Split Octonion Electrodynamics and Energy-Momentum Conservation Laws for Dyons

B. C. Chanyal, P. S. Bisht and O. P. S. Negi

Department of Physics
Kumaun University
S. S. J. Campus
Almora -263601 (U.K.) India

Email:- bcchanyal@gmail.com
ps_bisht123@rediffmail.com
ops_negi@yahoo.co.in

Abstract

Starting with the usual definations of octonions and split octonions in terms of Zorn vector matrix realization, we have made an attempt to write the consistent form of generalized Maxwell's equations in presence of electric and magnetic charges. We have thus written the generalized split octonion potential wave equations and the generalized fields equation of dyons in split octonions. Accordingly the split octonion forms of generalized Dirac Maxwell's equations are obtained in compact and consistent manner. Accordingly, we have made an attempt to investigate the work energy theorem or "Poynting Theorem", Maxwell stress tensor and Lorentz invariant for generalized fields of dyons in split octonion electrodynamics. Our theory of dyons in split octonion formulations is discussed in term of simple and compact notations. This theory reproduces the dynamic of electric (magnetic) in the absence of magnetic (electric) charges.

1 Introduction

The relationship between mathematics and physics has long been an area of interest and speculation. Magnetic monopoles [1] where advocated to symmeterize Maxwell's equations in a manifest way that the mere existence of an isolated magnetic charge implies the quantization of electric charge and accordingly the considerable literature [2-7] has come in force. The fresh interests are enhanced with idea of 't Hooft [8] and Polyakov [9] that the classical solutions having the properties of magnetic monopoles may be found in Yang-Mills gauge theories. Julia and Zee [10] extended it to construct the theory of non-Abelian dyons (particles [2, 3] carrying simultaneously electric and magnetic charge). In view of the explanation of CP-violation in terms of non-zero vacuum angle of world [11], the monopole are necessary dyons and Dirac quantization condition permits dyons to have analogous electric charge. The quantum mechanical excitation of fundamental monopoles include dyons which are automatically arisen [5, 7] from the semi-classical quantization of global charge rotation degree of freedom of monopoles. Accordingly, the self-consistent and manifestly covariant theory of generalized electromagnetic fields associated with dyons (particle carrying electric and magnetic charge) has been discussed [12, 13].

The close analogy between Newton's gravitation law and Coulomb's law of electricity led many authors to investigate further similarities, such as the possibility that the motion of mass-charge could generate the analogous of a magnetic field which is produced by the motion of electric-charge, i.e. the electric current. So, there should be the mass current would produced a magnetic type field namely 'gravitomagnetic' field. Maxwell [14] in one of his fundamental works on electromagnetism , turned his attention to the possibility of formulating the theory of gravitation in a form corresponding to the electromagnetic equations. In 1893 Heaviside [15] investigated the analogy between gravitation and electromagnetism where he explained the propagation of energy in a gravitational field, in terms of a gravito-electromagnetic Poynting vector, even though he (just as Maxwell did) considered the nature of gravitational energy a mystery. The analogy has also been explored by Einstein [16], in the framework of General Relativity, and then by Thirring [17] and Lense and Thirring [18], that a rotating mass generates a gravito magnetic field causing a precession of planetary orbits. Exploding the basics of the gravito electromagnetic form of the Einstein equations, theory of gravitomagnetism has also been reviewed by Ruggiero-Tartaglia [19].

Decomposition of four algebras, in view of celebrated Hurwitz theorem, has been characterized from Cayley Dickson process over the field of real numbers of dimensions N=1, N=2, N=4 and N=8 respectively for real, complex, quaternion and octonion algebras. Split octonion electrodynamics [20] has been developed interms of Zorn's vector matrix realization and corresponding field equation. Split octonion formulation of dyon field has has been carried out to reformulate the generalized four-potential, current equations, field equations, and electro-magnetic fields of dyons. So, there has been a revival in the formulation of natural laws so that there exists [21] four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathcal{O}$). All four algebra's are alternative with totally anti symmetric associators. Quaternions [22, 23] were very first example of hyper complex numbers have been widely used [24-30] to the various applications of mathematics and physics. Since octonions [31] share with complex numbers and quaternions, many attractive mathematical properties, one might except that they would be equally as useful as others. Octonion [31] analysis has been widely discussed by Baez [32]. It has also played an important role in the context of various physical problems [33-36] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [37-40] towards the developments of wave equation and octonion form of Maxwell's equations. Bisht et al. [41, 42] have also studied octonion electrodynamics, dyonic field equation and octonion gauge analyticity of dyons consistently and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation.

2 Octonion Definition

An octonion x is expressed [38-44] as a set of eight real numbers

$$\begin{aligned} x=(x_0,\,x_1,....,\,x_7) &= x_0e_0+x_1e_1+x_2e_2+x_3e_3+x_4e_4+x_5e_5+x_6e_6+x_7e_7 \\ &= +x_0e_0+\sum_{A=1}^{7}x_Ae_A \qquad (A=1,2,.....,7) \end{aligned} \tag{1}$$

where $e_A(A=1,2,.....,7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules

$$e_0=1, \qquad e_0e_A=e_Ae_0=e_A \qquad e_Ae_B=-\delta_{AB}e_0+f_{ABC}\,e_C. \quad (A,B,C=1,2,......7) \tag{2}$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 i.e. $f_{ABC}=+1=(123),\,(471),\,(257),\,(165),\,(624),\,(543),\,(736)$. Here the octonion algebra $\mathcal{O}$ is described over the algebra of rational numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by equations (2,3) are then generalized in the following table [43]:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Table 1: Octonion Multiplication table

Hence, we get the following relations among octonion basis elements i.e.

$$[e_A,\,e_B] = 2f_{ABC}e_C; \quad \{e_A,\,e_B\}=-\delta_{AB}e_0; \quad e_A(e_Be_C)\neq(e_Ae_B)e_C; \tag{3}$$

where brackets [ ] and { } are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol.Octonion conjugate is thus defined as,

$$\begin{aligned} \bar{x} &= x_0e_0-x_1e_1-x_2e_2-x_3e_3-x_4e_4-x_5e_5-x_6e_6-x_7e_7 \\ &= x_0e_0 - \sum_{A=1}^{7}x_Ae_A \qquad (A=1,2,.....,7). \end{aligned} \tag{4}$$

An Octonion can be decomposed in terms of its scalar ($Sc(x)$) and vector ($Vec(x)$) parts as

$$Sc(x) = \frac{1}{2}(x+\bar{x}) = x_0; \quad Vec(x) = \frac{1}{2}(x-\bar{x}) = \sum_{A=1}^{7} x_A e_A \tag{5}$$

Conjugates of product of two octonions and its own are described as

$$(\overline{xy}) = \overline{y}\,\overline{x}\ ; \quad \overline{(\bar{x})} = x \tag{6}$$

while the scalar product of two octonions is defined as

$$\langle x\,,\,y\rangle = \sum_{\alpha=0}^{7} x_\alpha y_\alpha = \frac{1}{2}(x\,\bar{y}+y\,\bar{x}) = \frac{1}{2}(\bar{x}\,y+\bar{y}\,x) \tag{7}$$

which can be written in terms of octonion units as

$$\langle e_A\,,\,e_B\rangle = \frac{1}{2}(e_A\overline{e_B}+e_B\overline{e_A}) = \frac{1}{2}(\overline{e_A}e_B+\overline{e_B}e_A) = \delta_{AB}. \tag{8}$$

