Proposed Quantum Mechanical Connection

E-mail: drutherford@softcom.net
http://www.softcom.net/users/der555/quantum.pdf

December 13, 2006

Abstract

We derive the equations of relativistic quantum mechanics from a modified version of classical electrodynamics, where probability is replaced by potential. As a result, a particle is not a localized entity, in the classical sense, but has a localized energy extremum. The particle/wave aspect of matter is inherent in the particle/wave equation describing elementary particles. Furthermore, the Heisenberg uncertainty and Planck-Einstein-de Broglie relations, and the Klein-Gordon, Dirac and Proca equations follow naturally from the particle/wave equation. In addition, we incorporate a new and more physical interpretation of spin angular momentum.

This is a proposed connection between my theory [1] and relativistic quantum mechanics (RQM). In order to show the relationship between the two, I have attempted to derive the major equations of RQM (Klein-Gordon, Dirac, Proca) from the equations in the article above.

The scalar electric potential ϕ [2] at the event $P(x, y, z, t)$, or P, due to a stationary 'point' charge q at the origin, in SI units is

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{s} \tag{1}$$

where s is the spacetime interval [3] from P to the origin

$$s = \sqrt{x^2 + y^2 + z^2 + c^2t^2} \tag{2}$$

where c is the speed of light in vacuo. For a stationary charge density ρ, we have the four-dimensional Poisson's equation [4]

$$\nabla^2\phi + \frac{\partial^2\phi}{c^2\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{4}$$

Setting $r^2 = x^2 + y^2 + z^2$, where r is the spatial distance between P and the origin, and $a^2 = c^2t^2$ in (2), where a is an unspecified (nonzero) constant, we can write (1) as

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2 + a^2}} \tag{5}$$

I intend now to show that (5) satisfies (3). Inserting (5) into the left-hand side of (3), we have, for the first term,

$$\begin{aligned} \nabla^2\phi &= \frac{q}{4\pi\epsilon_0} \nabla^2 \left(\frac{1}{\sqrt{r^2 + a^2}} \right) \\ &= -\frac{q}{4\pi\epsilon_0} \left(\frac{3a^2}{(r^2 + a^2)^{5/2}} \right) \end{aligned} \tag{6}$$

and for the second term,

$$\frac{\partial^2\phi}{c^2\partial t^2} = 0 \tag{7}$$

Adding (6) and (7) we have

$$\nabla^2\phi + \frac{\partial^2\phi}{c^2\partial t^2} = -\frac{q}{4\pi\epsilon_0} \left(\frac{3a^2}{(r^2 + a^2)^{5/2}} \right) \tag{8}$$

Equation (8) is well-behaved and, in general, nonzero everywhere.

Now, in order to show that (5) satisfies (3), we must show that the right-hand sides of (8) and (3) are equivalent, in other words, we must show that

$$-\frac{q}{4\pi\epsilon_0} \left(\frac{3a^2}{(r^2 + a^2)^{5/2}} \right) = -\frac{\rho}{\epsilon_0} \tag{9}$$

To do this, we start by noting that integrating ρdV over all space, where dV is the volume element, must result in the magnitude of the charge q. Solving (9) for ρ, we get

$$\rho = \frac{3a^2q}{4\pi(r^2 + a^2)^{5/2}} \tag{10}$$

In order to show that (9) is true, we must be able to substitute the right-hand side of (10) for ρ in our integration of ρdV, and obtain the result q. Carrying out the integration, using (10) and the volume element $dV = 4\pi r^2 dr$, we find

$$\int_{\text{all space}} \rho dV = \int_0^\infty \frac{3a^2q}{4\pi(r^2 + a^2)^{5/2}} \, 4\pi r^2 dr = q \tag{11}$$

showing that (5) does indeed satisfy (3). Note that, since the right-hand side of (10) is nonzero everywhere, the charge density ρ on the left-hand side of (10) must, also, be nonzero everywhere.

