

Octonion Electrodynamics in Isotropic and Chiral medium

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Abstract

Starting with the Dirac-Maxwell's equations in presence of electric and magnetic sources in an isotropic medium of dyons, we have derived the generalized octonion Maxwell's equations in isotropic medium. And the octonion formulation of generalized electromagnetic fields in chiral medium has also been developed in compact, simple and consistent manner.

Key Words: Octonions, dyons, isotropic and chiral medium, GDM equations etc.

1 Introduction

There has been a revival in the formulation of natural laws so that there exists [1] four division algebras consisting the algebra of real numbers, complex numbers, quaternions and octonions. Octonion analysis has been widely discussed by Baez [2]. It has also played an important role in the context of various physical problems of higher dimensional supersymmetry, supergravity and super strings etc. Few interest in the subject of monopoles and dyons was enhanced by the work of t' Hooft [3] and Polyakov [4] and its extension by Julia and Zee [5]. keeping in view the potential importance of monopole and the results of Witten [6] that monopoles are necessarily dyon, Bisht et. al. [7, 8] constructed a self-consistent co-variant theory of generalized electromagnetic fields associated with dyons each carrying the generalized charge as complex quantity with its real and imaginary part as electric and magnetic constituents. Kravchenko and co-authors [9, 10] discussed the Maxwell's equations in homogeneous media and accordingly developed [11] the quaternionic reformulation of the time-dependent Maxwell's equations along with the classical solution of a moving source i.e. electron. Kravchenko

et al have also demonstrated [12] the electromagnetic fields in chiral media and their quaternionic form in a simple and consistent manner. Recently, Bisht et. al. have developed the generalized Dirac Maxwell's equations in homogeneous (isotropic) medium [13] while their quaternionic forms in a unique and consistent way are also developed [14]. They have already obtained the solution of time independent generalized Dirac Maxwell's (GDM) equations in presence of electric and magnetic sources in chiral media and inhomogeneous media [14][15]. The quaternionic reformulation of generalized electromagnetic fields of dyons in chiral and inhomogeneous media has also been developed and the corresponding quaternionic equations are derived in compact, simple and consistent manner [15]. In this paper we have derived the various quantum equations of generalized electromagnetic fields of dyons (particles carrying simultaneously electric and magnetic charge) in isotropic medium in consistent and manifest co-variant ways. It has been shown that the present theory of dyons remains invariant under the duality transformations in isotropic homogeneous and chiral medium. The octonions analysis of time dependent Maxwell's equations in presence of electric and magnetic charges are obtained in unique, simple and consistent manner.

2 Generalized Electrodynamics of Dyons in Isotropic Medium

In order to write the various quantum equations of dyons in isotropic medium, we start with the definition of homogeneous (isotropic) medium [10-13] in the generalized electromagnetic fields as,

$$\begin{aligned}\vec{D} &= \epsilon \vec{\mathcal{E}} \quad (\epsilon = \epsilon_0 \epsilon_r), \\ \vec{B} &= \mu \vec{H} \quad (\mu = \mu_0 \mu_r); \end{aligned} \quad (1)$$

where \vec{D} and \vec{B} are respectively the electric and magnetic induction vectors while $\vec{\mathcal{E}}$ and \vec{H} are generalized electromagnetic fields. Here ϵ_0 is the free space permittivity, μ_0 is the permeability of free space, ϵ_r and μ_r are defined respectively as relative permittivity and permeability associated with electric and magnetic fields. Applying the above relations in Dirac [16] Maxwell's field equations and establish the dual invariance between electric and magnetic in isotropic medium as [15],

$$\begin{aligned}\vec{\nabla} \cdot \vec{\mathcal{E}} &= \frac{\rho}{\epsilon}; \\ \vec{\nabla} \cdot \vec{B} &= \mu \varrho; \\ \vec{\nabla} \times \vec{\mathcal{E}} &= -\frac{\partial \vec{B}}{\partial t} - \frac{\vec{k}}{\epsilon}; \\ \vec{\nabla} \times \vec{B} &= \frac{1}{v^2} \frac{\partial \vec{\mathcal{E}}}{\partial t} + \mu \vec{j}; \end{aligned} \quad (2)$$

where ρ and ϱ are respectively the electric and magnetic charge densities while \vec{j} and \vec{k} are the corresponding current densities. The electric and magnetic fields of dyons are expressed in following differential form in isotropic medium in term of two four - potentials [15] i.e.