The norm of the octonion $N(x)$ is defined as

$$N(x) = \overline{x}x = x\,\bar{x} = \sum_{\alpha=0}^{7} x_\alpha^2 e_0 \tag{9}$$

which is zero if $x=0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x). \tag{10}$$

As such, for a nonzero octonion x , we define its inverse as

$$x^{-1} = \frac{\bar{x}}{N(x)} \tag{11}$$

which shows that

$$x^{-1}x = xx^{-1} = 1.e_0; \quad (xy)^{-1} = y^{-1}x^{-1}. \tag{12}$$

3 Split-Octonions Definitions

The split octonions [44] are the non associative extension of quaternions (or the split quaternions). They differ form the octonion in the signature of quadratic form. The split octonions have a signature (4,4) whereas the octonions have positive signature (8,0). The Caylay algebra of octonions over the field of complex numbers $\mathbb{C}_{\mathbb{C}} = \mathbb{C}\otimes C$ is visualized as the algebra of split octonions with its following basis elements.

$$\begin{aligned}
u_0 &= \frac{1}{2}(e_0+ie_7)\,,\ u_0^* = \frac{1}{2}(e_0-ie_7)\,;\\
u_1 &= \frac{1}{2}(e_1+ie_4)\,,\ u_1^* = \frac{1}{2}(e_1-ie_4)\,;\\
u_2 &= \frac{1}{2}(e_2+ie_5)\,,\ u_2^* = \frac{1}{2}(e_2-ie_5)\,;\\
u_3 &= \frac{1}{2}(e_3+ie_6)\,,\ u_3^* = \frac{1}{2}(e_3-ie_6)\,;
\end{aligned} \tag{13}$$

where $(\star)$ is used for complex conjugation and $(i=\sqrt{1})$ commutes with all seven octonion imaginary unit e_A $(A=1,2,......,7)$. The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units e_A $(A=1,2,......,7)$ fall into its 7-dimensional representation. Under the $SU(3)_c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* transform like singlets, while u_j and u_j^* $(\forall\, j=1,2,3)$ transform like a triplet and anti-triplet respectively.

The split octonion basis element satisfy the following multiplication rule

$$\begin{aligned}u_iu_j = \epsilon_{ijk}u_k^*,\ u_iu_j^* = -\delta_{ij}u_0,\ u_iu_0 = 0,\\ u_i^*u_j = -\delta_{ij}u_0,\ u_iu_0^* = u_0,\ u_i^*u_0^* = 0,\\ u_0u_i = u_i,\ u_i^*u_0 = u_i^*,\ u_0^*u_i^* = u_i,\\ u_0^2 = u_0,\ u_0^{*2} = u_0^*, u_0u_0^* = u_0^*u_0 = 0.\ (\forall i,j,k = 1,2,3)\end{aligned} \tag{14}$$

The multiplication table [33] can now be written in a manifestly $SU(3)_c$ invariant manner as

$\cdot$	u_0^*	u_1^*	u_2^*	u_3^*	u_0	u_1	u_2	u_3
u_0^*	u_0^*	u_1^*	u_2^*	u_3^*	0	0	0	0
u_1^*	0	0	u_3	$-u_2$	u_1^*	$-u_0^*$	0	0
u_2^*	0	$-u_3$	0	u_1	u_2^*	0	$-u_0^*$	0
u_3^*	0	u_2	$-u_1$	0	u_3^*	0	0	$-u_0^*$
u_0	0	0	0	0	u_0	u_1	u_2	u_3
u_1	u_1	$-u_0$	0	0	0	0	u_3^*	$-u_2^*$
u_2	u_2	0	$-u_0$	0	0	$-u_3^*$	0	u_1^*
u_3	u_3	0	0	$-u_0$	0	u_2^*	$-u_1^*$	0

Table 2: Split-Octonion Multiplication Table

From the multiplication rules (14), we may obtain

$$(u_iu_j)u_k = -\ \epsilon_{ijk}u_0^*, \tag{15}$$

so that, we may put together the compactified multiplication table for the split octonion units as [33]

$\cdot$	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki}u_i^*$	$-\delta_{jk}u_0$
u_j^*	u_j^*	0	$-\delta_{jk}u_0^*$	$\epsilon_{jki}u_i$

Table 3: Compactified Split-Octonion Multiplication Table

Thus, one can relate u_j and u_j^* with fermionic annihilation and creation operators as

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ij}. \tag{16}$$

This fermionic Heisenberg algebra shows the three split unit u_i to be Grassmann numbers. Being non-associative, these split units give rise to an exceptional Grassmann algebra. Operators u_i, unlike ordinary fermion operators, are non associative. We also have

$$\frac{1}{2}[u_i, u_j] = \epsilon_{ijk}u_k^*. \tag{17}$$

The Jacobi identity does not hold since

$$[u_i, [u_i, u_k]] = -\ ie_7 \neq 0; \tag{18}$$

where e_7, anti commute with u_i and u_i^*. It is to be noticed that, like the imaginary units e_A, the split units cannot be represented by matrices. Unlike the octonion algebra, the split octonion algebra contains zero divisors and is therefore not a division algebra.

The associators of split octonion units are given below:

$$
\begin{aligned}
[u_i, u_j, u_k] =& \epsilon_{ijk}(u_0^* - u_0),\\
[u_i^*, u_j^*, u_k^*] =& \epsilon_{ijk}(u_0 - u_0^*),\\
[u_i, u_j, u_0] =& -\epsilon_{ijk}u_k^*,\\
[u_i, u_j, u_0^*] =& \epsilon_{ijk}u_k^*,\\
[u_i, u_j, u_k^*] =& \delta_{jk}u_i - \delta_{ik}u_j,\\
[u_i, u_j^*, u_k^*] =& \delta_{ik}u_j^* - \delta_{ij}u_k^*,\\
[u_i^*, u_j^*, u_0] =& \epsilon_{ijk}u_k,\\
[u_i^*, u_j^*, u_0^*] =& -\epsilon_{ijk}u_k,\\
[u_i, u_j^*, u_0] =& 0,\\
[u_i, u_j^*, u_0^*] =& 0.
\end{aligned}
\tag{19}
$$

So, the convenient realization for the basis elements (u_0, u_j, u_0^*, u_j^*) in term of Pauli spin matrices may now be introduced as

$$
\begin{aligned}
u_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \qquad & u_0^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\\
u_j = \begin{bmatrix} 0 & 0 \\ e_j & 0 \end{bmatrix}; \qquad & u_j^* = \begin{bmatrix} 0 & -e_j \\ 0 & 0 \end{bmatrix}; \qquad (\forall j = 1, 2, 3)
\end{aligned}
\tag{20}
$$

The split Cayley (octonion) algebra is thus expressed in terms of 2×2 Zorn's vector matrices components of which are scalar and vector parts of a quaternion i.e.

$$
\mathcal{O} = \{ \begin{pmatrix} m & \vec{p} \\ \vec{q} & n \end{pmatrix}; \quad m, n \in Sc(H); \quad \& \, \vec{p}, \vec{q} \in Vec(H) \}. \tag{21}
$$

