^{-z} = -z\,x^{-z-1}\,4\sum_{k=1}^{\infty}\frac{B_{2k}\left(2^{2k}-1\right)}{(2k)!}\frac{(z+2k-2)!}{z!}x^{-2k+2} \tag{4.35}$$

We recognize the final result as the well known Euler-Maclaurin expansion of the Riemann ζ function [Ri59, Ed74].

4.12 Miscellaneous Functions

For the square root we have no closed solution. In the expansions below we define the shortcut $t := \frac{1}{4x}$.

$$\sum \sqrt{x} = 2x^{3/2} \sum_{k=0}^{\infty} \frac{B_{2k}\,(4k)!}{(2k)!^2\,(1-4k)\,(3-4k)} (4x)^{-2k} \approx \tag{4.36}$$

$$\approx \frac{2}{3} x^{3/2} \left(1 + t^2 - \frac{1}{5}t^4 + \frac{2}{3}t^6 - \frac{33}{5}t^8 + 130t^{10} - \frac{446386}{105}t^{12} + \cdots\right)$$

$$\sum \frac{1}{\sqrt{x}} = 2\sqrt{x} \sum_{k=0}^{\infty} \frac{B_{2k}\,(4k)!}{(2k)!^2\,(1-4k)} (4x)^{-2k} \approx \tag{4.37}$$

$$\approx 2\sqrt{x}\left(1 - \frac{1}{3}t^2 + \frac{1}{3}t^4 - 2t^6 + \frac{143}{5}t^8 - \frac{2210}{3}t^{10} + \frac{446386}{15}t^{12} - \cdots\right)$$

4.13 Trigonometric Products

The following trigonometric products can either be calculated by simply using the corresponding linearized trigonometric forms or directly by using the product difference (3.35) and sum (3.36) formulae.

$$\Delta \cos ax \cdot \cos bx = -\tan\frac{a+b}{2}\sin(a+b)x - \tan\frac{a-b}{2}\sin(a-b)x$$

$$\Delta \cos ax \cdot \sin bx = +\tan\frac{a+b}{2}\cos(a+b)x - \tan\frac{a-b}{2}\cos(a-b)x$$

$$\Delta \sin ax \cdot \cos bx = +\tan\frac{a+b}{2}\cos(a+b)x + \tan\frac{a-b}{2}\cos(a-b)x$$

$$\Delta \sin ax \cdot \sin bx = +\tan\frac{a+b}{2}\sin(a+b)x - \tan\frac{a-b}{2}\sin(a-b)x$$

$$\Delta \cos^2 ax = -\tan a \cdot \sin 2ax$$

$$\Delta \sin^2 ax = +\tan a \cdot \sin 2ax$$

$$\Delta \cos ax \cdot \sin ax = +\tan a \cdot \cos 2ax$$

$$
\begin{aligned}
\sum \cos ax \cdot \cos bx &= +\frac{1}{4}\cot\frac{a+b}{2}\sin(a+b)x \;+\; \frac{1}{4}\cot\frac{a-b}{2}\sin(a-b)x \\
\sum \cos ax \cdot \sin bx &= -\frac{1}{4}\cot\frac{a+b}{2}\cos(a+b)x \;+\; \frac{1}{4}\cot\frac{a-b}{2}\cos(a-b)x \\
\sum \sin ax \cdot \cos bx &= -\frac{1}{4}\cot\frac{a+b}{2}\cos(a+b)x \;-\; \frac{1}{4}\cot\frac{a-b}{2}\cos(a-b)x \\
\sum \sin ax \cdot \sin bx &= -\frac{1}{4}\cot\frac{a+b}{2}\sin(a+b)x \;+\; \frac{1}{4}\cot\frac{a-b}{2}\sin(a-b)x \\
\sum \cos^2 ax &= \frac{1}{2} \;+\; \frac{1}{4}\cot a \cdot \sin 2ax \\
\sum \sin^2 ax &= \frac{1}{2} \;-\; \frac{1}{4}\cot a \cdot \sin 2ax \\
\sum \cos ax \cdot \sin ax &= -\frac{1}{4}\cot a \cdot \cos 2ax
\end{aligned}
$$

References

[Ab72] Milton Abramowitz, Irene A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1972

[Ap99] Tom M. Apostol, *An Elementary View of Euler's Summation Formula*, The American Mathematical Monthly, Vol. 106, No. 5. (May, 1999), pp. 409-418

[Ed74] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, New York - London 1974

[GKP95] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete mathematics*, Addison-Wesley, Reading, MA, 1995

[Go78] R.W. Gosper, *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA **75**, 1978, 40–42

[Ko94] W. Koepf, *Algorithms for the indefinite and definite summation*, Konrad-Zuse-Zentrum Berlin (ZIB), Preprint SC 94-33, 1994

[Jo65] Charles Jordan, *Calculus of Finite Differences*, Chelsea Publishing Company, 1965 (First Edition, Budapest, 1939)

[NI10] NIST, *Handbook of Mathematical Functions*, National Institute of Standards and Technology and Cambridge University Press, 2010, ISBN 978-0-521-19225-5, `http://dlmf.nist.gov`

[Ri59] Bernhard Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Größe*, Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, Nov. 1859, 671

[Sy] SymPy, a Python library for symbolic mathematics, `http://sympy.org`

[WP1] Wikipedia, *Indefinite sum*, `http://en.wikipedia.org/Indefinite_sum.html`

[Wi90] H.S. Wilf, *Generatingfunctionology*, Academic Press, Boston, 1990.

[Ze91] D. Zeilberger, *The method of creative telescoping*, J. Symbolic Computation **11**, 1991, 195–204