$$\vec{\mathcal{E}} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi - \vec{\nabla} \times \vec{B}; \quad (3)$$

$$\vec{\mathcal{B}} = -\frac{1}{v^2} \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \varphi + \vec{\nabla} \times \vec{A}; \quad (4)$$

where $\{A^\mu\} = \{\phi, v\vec{A}\}$ and $\{B^\mu\} = \{v\varphi, \vec{B}\}$ are the two four - potentials associated with electric and magnetic charges. Defining the complex vector field $\vec{\psi}$ as

$$\vec{\psi} = \vec{\mathcal{E}} - iv\vec{\mathcal{B}} \quad (5)$$

using the equations (3) and (4), we get the following relations between generalized field $\vec{\psi}$ and the components of four - potential as

$$\vec{\psi} = -\frac{\partial \vec{V}}{\partial t} - \vec{\nabla} \Phi - iv(\vec{\nabla} \times \vec{V}) \quad (6)$$

where $\{V_\mu\}$ is the generalized four - potential of dyons in isotropic medium and is defined as

$$V_\mu = \{\Phi, \vec{V}\}; \quad (7)$$

where

$$\Phi = \phi - iv\varphi; \quad (8)$$

and

$$\vec{V} = \vec{A} - \frac{i}{v}\vec{B}. \quad (9)$$

Maxwell's field equations (2) may then be written in term of generalized field $\vec{\psi}$ as

$$\vec{\nabla} \cdot \vec{\psi} = \frac{\rho}{\epsilon}; \quad (10)$$

$$\vec{\nabla} \times \vec{\psi} = -iv\left(\mu\vec{J} + \frac{1}{v^2}\frac{\partial \vec{\psi}}{\partial t}\right); \quad (11)$$

where ρ and \vec{J} are the generalized charge and current source densities of dyons in isotropic medium, given as

$$\boldsymbol{\rho} = \left(\rho - i \frac{\rho}{v} \right); \quad (12)$$

$$\vec{\mathcal{J}} = \left(\vec{j} - iv \vec{k} \right). \quad (13)$$

Here we introduce a new parameter $\vec{\mathcal{S}}$ (called field current) in the following form in term of source densities i. e.

$$\vec{\mathcal{S}} = \square \vec{\psi} = -\mu \frac{\partial \vec{\mathcal{J}}}{\partial t} - \frac{1}{\epsilon} \vec{\nabla} \rho - iv\mu \left(\vec{\nabla} \times \vec{\mathcal{J}} \right). \quad (14)$$

where \square is the D' Alembertian operator. In term of complex potential the Generalized Dirac - Maxwell's (GDM) equations of dyons in isotropic medium are written in the following form

$$\square \Phi = v\mu \rho; \quad (15)$$

$$\square \vec{V} = \mu \vec{\mathcal{J}}. \quad (16)$$

3 Generalized Octonionic Maxwell's Equations in Isotropic Medium

An octonion \mathcal{O} is expressed as [17] a set of eight real numbers

$$\begin{aligned} \mathcal{O} &= \mathcal{O}_0 e_0 + \mathcal{O}_1 e_1 + \mathcal{O}_2 e_2 + \mathcal{O}_3 e_3 + \mathcal{O}_4 e_4 + \mathcal{O}_5 e_5 + \mathcal{O}_6 e_6 + \mathcal{O}_7 e_7 \\ &= \mathcal{O}_0 e_0 + \sum_{A=1}^7 \mathcal{O}_A e_A. \quad (A = 1, 2, \dots, 7) \end{aligned} \quad (17)$$

where $e_A (A = 1, 2, \dots, 7)$ are imaginary octonion units and e_0 is the multiplicative unit element.

The octet $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis and its elements satisfy the following multiplication rules

$$e_0 = 1, \quad e_0 e_A = e_A e_0 = e_A, \quad e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C. \quad (A, B, C = 1, 2, \dots, 7) \quad (18)$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 i. e. $f_{ABC} = +1 = (123), (471), (257), (165), (624), (543), (736)$.