As such, we may write an arbitrary split octonion A in terms of following 2×2 Zorn's vector matrix realization as

$$
A = au_0^* + bu_0 + x_i u_i^* + y_i u_i = \begin{pmatrix} a & -\overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix}, \tag{22}
$$

where a and b are scalars and $\vec{x}$ and $\vec{y}$ are three vectors. Thus the product of two octonions in terms of following 2×2 Zorn's vector matrix realization is expressed as

$$
\begin{pmatrix} a & \overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix} * \begin{pmatrix} c & \overrightarrow{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} ac + (\overrightarrow{x}.\overrightarrow{v}) & a\vec{u} + d\vec{x} + (\overrightarrow{y} \times \vec{v}) \\ c\overrightarrow{y} + b\vec{v} - (\vec{x} \times \overrightarrow{u}) & bd + (\overrightarrow{y}.\overrightarrow{u}) \end{pmatrix} \tag{23}
$$

where $(\times)$ denotes the usual vector product, e_j $(j = 1, 2, 3)$ with $e_j \times e_k = \epsilon_{jkl}e_l$ and $e_j e_k = -\delta_{jk}$.
Octonion conjugate of equation (22) in terms of 2×2 Zorn's vector matrix realizations is now defined as

$$
\overline{A} \quad = au_0 + bu_0^\star - x_i u_i^\star - y_i u_i \quad = \begin{pmatrix} b & \overrightarrow{x} \\ -\overrightarrow{y} & a \end{pmatrix}. \tag{24}
$$

The norm of A is defined as

$$
N(A) = \overline{A}A = A\overline{A} = (ab + \overrightarrow{x}.\overrightarrow{y})\hat{1} = n(A)\hat{1}, \tag{25}
$$

where $\hat{1}$ is the identity elements of matrix order 2×2, and the expression $n(A) = (ab + \overrightarrow{x}.\overrightarrow{y})$ defines the quadratic form which admits the composition as

$$
n(\vec{A} \cdot \vec{B}) = n(\vec{A})\, n(\vec{B}), \quad (\forall \, \vec{A}, \vec{B} \in \mathcal{O}) \tag{26}
$$

As such, we may easily express the Euclidean or Minkowski four vector in split octonion formulation in terms of 2×2 Zorn's vector matrix realizations. So, any four - vector A_μ (complex or real) can equivalently be written in terms of the following Zorn matrix realization as

$$
Z(A) \quad = \quad \begin{pmatrix} x_4 & -\overrightarrow{x} \\ \overrightarrow{y} & y_4 \end{pmatrix}; \; Z(\overline{A}) = \begin{pmatrix} x_4 & \overrightarrow{x} \\ -\overrightarrow{y} & y_4 \end{pmatrix}. \tag{27}
$$

Hence, we may define the split octonion equivalent of space - time four differential operator $\boxdot$ may be written in terms of 2×2 Zorn's vector matrix as [44]

$$\begin{aligned}\boxdot = & \partial_t u_0^* - \partial_t u_0 + \overrightarrow{\nabla} u_i^* + \overrightarrow{\nabla} u_i \\ \cong & \begin{pmatrix} \partial_t & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & -\partial_t \end{pmatrix};\end{aligned} \tag{28}$$

where $\partial_t = \frac{\partial}{\partial t}$, we have taken other components like $\partial_0, \partial_4, \partial_5, \partial_6$ of equation vanishing. Accordingly, split octonion conjugate $\overline{\boxdot}$ of four differential operator may be written in terms of 2×2 Zorn's vector matrix as

$$\begin{aligned}\overline{\boxdot} = & - \partial_t u_0^* + \partial_t u_0 - \overrightarrow{\nabla} u_i^* - \overrightarrow{\nabla} u_i \\ \cong & \begin{pmatrix} -\partial_t & \overrightarrow{\nabla} \\ -\overrightarrow{\nabla} & \partial_t \end{pmatrix};\end{aligned} \tag{29}$$

As such , we get

$$\boxdot\overline{\boxdot} = \overline{\boxdot}\boxdot = \begin{pmatrix} \nabla^2 - \frac{\partial^2}{\partial t^2} & 0 \\ 0 & \nabla^2 - \frac{\partial^2}{\partial t^2} \end{pmatrix}, \tag{30}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{\partial^2}{\partial t^2}$ (d' Alembert operator).

4 Generalized Split Octonion Electrodynamics

In order to write the various quantum equations of dyons in split octonion formulation, we start with octonion form of generalized potential [42] of dyons. Using the definitions of split octonions and their connection with Zorn's vector matrix realization, it is easy to write the split octonion form of generalized four potential as [44],

$$\mathbb{V} = \begin{pmatrix} \varnothing_- & -\overrightarrow{\varnothing_+} \\ \overrightarrow{\varnothing_-} & \varnothing_+ \end{pmatrix} = \begin{pmatrix} (\varphi - \phi) & -\left(\overrightarrow{A} + \overrightarrow{B}\right) \\ \left(\overrightarrow{A} - \overrightarrow{B}\right) & (\varphi + \phi) \end{pmatrix}. \tag{31}$$

Here $\left[\varnothing_- \to (\varphi - \phi)\,,\ \varnothing_+ \to (\varphi + \phi)\,,\ \overrightarrow{\varnothing_-} \to \left(\overrightarrow{A} - \overrightarrow{B}\right)\ \overrightarrow{\varnothing_+} \to \left(\overrightarrow{A} + \overrightarrow{B}\right)\right]$. Now operating $\overline{\boxdot}$ given by the equation (29) to octonion potential $\mathbb{V}$ (31), we get [44];

$$\begin{aligned}\boxdot\mathbb{V} = & \begin{pmatrix} \frac{\partial}{\partial t} & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & -\frac{\partial}{\partial t} \end{pmatrix} * \begin{pmatrix} (\varphi - \phi) & -\left(\overrightarrow{A} + \overrightarrow{B}\right) \\ \left(\overrightarrow{A} - \overrightarrow{B}\right) & (\varphi + \phi) \end{pmatrix} \\ = & \begin{pmatrix} -\frac{\partial\varphi}{\partial t} + \frac{\partial\phi}{\partial t} + \overrightarrow{\nabla}\cdot\overrightarrow{A} - \overrightarrow{\nabla}\cdot\overrightarrow{B} & \begin{matrix}\frac{\partial\overrightarrow{A}}{\partial t} + \frac{\partial\overrightarrow{B}}{\partial t} + \overrightarrow{\nabla}\varphi + \overrightarrow{\nabla}\phi \\ -\overrightarrow{\nabla}\times\overrightarrow{A} + \overrightarrow{\nabla}\times\overrightarrow{B}\end{matrix} \\ \begin{matrix}-\overrightarrow{\nabla}\varphi + \overrightarrow{\nabla}\phi + \frac{\partial\overrightarrow{A}}{\partial t} - \frac{\partial\overrightarrow{B}}{\partial t} \\ +\overrightarrow{\nabla}\times\overrightarrow{A} + \overrightarrow{\nabla}\times\overrightarrow{B}\end{matrix} & \overrightarrow{\nabla}\cdot\overrightarrow{A} + \overrightarrow{\nabla}\cdot\overrightarrow{B} + \frac{\partial\varphi}{\partial t} + \frac{\partial\phi}{\partial t} \end{pmatrix}.\end{aligned} \tag{32}$$