At $r = 0$, (8) becomes

$$\begin{aligned} \nabla^2\phi + \frac{\partial^2\phi}{c^2\partial t^2} &= -\frac{3q}{4\pi\epsilon_0 a^3} \\ &= -\frac{3}{a^2} \frac{q}{4\pi\epsilon_0 a} \end{aligned} \tag{12}$$

On the right-hand side of (12), we see that $q/4\pi\epsilon_0 a$ is equivalent to the scalar potential ϕ from (5) at $r=0$. Thus, setting $k^2 = 3/a^2$, we can write the right-hand side of (12) as

$$-\frac{3}{a^2}\frac{q}{4\pi\epsilon_0 a} = -k^2\phi \tag{13}$$

for $a > 0$. Using (13), we can now write (12) as

$$\nabla^2\phi + \frac{\partial^2\phi}{c^2\partial t^2} = -k^2\phi \tag{14}$$

or

$$\left(\partial^2 + k^2\right)\phi = 0 \tag{15}$$

where

$$\partial^2 = \nabla^2 + \frac{\partial^2}{c^2\partial t^2} \tag{16}$$

with general solution

$$\phi = A\cos(\mathbf{k}\cdot\mathbf{x}) + B\sin(\mathbf{k}\cdot\mathbf{x}) \tag{17}$$

where A and B are arbitrary constants, $\mathbf{k} = (k_x, k_y, k_z, k_t)$, $\mathbf{x} = (x, y, z, ct)$, and $k^2 = k_x^2 + k_y^2 + k_z^2 + k_t^2$.[1]

If $a < 0$, with $k^2 = 3/a^2$, (12) becomes

$$\nabla^2\phi + \frac{\partial^2\phi}{c^2\partial t^2} = k^2\phi \tag{18}$$

or

$$\left(\partial^2 - k^2\right)\phi = 0 \tag{19}$$

with general solution

$$\phi = C\exp(\mathbf{k}\cdot\mathbf{x}) + D\exp(-\mathbf{k}\cdot\mathbf{x}) \tag{20}$$

where C and D are arbitrary constants.[2]

We can form still another solution to (18) if we multiply the two solutions (17) and (20) together. In order to differentiate between the two, I will refer to solution (17) as ϕ_+, since $a > 0$, and solution (20) as ϕ_-, since $a < 0$. Note that the $\mathbf{k}$'s in (17) and (20) need not be equivalent. Allowing for this possibility, I will replace the $\mathbf{k}$ associated with (20) with $\mathbf{p} = (p_x, p_y, p_z, p_t)$, where $p^2 = p_x^2 + p_y^2 + p_z^2 + p_t^2$. The combined potential ϕ is, thus,

$$\phi = \phi_+\phi_- = \left(A\cos(\mathbf{k}\cdot\mathbf{x}) + B\sin(\mathbf{k}\cdot\mathbf{x})\right)\left(C\exp(\mathbf{p}\cdot\mathbf{x}) + D\exp(-\mathbf{p}\cdot\mathbf{x})\right) \tag{21}$$

Expanding the right-hand side of (21) produces the sum $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$, where

$$\begin{aligned} \phi_1 &= A_1\exp(\mathbf{p}\cdot\mathbf{x})\cos(\mathbf{k}\cdot\mathbf{x}) \\ \phi_2 &= A_2\exp(\mathbf{p}\cdot\mathbf{x})\sin(\mathbf{k}\cdot\mathbf{x}) \\ \phi_3 &= A_3\exp(-\mathbf{p}\cdot\mathbf{x})\cos(\mathbf{k}\cdot\mathbf{x}) \\ \phi_4 &= A_4\exp(-\mathbf{p}\cdot\mathbf{x})\sin(\mathbf{k}\cdot\mathbf{x}) \end{aligned} \tag{22}$$

[1] An equally valid solution to (14) is $\phi = A\exp(\mathbf{k}\cdot\mathbf{x}) + B\exp(-\mathbf{k}\cdot\mathbf{x})$ with the basis vectors $\mathbf{e}_\mu$ [5] satisfying the relations $\mathbf{e}_\mu\mathbf{e}_\nu = -\mathbf{e}_4$ for $\mu = \nu$, where $\mu, \nu = 1, 2, 3, 4$.

[2] Here, in (20), the basis vectors $\mathbf{e}_\mu$ [5] satisfy the relations $\mathbf{e}_\mu\mathbf{e}_\nu = \mathbf{e}_4$ for $\mu = \nu$, where $\mu, \nu = 1, 2, 3, 4$.

and $A_1 = AC$, $A_2 = BC$, $A_3 = AD$ and $A_4 = BD$. If we set $A = B = C = D = 1$ and insert (21) into the left-hand side of (18), we get

$$\nabla^2 \phi + \frac{\partial^2 \phi}{c^2 \partial t^2} = \left(\mathbf{p}^2 - \mathbf{k}^2\right) \phi \tag{23}$$

Then, due to the multiplication rules for the orthonormal basis vectors $\mathbf{e}_\mu$ [5], we can write $(\mathbf{p}^2 - \mathbf{k}^2) = (\mathbf{p} - \mathbf{k})^2$ so that (23) becomes

$$\nabla^2 \phi + \frac{\partial^2 \phi}{c^2 \partial t^2} = (\mathbf{p} - \mathbf{k})^2 \phi \tag{24}$$