In order to write the quantum equations of dyons in isotropic media in terms of compact notations of octonions, let us start with the four dimensional representation of differential operator [17, 18] expressed in terms of octonion units as

$$\square = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} - \frac{i}{v} e_7 \frac{\partial}{\partial t}. \quad (19)$$

where the other components like $\partial_0, \partial_4, \partial_5, \partial_6$ are vanishing. The octonion conjugate differential operator in isotropic medium is defined as

$$\bar{\square} = -e_1 \frac{\partial}{\partial x} - e_2 \frac{\partial}{\partial y} - e_3 \frac{\partial}{\partial z} + \frac{i}{v} e_7 \frac{\partial}{\partial t}. \quad (20)$$

from equations (19) and (20) we get

$$\begin{aligned} \square \bar{\square} &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \\ &= \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} = \square = \bar{\square} \square. \end{aligned} \quad (21)$$

In terms of the components of electric $\{A_\mu\}$ and magnetic $\{B_\mu\}$ four - potentials of dyons (particles carrying simultaneously the electric and magnetic charges), the octonionic potential is expressed in the isotropic medium as [17, 18],

$$\begin{aligned} \mathbb{V} &= e_1(A_x + ie_7 \frac{B_x}{v}) + e_2(A_y + ie_7 \frac{B_y}{v}) + e_3(A_z + ie_7 \frac{B_z}{v}) + (\varphi + ie_7 \frac{\phi}{v}) \\ &= e_1 V_x + e_2 V_y + e_3 V_z + ie_7 \Phi. \end{aligned} \quad (22)$$

where $(\Phi, V_x, V_y, V_z) = (\Phi, \vec{V}) = \{V_\mu\}$ are described as the components of generalized four - potential $\{V_\mu\}$ associated with dyons in presence of generalized charge $(q = e + i v g)$ (where e and g are respectively known as electric and magnetic charges) of dyons. Now operating $\bar{\square}$ to octonion potential \mathbb{V} , we get

$$\begin{aligned} \bar{\square} \mathbb{V} &= e_1 \left(-\frac{\partial \varphi}{\partial x} + \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - \frac{1}{v^2} \frac{\partial B_x}{\partial t} \right) + e_2 \left(-\frac{\partial \varphi}{\partial y} + \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - \frac{1}{v^2} \frac{\partial B_y}{\partial t} \right) \\ &+ e_3 \left(-\frac{\partial \varphi}{\partial z} + \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - \frac{1}{v^2} \frac{\partial B_z}{\partial t} \right) - ie_4 \frac{1}{v} \left(-\frac{\partial \phi}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} - \frac{\partial A_x}{\partial t} \right) \\ &- ie_5 \frac{1}{v} \left(-\frac{\partial \phi}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} - \frac{\partial A_y}{\partial t} \right) - ie_6 \frac{1}{v} \left(-\frac{\partial \phi}{\partial z} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} - \frac{\partial A_z}{\partial t} \right). \end{aligned} \quad (23)$$

under the Lorentz Gauge condition

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{v^2} \frac{\partial \phi}{\partial t} = 0; \quad (24)$$

$$\vec{\nabla} \cdot \vec{B} + \frac{1}{v^2} \frac{\partial \varphi}{\partial t} = 0; \quad (25)$$

then equation (23) reduced in the following octonion form

$$\overline{\square} \nabla = \mathbb{F}, \quad (26)$$

where \mathbb{F} is generalized electromagnetic field of octonion of the dyon in isotropic medium

$$\mathbb{F} = e_0 F_0 + e_1 F_1 + e_2 F_2 + e_3 F_3 + e_4 F_4 + e_5 F_5 + e_6 F_6 + e_7 F_7 \quad (27)$$

where

$$\begin{aligned} F_0 &= F_7 = 0; \\ F_1 &= \left(-\frac{\partial \varphi}{\partial x} + \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - \frac{1}{v^2} \frac{\partial B_x}{\partial t}\right); \\ F_2 &= \left(-\frac{\partial \varphi}{\partial y} + \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - \frac{1}{v^2} \frac{\partial B_y}{\partial t}\right); \\ F_3 &= \left(-\frac{\partial \varphi}{\partial z} + \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - \frac{1}{v^2} \frac{\partial B_z}{\partial t}\right); \\ F_4 &= -\frac{i}{v} \left(-\frac{\partial \phi}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} - \frac{\partial A_x}{\partial t}\right); \\ F_5 &= -\frac{i}{v} \left(-\frac{\partial \phi}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} - \frac{\partial A_y}{\partial t}\right); \\ F_6 &= -\frac{i}{v} \left(-\frac{\partial \phi}{\partial z} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} - \frac{\partial A_z}{\partial t}\right); \end{aligned} \quad (28)$$