On applying the Lorentz gauge conditions, respectively for the dynamics of electric and magnetic charges of dyons as

$$\begin{aligned}\overrightarrow{\nabla}\cdot\overrightarrow{A} + \frac{\partial\phi}{\partial t} &= 0, \\ \overrightarrow{\nabla}\cdot\overrightarrow{B} + \frac{\partial\varphi}{\partial t} &= 0;\end{aligned} \tag{33}$$

We get the reduced form of equation (32) as

$$\boxdot\mathbb{V} = \mathbb{F}; \tag{34}$$

where $\mathbb{F}$ is also an octonion describing the generalized electromagnetic fields of dyons. The split octonion equivalent in terms of 2×2 Zorn's vector matrix realization may be written as [44]

$$\mathbb{F}=\left(\begin{array}{cc} 0 & -\overrightarrow{F_{+}} \\ \overrightarrow{F_{-}} & 0 \end{array}\right)=\left(\begin{array}{cc} 0 & -\left(\overrightarrow{F_{g}}+\overrightarrow{F_{e}}\right) \\ \left(\overrightarrow{F_{g}}-\overrightarrow{F_{e}}\right) & 0 \end{array}\right), \tag{35}$$

where $\overrightarrow{F_{+}}\rightarrow\overrightarrow{F_{g}}+\overrightarrow{F_{e}}$, $\overrightarrow{F_{-}}\rightarrow\overrightarrow{F_{g}}-\overrightarrow{F_{e}}$; and

$$\begin{aligned}
\overrightarrow{F_{g}} &= -\frac{\partial\overrightarrow{B}}{\partial t}-\overrightarrow{\nabla}\varphi+\overrightarrow{\nabla}\times\overrightarrow{A}. \longrightarrow \overrightarrow{H}; \\
\overrightarrow{F_{e}} &= -\frac{\partial\overrightarrow{A}}{\partial t}-\overrightarrow{\nabla}\phi-\overrightarrow{\nabla}\times\overrightarrow{B}; \longrightarrow \overrightarrow{E}.
\end{aligned} \tag{36}$$

Here $\overrightarrow{E}$ and $\overrightarrow{H}$ are respectively denoted as generalized electric and magnetic fields of dyons. As such, we may write the generalized electromagnetic field vector $\mathbb{F}$ of dyons in term of following split octonionic representation as

$$\mathbb{F}=\left(\begin{array}{cc} 0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right) \\ \overrightarrow{H}-\overrightarrow{E} & 0 \end{array}\right)=\left(\begin{array}{cc} 0 & -\overrightarrow{\psi_{+}} \\ \overrightarrow{\psi_{-}} & 0 \end{array}\right), \tag{37}$$

where $\overrightarrow{\psi_{+}}=\overrightarrow{H}+\overrightarrow{E}$ and $\overrightarrow{\psi_{-}}=\overrightarrow{H}-\overrightarrow{E}$ are the generalized electromagnetic vector fields of dyons. Now applying the differential operator $\boxdot$ given by (28) to equation (35) for generalized fields of dyon, we get

$$\boxdot\,\mathbb{F}=\left(\begin{array}{cc} \overrightarrow{\nabla}\cdot\overrightarrow{F_{g}}-\overrightarrow{\nabla}\cdot\overrightarrow{F_{e}} & \frac{\partial\overrightarrow{F_{g}}}{\partial t}+\frac{\partial\overrightarrow{F_{e}}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{F_{g}}+\overrightarrow{\nabla}\times\overrightarrow{F_{e}} \\ \frac{\partial\overrightarrow{F_{g}}}{\partial t}-\frac{\partial\overrightarrow{F_{e}}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{F_{g}}+\overrightarrow{\nabla}\times\overrightarrow{F_{e}} & \overrightarrow{\nabla}\cdot\overrightarrow{F_{g}}+\overrightarrow{\nabla}\cdot\overrightarrow{F_{e}} \end{array}\right), \tag{38}$$

which is further reduced to the following wave equation in split octonion form as

$$\boxdot\mathbb{F}=-\mathbb{J}. \tag{39}$$

Here $\mathbb{J}$ is the split octonion equivalent of generalized four current of dyons and may be written in terms of 2×2 Zorn's vector matrix realization as [44],

$$\mathbb{J}=\left(\begin{array}{cc} (\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right) \\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho) \end{array}\right)=\left(\begin{array}{cc} \jmath_{-} & -\vec{\jmath}_{+} \\ \vec{\jmath}_{-} & \jmath_{+} \end{array}\right); \tag{40}$$

where $\jmath_{-}\rightarrow(\varrho-\rho)$, $\vec{\jmath}_{-}\rightarrow\left(\overrightarrow{j}-\overrightarrow{k}\right)$, $\jmath_{+}\rightarrow(\varrho+\rho)$, $\vec{\jmath}_{+}\rightarrow\left(\overrightarrow{j}+\overrightarrow{k}\right)$. Here $(\rho,\,\overrightarrow{j})=\{\mathrm{j}_{\mu}\}$, $(\varrho,\,\overrightarrow{j})=\{\mathrm{k}_{\mu}\}$ and $(\mathrm{J}_{0},\,\overrightarrow{J})=\{\mathrm{J}_{\mu}\}$ are respectively the four currents associated with electric charge, magnetic monopole and generalized fields of dyons.

Equations (39) contains the following differential equations

$$\begin{aligned}
(\overrightarrow{\nabla}\cdot\overrightarrow{F_{e}}) &= \rho; \\
(\overrightarrow{\nabla}\times\overrightarrow{F_{e}}) &= -\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{k}; \\
(\overrightarrow{\nabla}\times\overrightarrow{F_{g}}) &= \frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j}; \\
(\overrightarrow{\nabla}\cdot\overrightarrow{F_{g}}) &= \varrho.
\end{aligned} \tag{41}$$

Replacing $\overrightarrow{F_{e}}\longrightarrow\overrightarrow{E}$, $\overrightarrow{F_{g}}\longrightarrow\overrightarrow{H}$, equation (41) is changed to following form of Maxwell's equations

$$\begin{aligned}
(\overrightarrow{\nabla}\cdot\overrightarrow{E}) &= \rho; \\
(\overrightarrow{\nabla}\times\overrightarrow{E}) &= -\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{k}; \\
(\overrightarrow{\nabla}\times\overrightarrow{H}) &= \frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j}; \\
(\overrightarrow{\nabla}\cdot\overrightarrow{H}) &= \varrho.
\end{aligned} \tag{42}$$

Which are the Generalized Dirac-Maxwell's (GDM) equations of dyons.