Therefore, (21) is also a solution to (18) with $k = \mathbf{p} - \mathbf{k}$. Furthermore, it can be shown that $k = \mathbf{p} + \mathbf{k}$, $k = -\mathbf{p} - \mathbf{k}$, and $k = -\mathbf{p} + \mathbf{k}$ also satisfy $k^2 = \mathbf{p}^2 - \mathbf{k}^2$ and, therefore, $k^2 = (\mathbf{p} - \mathbf{k})^2$. Note, also, that if $\mathbf{p} < \mathbf{k}$, then $k^2 = \mathbf{p}^2 - \mathbf{k}^2 < 0$, thus, (21) is also a solution to (14). This would seem to suggest that whether k^2 is positive, negative, or zero depends on whether $\mathbf{p}$ is greater than, less than, or equal to $\mathbf{k}$, respectively.

Evidently, from (21), a particle is *not* a localized entity, in the classical sense (i.e., it's not a little round ball) - it extends over all space. It does, however, have a localized energy maximum (or minimum) at $r = 0$. The localized energy maximum at $r = 0$ is associated with the $\exp(-\mathbf{p} \cdot \mathbf{x})$ terms, and the localized energy minimum is associated with the $\exp(\mathbf{p} \cdot \mathbf{x})$ terms. I hesitate, however, to disregard the $\exp(\mathbf{p} \cdot \mathbf{x})$ terms on the basis that they 'blow up' as $r \to \infty$. I suspect they describe physically valid quantities - possibly antiparticles. The terms $\cos(\mathbf{k} \cdot \mathbf{x})$ and $\sin(\mathbf{k} \cdot \mathbf{x})$ may describe the field of the particle or antiparticle described by $\exp(-\mathbf{p} \cdot \mathbf{x})$ or $\exp(\mathbf{p} \cdot \mathbf{x})$.[3]

It would seem natural to attribute the sinusoidal parts of (21) to the wave aspect, and the exponential parts to the particle aspect, of matter. In order to make the correspondence with the equations of RQM, I will associate $\mathbf{p}$ with the momentum four-vector $\mathbf{p} = (p_x, p_y, p_z, E/c)$ (in which case we would need to divide by $\hbar$) and $\mathbf{k}$ with the wave four-vector $\mathbf{k} = (k_x, k_y, k_z, \omega/c)$.

Using these definitions for $\mathbf{p}$ and $\mathbf{k}$, if we set $k = (\mathbf{p}/\hbar) - \mathbf{k} = 0$, after rearranging, we arrive at

$$\mathbf{p} = \hbar \mathbf{k} \tag{25}$$

which contains the Planck-Einstein-de Broglie relations.[4] Analogously, if $k = (\mathbf{p}/\hbar) - \mathbf{k} > 0$, then

$$\mathbf{p} > \hbar \mathbf{k} \tag{26}$$

Combining (25) and (26), we get

$$\mathbf{p} \geq \hbar \mathbf{k} \tag{27}$$

which leads to the Heisenberg uncertainty relations $\Delta\mathbf{p}\Delta\mathbf{x} \geq \hbar$.[5] If $k = (\mathbf{p}/\hbar) - \mathbf{k} < 0$, then

$$\mathbf{p} < \hbar \mathbf{k} \tag{28}$$

[3]In my paper [1] the field of a particle (antiparticle) travelling 'forward' in time is its associated antiparticle (particle) travelling (relatively) 'backward' in time, thus, the particle/field 'pair' form an electric 'dipole', since every charged particle is accompanied by its field [6]. In this case, $\phi_+ = A\cos(\mathbf{k} \cdot \mathbf{x}) + B\sin(\mathbf{k} \cdot \mathbf{x})$ in (21) might better be replaced with $\phi_+ = A\exp(\mathbf{k} \cdot \mathbf{x}) + B\exp(-\mathbf{k} \cdot \mathbf{x})$ with the basis vectors $\mathbf{e}_\mu$ [5] satisfying the relations $\mathbf{e}_\mu \mathbf{e}_\nu = -\mathbf{e}_4$ for $\mu = \nu$, where $\mu, \nu = 1, 2, 3, 4$. In addition, if $k = 0$ in (14) or (18), or if $\mathbf{p} - \mathbf{k} = 0$ in (24), the implication is that there are no electric monopoles.

[4]Thus the Planck-Einstein-de Broglie relations imply no electric monopoles.