in term of electric and magnetic field, we may described the different components of \mathbb{F} as

$$\begin{aligned} F_1 &\mapsto \mathcal{B}_x; & F_4 &\mapsto -i \frac{\mathcal{E}_x}{v}; \\ F_2 &\mapsto \mathcal{B}_y; & F_5 &\mapsto -i \frac{\mathcal{E}_y}{v}; \\ F_3 &\mapsto \mathcal{B}_z; & F_6 &\mapsto -i \frac{\mathcal{E}_z}{v}. \end{aligned} \quad (29)$$

Thus, from equation (28) the generalized electromagnetic field of dyons \mathbb{F} in terms of the octonion in isotropic medium may be expressed as

$$\begin{aligned} \mathbb{F} &= e_1 \left(\mathcal{B}_x + i e_7 \frac{\mathcal{E}_x}{v}\right) + e_2 \left(\mathcal{B}_y + i e_7 \frac{\mathcal{E}_y}{v}\right) + e_3 \left(\mathcal{B}_z + i e_7 \frac{\mathcal{E}_z}{v}\right) \\ &= e_1 \Psi_x + e_2 \Psi_y + e_3 \Psi_z \end{aligned} \quad (30)$$

where $\vec{\Psi} = \vec{\mathcal{B}} + i e_7 \frac{\vec{\mathcal{E}}}{v}$ is the generalized vector field of dyon in in isotropic medium [17, 18]. Now applying the differential

operator to the equation (30), we get

$$\begin{aligned}
\Box \mathbb{F} = & -e_0(\vec{\nabla} \cdot \vec{\mathcal{B}}) + e_1[(\vec{\nabla} \times \vec{\mathcal{B}})_x - \frac{1}{v^2} \frac{\partial \mathcal{E}_x}{\partial t}] \\
& + e_2[(\vec{\nabla} \times \vec{\mathcal{B}})_y - \frac{1}{v^2} \frac{\partial \mathcal{E}_y}{\partial t}] + e_3[(\vec{\nabla} \times \vec{\mathcal{B}})_z - \frac{1}{v^2} \frac{\partial \mathcal{E}_z}{\partial t}] \\
& - ie_4 \frac{1}{v} [(\vec{\nabla} \times \vec{\mathcal{E}})_x - \frac{\partial \mathcal{B}_x}{\partial t}] - ie_5 \frac{1}{v} [(\vec{\nabla} \times \vec{\mathcal{E}})_y - \frac{\partial \mathcal{B}_y}{\partial t}] \\
& - ie_6 \frac{1}{v} [(\vec{\nabla} \times \vec{\mathcal{E}})_z - \frac{\partial \mathcal{B}_z}{\partial t}] + ie_7 \frac{1}{v} (\vec{\nabla} \cdot \vec{\mathcal{E}}).
\end{aligned} \tag{31}$$

the wave equation of dyons in isotropic medium in terms of octonion may be expressed as

$$\Box \mathbb{F} = \mathbb{J}; \tag{32}$$

where \mathbb{J} is the generalized octonion current. In the isotropic medium generalized octonion current \mathbb{J} is defined as

$$\mathbb{J} = \mu(e_0 \rho + e_1 j_x + e_2 j_y + e_3 j_z) - \frac{1}{\epsilon} \cdot \frac{i}{v} (e_4 k_x + e_5 k_y + e_6 k_z + e_7 \rho). \tag{33}$$

Equations (26) and (32) are represents the octonion form of field equations associated respectively with the generalized potential and current of dyons in isotropic medium. Thus the equation (32) leads to the generalized Dirac Maxwell's equations of dyons in isotropic medium.