5 Generalized Split-Octonion Potential Wave Equations

In order to write the split-octonion form of potential wave equations, let us use the equations (30) and (31) and we get

$$\boxdot\overline{\boxdot}\mathbb{V}=\overline{\boxdot}\boxdot\mathbb{V}=-\mathbb{J}; \tag{43}$$

which can be visualized in term of 2×2 Zorn's vector matrix as

$$\begin{aligned}\boxdot\overline{\boxdot}\mathbb{V}&=\begin{pmatrix}(\Box\varphi-\Box\phi) & -\left(\Box\overrightarrow{A}+\Box\overrightarrow{B}\right)\\ \left(\Box\overrightarrow{A}-\Box\overrightarrow{B}\right) & (\Box\varphi+\Box\phi)\end{pmatrix}\\ \Longrightarrow-\begin{pmatrix}\jmath_{-} & -\vec{\jmath}_{+}\\ \vec{\jmath}_{-} & \jmath_{+}\end{pmatrix}&=-\begin{pmatrix}(\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right)\\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho)\end{pmatrix}.\end{aligned} \tag{44}$$

This can further be reduced to following form of wave equations

$$\begin{aligned}\Box\phi&=-\rho,\\ \Box\varphi&=-\varrho,\\ \Box\mathrm{A}_{\mu}&=-j_{\mu},\\ \Box B_{\mu}&=-k_{\mu}.\end{aligned} \tag{45}$$

Using the definitions (30) , we get the following expanded form of equation (45) as

$$\begin{aligned}\frac{\partial^{2}\phi}{\partial t^{2}}-\nabla^{2}\phi&=\rho,\\ \frac{\partial^{2}\varphi}{\partial t^{2}}-\nabla^{2}\varphi&=\varrho,\\ \frac{\partial^{2}A_{\mu}}{\partial t^{2}}-\nabla^{2}A_{\mu}&=j_{\mu},\\ \frac{\partial^{2}B_{\mu}}{\partial t^{2}}-\nabla^{2}B_{\mu}&=k_{\mu};\end{aligned} \tag{46}$$

which are the potential wave equations for generalized fields of dyons.

6 Generalized Split-Octonion Current Wave Equations

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. Thus, in the case of split-octonion current wave equations, we operate $\overline{\boxdot}$ given by equation (29) to split octonion current $\mathbb{J}$ (40) in term of 2×2 Zorn's vector matrix as [44]

$$\begin{aligned}\overline{\boxdot}\,\mathbb{J}&=\begin{pmatrix}-\frac{\partial}{\partial t} & \overrightarrow{\nabla}\\ -\overrightarrow{\nabla} & \frac{\partial}{\partial t}\end{pmatrix}*\begin{pmatrix}(\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right)\\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho)\end{pmatrix}\\ &=\begin{pmatrix}-\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot\overrightarrow{j}-\overrightarrow{\nabla}\cdot\overrightarrow{k} & \begin{array}{c}\frac{\partial\overrightarrow{j}}{\partial t}+\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho\\ -\overrightarrow{\nabla}\times\vec{j}+\overrightarrow{\nabla}\times\overrightarrow{k}\end{array}\\ \begin{array}{c}-\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho+\frac{\partial\overrightarrow{j}}{\partial t}-\frac{\partial\overrightarrow{k}}{\partial t}\\ +\overrightarrow{\nabla}\times\vec{j}+\overrightarrow{\nabla}\times\overrightarrow{k}\end{array} & \overrightarrow{\nabla}\cdot\overrightarrow{j}+\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}\end{pmatrix};\end{aligned} \tag{47}$$

which can be compactified as

$$\overline{\boxdot}\,\mathbb{J}=\mathbb{S}, \tag{48}$$

where

$$\mathbb{S}=(\Im_m-\Im_e)\,u_0^*+(\Im_m+\Im_e)\,u_0+(\overrightarrow{r}-\overrightarrow{s})\,u_i^*+(\overrightarrow{r}+\overrightarrow{s})\,u_i$$
$$=\begin{pmatrix}(\Im_m-\Im_e) & -(\overrightarrow{r}+\overrightarrow{s})\\ (\overrightarrow{r}-\overrightarrow{s}) & (\Im_m+\Im_e)\end{pmatrix}\longmapsto\begin{pmatrix}\Im_- & \overrightarrow{S_+}\\ \overrightarrow{S_-} & \Im_+\end{pmatrix}. \quad (49)$$

Here $\Im_- \rightarrow (\Im_m-\Im_e)$, $\overrightarrow{S_-}\rightarrow(\overrightarrow{r}-\overrightarrow{s})$, $\Im_+\rightarrow(\Im_m+\Im_e)$, $\overrightarrow{S_+}\rightarrow(\overrightarrow{r}+\overrightarrow{s})$. So, equation (48) leads to following four equations

$$\Im_m=\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t};$$
$$\Im_e=\overrightarrow{\nabla}\cdot\overrightarrow{j}+\frac{\partial\rho}{\partial t};$$
$$\overrightarrow{r}=-\overrightarrow{\nabla}\rho-\frac{\partial\overrightarrow{j}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{k};$$
$$\overrightarrow{s}=-\overrightarrow{\nabla}\varrho-\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{j}; \quad (50)$$

which are the split octonion current wave equations for the components of generalized fields of dyons. In equation (50) $\Im_m$ and $\Im_e$ are vanishing due to Lorentz gauge conditions applied for the cases of electric and magnetic charges.

7 Generalized Split-Octonion Field Equations (Continuity Equation)

A continuity equation in physics is an equation that describes the transport of a conserved quantity. Since mass, energy, momentum, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using continuity equations. A continuity equation is a special case of the more general transport equation. In the case of split-octonion field equations, we operate $\overline{\boxdot}$ both sides to the equation (29) as

$$\overline{\boxdot}\,(\boxdot\mathbb{F})=\left(\overline{\boxdot}\boxdot\right)\mathbb{F}=-\,\overline{\boxdot}\mathbb{J}\longrightarrow-\mathbb{S}, \quad (51)$$

where $\mathbb{F}$, $\mathbb{J}$, and $\mathbb{S}$ are defined in equations (37),(40) and (49). So, in terms of 2×2 Zorn's vector matrix, the left hand side of equation (51) can be written as

$$\left(\overline{\boxdot}\boxdot\right)\mathbb{F}=\begin{pmatrix}\square & 0\\ 0 & \square\end{pmatrix}*\begin{pmatrix}0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right)\\ \overrightarrow{H}-\overrightarrow{E} & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & -\left(\square\overrightarrow{H}+\square\overrightarrow{E}\right)\\ \left(\square\overrightarrow{H}-\square\overrightarrow{E}\right) & 0\end{pmatrix}; \quad (52)$$

whereas the right hand side of equation (51) is expressed as

$$\overline{\boxdot}\mathbb{J}=\begin{pmatrix}-\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot\overrightarrow{j}-\overrightarrow{\nabla}\cdot\overrightarrow{k} & \begin{matrix}\frac{\partial\overrightarrow{j}}{\partial t}+\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho\\ -\overrightarrow{\nabla}\times\overrightarrow{j}+\overrightarrow{\nabla}\times\overrightarrow{k}\end{matrix}\\ \begin{matrix}-\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho+\frac{\partial\overrightarrow{j}}{\partial t}-\frac{\partial\overrightarrow{k}}{\partial t}\\ +\overrightarrow{\nabla}\times\overrightarrow{j}+\overrightarrow{\nabla}\times\overrightarrow{k}\end{matrix} & \overrightarrow{\nabla}\cdot\overrightarrow{j}+\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}\end{pmatrix}. \quad (53)$$