[5]$\Delta\mathbf{p}\Delta\mathbf{x}$ signifies the product of the uncertainties in the corresponding components of $\mathbf{p}$ and $\mathbf{x}$, i.e. $\Delta\mathbf{p}\Delta\mathbf{x} \geq \hbar$ signifies $\Delta p_x \Delta x \geq \hbar$, $\Delta p_y \Delta y \geq \hbar$, $\Delta p_z \Delta z \geq \hbar$, and $\Delta E \Delta t \geq \hbar$.

The meaning of (28) is not clear at this time, however, one implication is that $\Delta\mathbf{p}\Delta\mathbf{x} < \hbar$, providing $\mathbf{p} < \hbar\mathbf{k}$.

If we set $k^2 = m^2c^2/\hbar^2$, in (19), where m is the mass of the particle, equation (19) becomes

$$\left(\partial^2 - \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \tag{29}$$

This is the *Klein-Gordon equation*, up to a sign, which describes a particle with spin 0 and mass m.

If we specify that $k^2 = m^2c^2/\hbar^2 = 3/a^2$, the value of a would be $a = \pm\hbar\sqrt{3}/mc$. Note, however, that this value for a arises from the time interval t, *not* a spatial interval, since $a = ct$.

To get the Dirac equation, we start with (29), multiply by $\hbar^2$, then, due to the multiplication rules for the orthonormal basis vectors $\mathbf{e}_\mu$ [5] (remembering that ∂ is a four-vector [7]), we can factor, to obtain

$$(\hbar\partial + mc)(\hbar\partial - mc)\phi = 0 \tag{30}$$

Eliminating the first term in parentheses, by convention, we then have

$$(\hbar\partial - mc)\phi = 0 \tag{31}$$

This is the *Dirac equation*, up to a sign, which describes a particle with spin 1/2 and mass m.

Equation (15) can easily be made to describe a particle with spin 1 and mass m, by simply replacing the scalar potential ϕ with the potential four-vector A_μ, and setting $k^2 = m^2c^2/\hbar^2$, thus,

$$\left(\partial^2 + \left(\frac{mc}{\hbar}\right)^2\right)A_\mu = 0 \tag{32}$$

This is the *Proca equation*, up to a sign, where the term $\partial_\nu(\partial_\mu A_\nu)$ from Maxwell's equations has been cancelled by the time component of the electric field [8], thus it is unnecessary to invoke the Lorenz condition $\partial_\nu A_\nu = 0$. The free photon is described by setting $m = 0$ in (32).

We can incorporate angular momentum and spin [9] by replacing ϕ, ϕ_+, and ϕ_- in (21) with

$$\phi' = \phi'_+\phi'_- = (A\cos(\mathbf{kx}) + B\sin(\mathbf{kx}))\,(C\exp(\mathbf{px}/\hbar) + D\exp(-\mathbf{px}/\hbar)) \tag{33}$$

where $\mathbf{p}$ is the momentum four-vector, $\mathbf{k}$ is the wave four-vector, and $\mathbf{kx}$ and $\mathbf{px}$ are four-vector products [10].

References

[1] See "New Transformation Equations and the Electric Field Four-vector", at http://www.softcom.net/users/der555/newtransform.pdf.

[2] See "New Transformation Equations and the Electric Field Four-vector", Section 11.1, at http://www.softcom.net/users/der555/newtransform.pdf.

[3] See "New Transformation Equations and the Electric Field Four-vector", Section 2.2, at http://www.softcom.net/users/der555/newtransform.pdf.

[4] See "New Transformation Equations and the Electric Field Four-vector", Section 12.3, at http://www.softcom.net/users/der555/newtransform.pdf.

[5] See "New Transformation Equations and the Electric Field Four-vector", Section 4.1.1, at http://www.softcom.net/users/der555/newtransform.pdf.

[6] See "New Transformation Equations and the Electric Field Four-vector", Conclusions, at http://www.softcom.net/users/der555/newtransform.pdf.

[7] See "New Transformation Equations and the Electric Field Four-vector", Section 4.1.3, at http://www.softcom.net/users/der555/newtransform.pdf.

[8] See "New Transformation Equations and the Electric Field Four-vector", Section 12.1, at http://www.softcom.net/users/der555/newtransform.pdf.

[9] See "New Transformation Equations and the Electric Field Four-vector", Section 20, at http://www.softcom.net/users/der555/newtransform.pdf.

[10] See "New Transformation Equations and the Electric Field Four-vector", Section 4.1.2, at http://www.softcom.net/users/der555/newtransform.pdf.