4 Generalized Maxwell's field equations in Chiral Media

In the case of homogeneous (isotropic) medium given by equation (1), we have taken the electric field ($\vec{\mathcal{E}}$) and the magnetic field ($\vec{\mathcal{B}}$) are time harmonic. The electric and magnetic fields in complex space can be defined as [15],

$$\begin{aligned}
\vec{\mathcal{E}}(x, t) &= Re\{\vec{\mathcal{E}}(x) e^{-i\omega t}\}; \\
\vec{\mathcal{B}}(x, t) &= Re\{\vec{\mathcal{B}}(x) e^{-i\omega t}\}.
\end{aligned} \tag{34}$$

Thus, the generalized Maxwell's equations (2) for dyons in homogeneous (isotropic) medium may be written as

$$\begin{aligned}
\vec{\nabla} \cdot \vec{\mathcal{E}} &= \frac{\rho(x)}{\epsilon}; \\
\vec{\nabla} \cdot \vec{H} &= \varrho(x); \\
\nabla \times \vec{\mathcal{E}}(x) &= i\omega \vec{B}(x) - \frac{\tilde{\mathbf{k}}(x)}{\epsilon}; \\
\vec{\nabla} \times \vec{H}(x) &= -i\omega \vec{D}(x) + \mu \vec{j}(x).
\end{aligned} \tag{35}$$

In chiral medium [19], the electric and magnetic fields are paired with each other. As such that the constitutive relations $\vec{D} = \epsilon \vec{\mathcal{E}}$ and $\vec{B} = \mu \vec{H}$ will become in paired form as

$$\vec{D} = \epsilon \vec{\mathcal{E}} + \epsilon' \vec{H} \quad \Longrightarrow \quad \vec{D} = \epsilon \left(\vec{\mathcal{E}} + \beta \left(\nabla \times \vec{\mathcal{E}} \right) \right); \tag{36}$$

$$\vec{B} = \mu \vec{H} + \mu' \vec{\mathcal{E}} \quad \Longrightarrow \quad \vec{B} = \mu \left(\vec{H} + \beta \left(\nabla \times \vec{H} \right) \right). \tag{37}$$

where β is chiral parameter and ϵ' and μ' are pairing constants and the relation (36) and (37) are known as Drude - Born - Fedorov constitutive relations [20, 21]. The cartesian field components of the equations (36), (37) are given by

$$\begin{aligned}
\vec{D}_x &= \epsilon \mathcal{E}_x - \epsilon \beta (\nabla \times \mathcal{E})_x \\
\vec{D}_y &= \epsilon \mathcal{E}_y - \epsilon \beta (\nabla \times \mathcal{E})_y \\
\vec{D}_z &= \epsilon \mathcal{E}_z - \epsilon \beta (\nabla \times \mathcal{E})_z \\
\vec{B}_x &= \mu H_x - \mu \beta (\nabla \times H)_x \\
\vec{B}_y &= \mu H_y - \mu \beta (\nabla \times H)_y \\
\vec{B}_z &= \mu H_z - \mu \beta (\nabla \times H)_z.
\end{aligned} \tag{38}$$

using the Drude - Born - Fedorov constitutive relations (36) (37), the generalized Maxwell's equations (2) may be written as

$$\begin{aligned}
\vec{\nabla} \cdot \vec{\mathcal{E}} &= \frac{\rho(x)}{\epsilon}; \\
\vec{\nabla} \cdot \vec{H} &= \varrho(x); \\
\nabla \times \vec{\mathcal{E}}(x) &= +i\omega \mu \left(\vec{H}(x) + \beta \left(\nabla \times \vec{H}(x) \right) \right) - \frac{\tilde{\mathbf{k}}(x)}{\epsilon}; \\
\vec{\nabla} \times \vec{H}(x) &= -i\omega \epsilon \left(\vec{\mathcal{E}}(x) + \beta \left(\nabla \times \vec{\mathcal{E}}(x) \right) \right) + \mu \vec{j}(x).
\end{aligned} \tag{39}$$

The equation (39) represents the Generalized Dirac-Maxwell's (GDM) field equations of dyons in chiral media.

5 Octonion Electrodynamic in Chiral Medium

In the case of octonions the Born - Fedorov constitutive relations given by equation (36) (37) may be written as

$$\vec{\mathcal{B}} = (\vec{\mathcal{B}}_x, \vec{\mathcal{B}}_y, \vec{\mathcal{B}}_z), \quad \vec{\mathcal{D}} = (\vec{\mathcal{D}}_x, \vec{\mathcal{D}}_y, \vec{\mathcal{D}}_z) \quad (40)$$

where

$$\begin{aligned} \vec{\mathcal{B}}_x &= \mu H_x e_1 - e_7 \mu \beta (\nabla \times H)_x \cdot e_4; \\ \vec{\mathcal{B}}_y &= \mu H_y e_2 - e_7 \mu \beta (\nabla \times H)_y \cdot e_5; \\ \vec{\mathcal{B}}_z &= \mu H_z e_3 - e_7 \mu \beta (\nabla \times H)_z \cdot e_6; \end{aligned} \quad (41)$$