So equation (53) thus leads to following differential equations

$$\Im_m\longrightarrow\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t}\Rightarrow 0,$$
$$\Im_e\longrightarrow\overrightarrow{\nabla}\cdot\overrightarrow{j}+\frac{\partial\rho}{\partial t}\Rightarrow 0; \quad (54)$$

which are the "*Continuity equations*" of generalized fields of dyons in split octonion formulation for moving dyons. Accordingly, from equation (50), we get

$$\overrightarrow{r} \longrightarrow -\overrightarrow{\nabla}\rho - \frac{\partial \overrightarrow{j}}{\partial t} - \overrightarrow{\nabla} \times \overrightarrow{k} = \Box \overrightarrow{E};$$
$$\overrightarrow{s} \longrightarrow -\overrightarrow{\nabla}\varrho - \frac{\partial \overrightarrow{k}}{\partial t} + \overrightarrow{\nabla} \times \overrightarrow{j} = \Box \overrightarrow{H}; \tag{55}$$

which may be reduced to

$$\nabla^2 \overrightarrow{H} - \frac{\partial^2 \overrightarrow{H}}{\partial t^2} = \overrightarrow{\nabla}\varrho + \frac{\partial \overrightarrow{k}}{\partial t} - \overrightarrow{\nabla} \times \overrightarrow{j} \longrightarrow \overrightarrow{s}$$
$$\nabla^2 \overrightarrow{E} - \frac{\partial^2 \overrightarrow{E}}{\partial t^2} = \overrightarrow{\nabla}\rho + \frac{\partial \overrightarrow{j}}{\partial t} + \overrightarrow{\nabla} \times \overrightarrow{k} \longrightarrow \overrightarrow{r} \tag{56}$$

These are generalized wave equations of dyons in split octonions.

8 Energy-Momentum Conservation in Split Octonion Electrodynamics

The laws of energy and momentum conservation [45] are probably the most frequently quoted laws in physics. The law of conservation of energy is a law of physics, i.e. the total amount of energy in an isolated system remains constant over time. The total energy is said to be conserved over time. For an isolated system, this law means that energy can change its location within the system, and that it can change form within the system, for instance chemical energy can become kinetic energy, but that energy can be neither created nor destroyed. And the other hand the law of conservation of momentum is a fundamental law of nature, and it states that if no external force acts on a closed system of objects, the momentum of the closed system remains constant. Conservation of momentum is a mathematical consequence of the homogeneity (shift symmetry) of space (position in space is the canonical conjugate quantity to momentum). That is, conservation of momentum is equivalent to the fact that the physical laws do not depend on position.

In the case of split octonion electrodynamics, we have used the equation (39). Operating $\overline{\mathbb{F}}$ on both sides of equation (39) as

$$\overline{\mathbb{F}}\,(\boxdot\mathbb{F}) = -\overline{\mathbb{F}}\,\mathbb{J}, \tag{57}$$

where the left hand side of equation (57) , i.e. $\overline{\mathbb{F}}\,(\boxdot\mathbb{F})$ may now be expressed in terms of 2×2 Zorn's vector matrix as

$$\overline{\mathbb{F}}\,(\boxdot\mathbb{F}) = \begin{pmatrix} 0 & \overrightarrow{H} - \overrightarrow{E} \\ -\overrightarrow{H} - \overrightarrow{E} & 0 \end{pmatrix} * \begin{pmatrix} -\overrightarrow{\nabla}.\overrightarrow{H} + \overrightarrow{\nabla}.\overrightarrow{E} & \begin{matrix} -\frac{\partial \overrightarrow{H}}{\partial t} - \frac{\partial \overrightarrow{E}}{\partial t} \\ +\overrightarrow{\nabla} \times \overrightarrow{H} - \overrightarrow{\nabla} \times \overrightarrow{E} \end{matrix} \\ \begin{matrix} -\frac{\partial \overrightarrow{H}}{\partial t} + \frac{\partial \overrightarrow{E}}{\partial t} \\ -\overrightarrow{\nabla} \times \overrightarrow{H} - \overrightarrow{\nabla} \times \overrightarrow{E} \end{matrix} & -\overrightarrow{\nabla}.\overrightarrow{H} - \overrightarrow{\nabla}.\overrightarrow{E} \end{pmatrix}$$
$$= \begin{pmatrix} B - A & -(C + D) \\ C - D & B + A \end{pmatrix}. \tag{58}$$

Here $A, B, C, D,$ the reduced forms of the matrix multiplication are described as

$$A = \overrightarrow{H}.\frac{\partial \overrightarrow{H}}{\partial t} + \overrightarrow{E}.\frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{H}.\left(\overrightarrow{\nabla} \times \overrightarrow{E}\right) - \overrightarrow{E}.\left(\overrightarrow{\nabla} \times \overrightarrow{H}\right);$$
$$B = \overrightarrow{H}.\frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{E}.\frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{H}.\left(\overrightarrow{\nabla} \times \overrightarrow{H}\right) + \overrightarrow{E}.\left(\overrightarrow{\nabla} \times \overrightarrow{E}\right);$$
$$C = \overrightarrow{H} \times \frac{\partial \overrightarrow{E}}{\partial t} - \overrightarrow{E} \times \frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{H} \times \left(\overrightarrow{\nabla} \times \overrightarrow{H}\right) - \overrightarrow{E} \times \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right)$$
$$+ \overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right) + \overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right);$$
$$D = \overrightarrow{E} \times \frac{\partial \overrightarrow{E}}{\partial t} - \overrightarrow{H} \times \frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{H} \times \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right) - \overrightarrow{E} \times \left(\overrightarrow{\nabla} \times \overrightarrow{H}\right)$$
$$+ \overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right) + \overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right); \tag{59}$$

whereas the right hand side of the equation (57) may also be expressed as

$$-\overline{\mathbb{F}}\mathbb{J} = -\begin{pmatrix} 0 & \left(\overrightarrow{H}-\overrightarrow{E}\right) \\ -\left(\overrightarrow{H}+\overrightarrow{E}\right) & 0 \end{pmatrix} * \begin{pmatrix} (\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right) \\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho) \end{pmatrix}$$
$$= \begin{pmatrix} B'-A' & -(C'+D') \\ C'-D' & B'+A' \end{pmatrix}, \tag{60}$$

where A', B', C', D', again the reduced forms, are described as

$$\begin{aligned} A' &= -\overrightarrow{H}.\overrightarrow{k}-\overrightarrow{E}.\overrightarrow{j}; \\ B' &= -\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k}; \\ C' &= -\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)-\left(\overrightarrow{E}\times\overrightarrow{k}\right); \\ D' &= -\rho\overrightarrow{H}+\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)+\left(\overrightarrow{E}\times\overrightarrow{j}\right). \end{aligned} \tag{61}$$

The above analysis shows that the left hand and right sides of equations (57) resemble to one another if the coefficients A, B, C, D and A', B', C', D' coincide to each other (i.e. $A \cong A'$, $B \cong B'$, $C \cong C'$, $D \cong D'$). Let us discuss the various consequences of the above analysis in following subsections.

8.1 Conservation of Energy of the Octonion Electrodynamics

In order to discuss the conservation of energy of the octonion electrodynamics, let us use equations (59) and (61) for A and A'. So, we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{H}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)-\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)=-\overrightarrow{H}.\overrightarrow{k}-\overrightarrow{E}.\overrightarrow{j}, \tag{62}$$

which reduces to

$$\frac{1}{2}\frac{\partial H^2}{\partial t}+\frac{1}{2}\frac{\partial E^2}{\partial t}+\overrightarrow{\nabla}.\left(\overrightarrow{E}\times\overrightarrow{H}\right)=-\overrightarrow{H}.\overrightarrow{k}-\overrightarrow{E}.\overrightarrow{j}. \tag{63}$$

This expression (63) may be visualized as the *"work-energy theorem"* or "*Poynting Theorem*" [45] to the case generalized octonion electrodynamics. Poynting's theorem is analogous to the work-energy theorem of classical mechanics reproducing the continuity equation, so that it relates the energy stored in generalized electromagnetic field to the work done on a charge distribution , through energy flux. It should be noted that the Poynting theorem is not valid in electrostatics and magnetostatics, since electric and magnetic fields change with time when electromagnetic energy flows. Equation (63) may then be reduced to

$$\frac{1}{2}\frac{\partial}{\partial t}\left(E^2+H^2\right)+\overrightarrow{\nabla}.\left(\overrightarrow{E}\times\overrightarrow{H}\right)+\left(\overrightarrow{H}.\overrightarrow{k}+\overrightarrow{E}.\overrightarrow{j}\right)=0, \tag{64}$$

where the energy due to electric field is given by

$$W_e = \frac{1}{2}\int E^2 d\tau, \tag{65}$$

whereas the energy due to magnetic field is discussed as

$$W_m = \frac{1}{2}\int H^2 d\tau, \tag{66}$$

So, the total energy stored in generalized electromagnetic fields of dyons is

$$W_{em} = \frac{1}{2}\int \left(E^2+H^2\right) d\tau. \tag{67}$$

As such, the energy density i.e. the energy per unit time, per unit area, transported by the fields is called the *Poynting vector* ($\overrightarrow{S}$) given by

$$\overrightarrow{S}=\left(\overrightarrow{E}\times\overrightarrow{H}\right), \tag{68}$$

which represents the directional energy flux density (the rate of energy transfer per unit area, in W/m^2) of an electromagnetic field. Thus, the *Poynting Theorem* also may be generalized as the conservation of energy i.e.

$$\frac{dW}{dt}=-\frac{\partial W_{em}}{\partial t}-\overrightarrow{\nabla}.\overrightarrow{S}-\left(\overrightarrow{H}.\overrightarrow{k}+\overrightarrow{E}.\overrightarrow{j}\right). \tag{69}$$

Similarly equating the coefficients B and B' of the equations (59) and (61), we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=-\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k}, \tag{70}$$

which is reduced to the following generalized Dirac-Maxwell's Equations (GDM) of dyons from octonion electrodynamics, i.e.

$$\begin{aligned}\frac{\partial\overrightarrow{H}}{\partial t}+(\overrightarrow{\nabla}\times\overrightarrow{E})=-\overrightarrow{k}\ &\longmapsto(\overrightarrow{\nabla}\times\overrightarrow{E})=-\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{k};\\ \frac{\partial\overrightarrow{E}}{\partial t}-(\overrightarrow{\nabla}\times\overrightarrow{H})=-\overrightarrow{j}\ &\longmapsto\overrightarrow{\nabla}\times\overrightarrow{H})=\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j}.\end{aligned} \tag{71}$$

8.2 Conservation of Momentum for Octonion Electrodynamics

The conservation of momentum is a fundamental concept of physics along with the conservation of energy and the conservation of mass. In order to understand the conservation of momentum for octonion electrodynamics, let us equate C and C' of the equations (59) and (61). So, we get

$$\begin{aligned}\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\=-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)-\left(\overrightarrow{E}\times\overrightarrow{k}\right).\end{aligned} \tag{72}$$

Now using the following identities

$$\begin{aligned}\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}&=-\frac{\partial}{\partial t}\left(\overrightarrow{E}\times\overrightarrow{H}\right)\Longrightarrow-\frac{\partial\overrightarrow{S}}{\partial t};\\ \overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)&=\frac{1}{2}\nabla\left(H^2\right)-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H};\\ \overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)&=\frac{1}{2}\nabla\left(E^2\right)-\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E};\end{aligned} \tag{73}$$

we get the following reduced form of equation (72), i.e.

$$\begin{aligned}\frac{\partial\overrightarrow{S}}{\partial t}+\frac{1}{2}\nabla\left(E^2+H^2\right)-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H}-\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E}-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)-\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\=\varrho\overrightarrow{H}+\rho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right),\end{aligned} \tag{74}$$

which gives rise the connection between electromagnetic energy and the force due to presence of electric and magnetic energy of dyons in the following manner

$$\begin{aligned}\overrightarrow{\boldsymbol{F}}=-\frac{\partial\overrightarrow{S}}{\partial t}-\frac{1}{2}\nabla\left(E^2+H^2\right)+\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H}+\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E}+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)\\+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\varrho\overrightarrow{H}+\rho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right).\end{aligned} \tag{75}$$

8.3 Maxwell Stress Tensor

The Maxwell Stress Tensor [45] is a second rank tensor used in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum. So, in order to get the solution for the electromagnetic force discussed by equation (75), we start with the following expression of Maxwell Stress Tensor T_{ij} for generalized electromagnetic field as [45]

$$T_{ij} = \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \left(H_i H_j - \frac{1}{2}\delta_{ij}H^2\right), \tag{76}$$

where the indices i and j refer to the coordinates x, y and z. So the stress tensor (76) has a total of nine components. Thus, we may also write (76) in terms of its following components

$$\begin{aligned} T_{xx} =& \frac{1}{2}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2}\left(H_x^2 - H_y^2 - H_z^2\right), \\ T_{xy} =& E_x E_y + H_x H_y, \end{aligned} \tag{77}$$

and so on. Maxwell Stress Tensor is usually denoted by a double arrow $\overleftrightarrow{T}$ to carry out two indices (i & j) where one of the indices represent vector.

So, the divergence of $\overleftrightarrow{T}$ is associated with its jth component as [45]

$$\begin{aligned} \left(\nabla.\overleftrightarrow{T}\right)_j =& \left[\left(\vec{\nabla}.\vec{E}\right)E_j + \left(\vec{E}.\vec{\nabla}\right)E_j - \frac{1}{2}\nabla_j E^2\right] \\ &+ \left[\left(\vec{\nabla}.\vec{H}\right)H_j + \left(\vec{H}.\vec{\nabla}\right)H_j - \frac{1}{2}\nabla_j H^2\right]. \end{aligned} \tag{78}$$

Hence the total octonionic representation of electromagnetic field given by (75) may be written as

$$\vec{\boldsymbol{F}} = \left(\nabla.\overleftrightarrow{T}\right) - \frac{\partial\vec{S}}{\partial t} + \vec{f}, \tag{79}$$

where $\vec{f}$ is given by

$$\begin{aligned} \vec{f} =& \varrho\vec{H} + \rho\vec{E} - \left(\vec{H}\times\vec{j}\right) + \left(\vec{E}\times\vec{k}\right) \\ =& \left(\rho\vec{E} + \vec{j}\times\vec{H}\right) + \left(\varrho\vec{H} - \vec{k}\times\vec{E}\right). \end{aligned} \tag{80}$$