equation (41) may be generalized as

$$\vec{\mathcal{B}} = \mu \vec{H} e_j - e_7 \mu \beta (\nabla \times H) \cdot e_{j+3}; \quad (j = 1, 2, 3). \quad (42)$$

similarly

$$\begin{aligned} \vec{\mathcal{D}}_x &= \epsilon \mathcal{E}_x e_1 - e_7 \epsilon \beta (\nabla \times \mathcal{E})_x \cdot e_4; \\ \vec{\mathcal{D}}_y &= \epsilon \mathcal{E}_y e_2 - e_7 \epsilon \beta (\nabla \times \mathcal{E})_y \cdot e_5; \\ \vec{\mathcal{D}}_z &= \epsilon \mathcal{E}_z e_3 - e_7 \epsilon \beta (\nabla \times \mathcal{E})_z \cdot e_6; \end{aligned} \quad (43)$$

equation (43) written in the following reduced form

$$\vec{\mathcal{D}} = \epsilon \vec{\mathcal{E}} e_j - e_7 \epsilon \beta (\nabla \times \mathcal{E}) \cdot e_{j+3}; \quad (j = 1, 2, 3). \quad (44)$$

generalized electromagnetic field of dyons $\vec{\mathcal{F}}$ of octonion in chiral medium then may be write as

$$\vec{\mathcal{F}} = \vec{\mathcal{B}} + ie_7 \vec{\mathcal{D}}. \quad (45)$$

in this field equation we substitute $\vec{\mathcal{B}}$ and $\vec{\mathcal{D}}$ value from equations (42) and (44) as

$$\vec{\mathcal{F}} = \left(\mu \vec{H} e_j - e_7 \mu \beta (\nabla \times H) \cdot e_{j+3} \right) + i e_7 \left(\epsilon \vec{\mathcal{E}} e_j - e_7 \epsilon \beta (\nabla \times \mathcal{E}) \cdot e_{j+3} \right); \quad (j = 1, 2, 3). \quad (46)$$

the equation (46) express as

$$\begin{aligned} \vec{\mathcal{F}} = & (\mu H_x + i e_7 \epsilon \mathcal{E}_x) \cdot e_1 + (\mu H_y + i e_7 \epsilon \mathcal{E}_y) \cdot e_2 \\ & + (\mu H_z + i e_7 \epsilon \mathcal{E}_z) \cdot e_3 - e_7 \beta (\mu (\nabla \times H)_x + i e_7 \epsilon (\nabla \times \mathcal{E})_x) \cdot e_4 \\ & - e_7 \beta (\mu (\nabla \times H)_y + i e_7 \epsilon (\nabla \times \mathcal{E})_y) \cdot e_5 - e_7 \beta (\mu (\nabla \times H)_z + i e_7 \epsilon (\nabla \times \mathcal{E})_z) \cdot e_6 \end{aligned} \quad (47)$$

thus we may write equation (47) simple form as

$$\begin{aligned} \vec{\mathcal{F}} = & \{ \mu H_x + \mu \beta (\nabla \times H)_x \} \cdot e_1 \\ & + \{ \mu H_y + \mu \beta (\nabla \times H)_y \} \cdot e_2 \\ & + \{ \mu H_z + \mu \beta (\nabla \times H)_z \} \cdot e_3 \\ & + \{ \epsilon \mathcal{E}_x + \epsilon \beta (\nabla \times \mathcal{E})_x \} \cdot e_4 \\ & + \{ \epsilon \mathcal{E}_y + \epsilon \beta (\nabla \times \mathcal{E})_y \} \cdot e_5 \\ & + \{ \epsilon \mathcal{E}_z + \epsilon \beta (\nabla \times \mathcal{E})_z \} \cdot e_6 \end{aligned} \quad (48)$$