Thus, equation (80) may be identified as an expression of *"generalized electromagnetic force"* of dyons. Let us use here the Newton's second law

$$\vec{\boldsymbol{F}} = \frac{\partial\overrightarrow{P_{mech}}}{\partial t} \qquad and \qquad \vec{f} = \frac{\partial\overrightarrow{P_{dyons}}}{\partial t}, \tag{81}$$

where $\overrightarrow{P_{mech}}$ and $\overrightarrow{P_{dyons}}$ are the mechanical and dyonic momentum respectively. So, the expression (81) describes the conservation of momentum in the following manner, i.e.

$$\begin{aligned} \frac{\partial\overrightarrow{P_{mech}}}{\partial t} &= \left(\nabla.\overleftrightarrow{T}\right) - \frac{\partial\vec{S}}{\partial t} + \frac{\partial\overrightarrow{P_{dyons}}}{\partial t} \\ &= \left(\nabla.\overleftrightarrow{T}\right) - \frac{\partial\left(\vec{S} - \overrightarrow{P_{dyons}}\right)}{\partial t}, \end{aligned} \tag{82}$$

which can further be reduced in the following form of *continuity equation as*

$$\frac{\partial}{\partial t}\left(\overrightarrow{P_{mech}} + \overrightarrow{P_{Gem}}\right) = \nabla.\overleftrightarrow{T}; \tag{83}$$

where $\overrightarrow{P_{Gem}} \longmapsto \left(\overrightarrow{S}-\overrightarrow{P_{dyons}}\right)$ is the total generalized electromagnetic momentum.
Comparing the coefficients D and D' of equations (59) and (61) respectively, we get

$$\begin{aligned}\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)-\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)\\ =-\rho\overrightarrow{H}+\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)+\left(\overrightarrow{E}\times\overrightarrow{j}\right),\end{aligned} \tag{84}$$

which can further be reduced to the complete set of Generalized Dirac Maxwell equations (42) of dyons.

9 Split-Octonionic realization for Lorentz invariants

Now instead of operating equation (39) by $\mathbb{F}$ from the left, let us operate it both sides from the left by $\mathbb{F}$. Accordingly, we get

$$\mathbb{F}\left(\boxdot\mathbb{F}\right)=-\mathbb{F}\left(\mathbb{J}\right). \tag{85}$$

The left hand side of the equation (85) may be written as

$$\begin{aligned}\mathbb{F}\left(\boxdot\mathbb{F}\right)=&\begin{pmatrix}0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right)\\ \overrightarrow{H}-\overrightarrow{E} & 0\end{pmatrix}*\begin{pmatrix}\overrightarrow{\nabla}.\overrightarrow{H}-\overrightarrow{\nabla}.\overrightarrow{E} & \frac{\partial\overrightarrow{H}}{\partial t}+\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{H}+\overrightarrow{\nabla}\times\overrightarrow{E}\\ \frac{\partial\overrightarrow{H}}{\partial t}-\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{H}+\overrightarrow{\nabla}\times\overrightarrow{E} & \overrightarrow{\nabla}.\overrightarrow{H}+\overrightarrow{\nabla}.\overrightarrow{E}\end{pmatrix}\\ =&\begin{pmatrix}\beta-\alpha & -\left(\gamma+\zeta\right)\\ \gamma-\zeta & \beta+\alpha\end{pmatrix};\end{aligned} \tag{86}$$

where the coefficients of the equation (86) expressed as

$$\begin{aligned}\alpha=&\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}+\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right);\\ \beta=&\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right);\\ \gamma=&\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)\\ &-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right);\\ \zeta=&-\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)\\ &-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right);\end{aligned} \tag{87}$$

Similarly, the right hand side of the equation (85) expressed as

$$\begin{aligned}-\mathbb{F}\left(\mathbb{J}\right)=&-\begin{pmatrix}0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right)\\ \overrightarrow{H}-\overrightarrow{E} & 0\end{pmatrix}*\begin{pmatrix}\left(\varrho-\rho\right) & -\left(\overrightarrow{j}+\overrightarrow{k}\right)\\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & \left(\varrho+\rho\right)\end{pmatrix}\\ =&\begin{pmatrix}\beta'-\alpha' & -\left(\gamma'+\zeta'\right)\\ \gamma'-\zeta' & \beta'+\alpha'\end{pmatrix},\end{aligned} \tag{88}$$

where

$$\begin{aligned}\alpha'=&\overrightarrow{H}.\overrightarrow{k}-\overrightarrow{E}.\overrightarrow{j};\\ \beta'=&\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k};\\ \gamma'=&-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right);\\ \zeta'=&-\rho\overrightarrow{H}-\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)-\left(\overrightarrow{E}\times\overrightarrow{j}\right).\end{aligned} \tag{89}$$

Using equations (87) and (89) for α and α', we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}+\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=\overrightarrow{H}.\overrightarrow{k}-\overrightarrow{E}.\overrightarrow{j}; \tag{90}$$

which is further reduced to

$$\frac{\partial}{\partial t}\left(\vec{E}^{2}-\vec{H}^{2}\right)=\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)-\overrightarrow{H}.\overrightarrow{k}+\overrightarrow{E}.\overrightarrow{j}. \tag{91}$$

This expression leads to the relation for first *Lorentz invariant* [46] of $\left(\vec{E}^{2}-\vec{H}^{2}\right)$. Similarly equating coefficients β and β' from equations (87) and (89), we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k}; \tag{92}$$

which is reduced to

$$\frac{\partial}{\partial t}\left(\overrightarrow{E}.\overrightarrow{H}\right)=\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)+\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k}. \tag{93}$$

This expression leads to the relation for second *Lorentz invariant* [46] of $\left(\overrightarrow{E}.\overrightarrow{H}\right)$. Accordingly equating the coefficients γ and γ' from equations (87) and (89), we get

$$\begin{aligned}\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\=-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right),\end{aligned} \tag{94}$$

which is further reduced to

$$\begin{aligned}\frac{1}{2}\overrightarrow{\nabla}\left(\vec{E}^{2}-\vec{H}^{2}\right)=\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E}-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H}+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)-\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\-\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right).\end{aligned} \tag{95}$$

This expression leads to the relation for the gradient of first *Lorentz invariant* [46] i.e. $\overrightarrow{\nabla}\left(\vec{E}^{2}-\vec{H}^{2}\right)$. Consequently equating the coefficients ζ and ζ' from equations (87) and (89), we get,

$$\begin{aligned}-\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)\\=-\rho\overrightarrow{H}-\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)-\left(\overrightarrow{E}\times\overrightarrow{j}\right),\end{aligned} \tag{96}$$

which is further simplified to

$$\begin{aligned}\overrightarrow{\nabla}\left(\overrightarrow{E}\cdot\overrightarrow{H}\right)=\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{H}-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{E}+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)-\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t}\\-\rho\overrightarrow{H}-\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)-\left(\overrightarrow{E}\times\overrightarrow{j}\right).\end{aligned} \tag{97}$$

These expression leads to the relation for the gradient of second *Lorentz invariant* [46] i.e. $\overrightarrow{\nabla}\left(\overrightarrow{E}\cdot\overrightarrow{H}\right)$. It should be noted that the theory of classical electrodynamics has been generalized consistently to the case of dyons by means of split octonions and it is shown that fore going analysis is compact, simpler and manifestly covariant.

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