now operate \square to the chiral octonion field $\vec{\mathcal{F}}$, we get

$$\begin{aligned} \square \mathcal{F} = & - \left[\mu \frac{\partial}{\partial x} H_x + \mu \beta \frac{\partial}{\partial x} (\nabla \times H)_x + \mu \frac{\partial}{\partial y} H_y + \mu \beta \frac{\partial}{\partial y} (\nabla \times H)_y + \mu \frac{\partial}{\partial z} H_z + \mu \beta \frac{\partial}{\partial z} (\nabla \times H)_z \right] \cdot e_0 \\ & + \left[\mu \frac{\partial}{\partial y} H_z + \mu \beta \frac{\partial}{\partial y} (\nabla \times H)_z - \mu \frac{\partial}{\partial z} H_y - \mu \beta \frac{\partial}{\partial z} (\nabla \times H)_y - \epsilon \frac{\partial}{\partial t} \mathcal{E}_x - \epsilon \beta \frac{\partial}{\partial t} (\nabla \times \mathcal{E})_x \right] \cdot e_1 \\ & + \left[\mu \frac{\partial}{\partial z} H_x + \mu \beta \frac{\partial}{\partial z} (\nabla \times H)_x - \mu \frac{\partial}{\partial x} H_z - \mu \beta \frac{\partial}{\partial x} (\nabla \times H)_z - \epsilon \frac{\partial}{\partial t} \mathcal{E}_y - \epsilon \beta \frac{\partial}{\partial t} (\nabla \times \mathcal{E})_y \right] \cdot e_2 \\ & + \left[\mu \frac{\partial}{\partial x} H_y + \mu \beta \frac{\partial}{\partial x} (\nabla \times H)_y - \mu \frac{\partial}{\partial y} H_x - \mu \beta \frac{\partial}{\partial y} (\nabla \times H)_x - \epsilon \frac{\partial}{\partial t} \mathcal{E}_z - \epsilon \beta \frac{\partial}{\partial t} (\nabla \times \mathcal{E})_z \right] \cdot e_3 \\ & + i \left[\epsilon \frac{\partial}{\partial z} \mathcal{E}_y + \epsilon \beta \frac{\partial}{\partial z} (\nabla \times \mathcal{E})_y - \epsilon \frac{\partial}{\partial y} \mathcal{E}_z - \epsilon \beta \frac{\partial}{\partial y} (\nabla \times \mathcal{E})_z - \mu \frac{\partial}{\partial t} H_x - \mu \beta \frac{\partial}{\partial t} (\nabla \times H)_x \right] \cdot e_4 \\ & + i \left[\epsilon \frac{\partial}{\partial x} \mathcal{E}_z + \epsilon \beta \frac{\partial}{\partial x} (\nabla \times \mathcal{E})_z - \epsilon \frac{\partial}{\partial z} \mathcal{E}_x - \epsilon \beta \frac{\partial}{\partial z} (\nabla \times \mathcal{E})_x - \mu \frac{\partial}{\partial t} H_y - \mu \beta \frac{\partial}{\partial t} (\nabla \times H)_y \right] \cdot e_5 \\ & + i \left[\epsilon \frac{\partial}{\partial y} \mathcal{E}_x + \epsilon \beta \frac{\partial}{\partial y} (\nabla \times \mathcal{E})_x - \epsilon \frac{\partial}{\partial x} \mathcal{E}_y - \epsilon \beta \frac{\partial}{\partial x} (\nabla \times \mathcal{E})_y - \mu \frac{\partial}{\partial t} H_z - \mu \beta \frac{\partial}{\partial t} (\nabla \times H)_z \right] \cdot e_6 \\ & + i \left[\epsilon \frac{\partial}{\partial x} \mathcal{E}_x + \epsilon \beta (\nabla \times \mathcal{E})_x + \epsilon \frac{\partial}{\partial y} \mathcal{E}_y + \epsilon \beta (\nabla \times \mathcal{E})_y + \epsilon \frac{\partial}{\partial z} \mathcal{E}_z + \epsilon \beta (\nabla \times \mathcal{E})_z \right] \cdot e_7 \end{aligned} \quad (49)$$

equation (49) may write in the following reduced form as

$$\begin{aligned}
\boxdot \mathcal{F} = & - \left[\mu(\vec{\nabla} \cdot \vec{H}) + \mu\beta\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) \right] \cdot e_1 \\
& + \left[\mu(\vec{\nabla} \times \vec{H}) - \epsilon\frac{\partial}{\partial t}\vec{\mathcal{E}} + \mu\beta\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) - \epsilon\beta\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathcal{E}}) - \mu\beta\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) \right] \cdot e_j \\
& + i \left[-\epsilon(\vec{\nabla} \times \vec{\mathcal{E}}) + \epsilon\beta\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathcal{E}}) - \mu\frac{\partial}{\partial t}\vec{H} - \mu\beta\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}) - \epsilon\beta\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathcal{E}}) \right] \cdot e_{j+3} \\
& + i \left[\epsilon(\vec{\nabla} \cdot \vec{\mathcal{E}}) + \epsilon\beta\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathcal{E}}) \right] \cdot e_7
\end{aligned} \tag{50}$$

The octonion GDM wave equation in Chiral medium of dyons may be written as

$$\boxdot \mathcal{F} = \mathcal{J}. \tag{51}$$

where octonion current source \mathcal{J} is

$$\begin{aligned}
\mathcal{J} = & e_0 J_0 + e_1 J_1 + e_2 J_2 + e_3 J_3 + e_4 J_4 + e_5 J_5 + e_6 J_6 + e_7 J_7 \\
= & -e_0 \rho + e_j \vec{j} - ie_{j+3} \vec{k} + ie_7 \rho
\end{aligned} \tag{52}$$

from the equation (51) we get following relation

$$\begin{aligned}
\epsilon(\vec{\nabla} \cdot \vec{\mathcal{E}}) & = \rho; \\
\mu(\vec{\nabla} \cdot \vec{H}) & = \rho; \\
\epsilon(\vec{\nabla} \times \vec{\mathcal{E}}) & = -\mu\frac{\partial}{\partial t}\vec{H} - \mu\beta\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}) - \vec{k}; \\
\mu(\vec{\nabla} \times \vec{H}) & = \epsilon\frac{\partial}{\partial t}\vec{\mathcal{E}} + \epsilon\beta\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathcal{E}}) + \vec{j}.
\end{aligned} \tag{53}$$

These equations are octonionic GDM equations in Chiral medium. If we use the time-derivative relations as

$$\frac{\partial}{\partial t}\vec{D} = \epsilon\frac{\partial}{\partial t}(\vec{\mathcal{E}} + \beta(\vec{\nabla} \times \vec{\mathcal{E}})) = \epsilon\frac{\partial}{\partial t}\vec{\mathcal{E}} + \epsilon\beta\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathcal{E}}); \tag{54}$$

$$\frac{\partial}{\partial t}\vec{B} = \mu\frac{\partial}{\partial t}(\vec{H} + \beta(\vec{\nabla} \times \vec{H})) = \mu\frac{\partial}{\partial t}\vec{H} + \mu\beta\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}); \tag{55}$$

and we have

$$\vec{\nabla} \cdot \vec{D} = \epsilon\vec{\nabla} \cdot (\vec{\mathcal{E}} + \beta(\vec{\nabla} \times \vec{\mathcal{E}})) = \epsilon(\vec{\nabla} \cdot \vec{\mathcal{E}}); \tag{56}$$

$$\vec{\nabla} \cdot \vec{B} = \mu\vec{\nabla} \cdot (\vec{H} + \beta(\vec{\nabla} \times \vec{H})) = \mu(\vec{\nabla} \cdot \vec{H}); \tag{57}$$

using above equations (54),(55) and equations (56),(57), we get

$$\begin{aligned}
\epsilon (\vec{\nabla} \cdot \vec{\mathcal{E}}) &= \rho; \\
\mu (\vec{\nabla} \cdot \vec{H}) &= \varrho; \\
\epsilon (\vec{\nabla} \times \vec{\mathcal{E}}) &= -\frac{\partial \vec{B} - \vec{k};}{\partial t} \\
\mu (\vec{\nabla} \times \vec{H}) &= \frac{\partial \vec{D} + \vec{j}.}{\partial t}
\end{aligned} \tag{58}$$

These equations are another form of Maxwell's equation in Chiral medium in case of generalized octonion electrodynamics.

6 Discussion

The equations ((42)) and ((44)) showing the constitutive relation and then describe the rich variety of physical phenomenon representing the properties and responses of the chiral medium and to the application of generalized octonionic electromagnetic field of dyons. The octonions field equations are described here in isotropic medium and chiral medium. The present theory of generalized octonions electrodynamics of dyons leads to the connection between the mechanical parameters with the chirality and dielectric properties. And we formulised octonions Maxwell's equation for electromagnetic wave theory, are being unique, compact, and consistent manner for isotropic and chiral medium.

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