Energy Density Correction

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Abstract

We show that the energy density of a continuous charge distribution must be twice the conventionally accepted value. This conclusion is qualified through logical argument and quantified using conventional mathematical methods.

1 The Energy Density of a Charge Distribution

The energy density u of a continuous charge distribution, according to popular belief, in SI units, is

$$u = \frac{\epsilon_0}{2} E^2 \tag{1}$$

where ϵ_0 is the permittivity constant and E is the conventional electric field strength. I intend to show, using conventional terminology, that this should instead be

$$u = \epsilon_0 E^2 \tag{2}$$

In order to simplify, I'll start with the energy of a point charge distribution. The energy of the distribution is just the work required to assemble the distribution. Let's start with a pair of identical charged particles, q_1 and q_2, at $+\infty$ and $-\infty$, respectively. The work W_{12} required to bring q_1 in from $+\infty$ to the origin against the field of q_2, is

$$W_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1\, q_2}{r_{12}} \tag{3}$$

where $(1/4\pi\epsilon_0)(q_2/r_{12})$ is the potential at the location of q_1 due to q_2, r_{12} is the distance between q_1 and q_2 after we're through bringing in q_1. But, since q_2 is not in the picture yet ($r_{12} = \infty$), the work required to bring q_1 in is zero, i.e., $W_{12} = 0$.

The work W_{21} necessary to bring in q_2 is

$$W_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_2\, q_1}{r_{21}} \tag{4}$$

However, as we bring in q_2, we must do work on q_1 against the field of q_2 in order to keep it in place at the origin.[1] The work necessary to keep q_1 in place, is the same as the work that would be required to bring q_1 in against the field of q_2, had we brought q_2 in first. But this is the same as (3) for $r_{12} = r_{21}$. So the total magnitude of the work W required to assemble the two particles is

$$W = W_{12} + W_{21} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1\,q_2}{r_{12}} + \frac{q_2\,q_1}{r_{21}}\right) \tag{5}$$

or

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{q_i q_j}{r_{ij}} \tag{6}$$

for $i \neq j$.

For three particles, we have to add the work required to bring in a third particle q_3 from infinity against the fields of both q_1 and q_2

$$W_{31} + W_{32} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_3\,q_1}{r_{31}} + \frac{q_3\,q_2}{r_{32}}\right) \tag{7}$$

to the work required to assemble q_1 and q_2 from (5). But again we have to include the work required to keep q_1 and q_2 in position as we bring in q_3. The work W_{13} and W_{23} required to keep q_1 and q_2 in position is

$$W_{13} = \frac{1}{4\pi\epsilon_0}\frac{q_1\,q_3}{r_{13}} \tag{8}$$

and

$$W_{23} = \frac{1}{4\pi\epsilon_0}\frac{q_2\,q_3}{r_{23}} \tag{9}$$

respectively. So the total work W required to assemble the three particles, from (5), (7), (8) and (9), is

$$W = W_{12} + W_{13} + W_{21} + W_{23} + W_{31} + W_{32} \tag{10}$$

or

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1\,q_2}{r_{12}} + \frac{q_1\,q_3}{r_{13}} + \frac{q_2\,q_1}{r_{21}} + \frac{q_2\,q_3}{r_{23}} + \frac{q_3\,q_1}{r_{31}} + \frac{q_3\,q_2}{r_{32}}\right) \tag{11}$$

which we can write as

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{q_i q_j}{r_{ij}} \tag{12}$$

for $i \neq j$. So for any number of particles n, we can apparently write

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{r_{ij}} \tag{13}$$

[1]This contradicts the conventional definition of work in which work can only be done on a body that undergoes a displacement, however, I believe this definition is incomplete. Please refer to Section 19 in http://www.softcom.net/users/der555/newtransform.pdf for my definition of work density.

for $i \neq j$. We can also write this as

$$W = \sum_{i=1}^{n} q_i \left(\sum_{j=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) \tag{14}$$

for $i \neq j$. The term in brackets is the potential $\phi(\mathbf{r}_i)$ at the point $\mathbf{r}_i$ (the position of q_i) due to all other charges. Thus, we can write (14) as

$$W = \sum_{i=1}^{n} q_i \phi(\mathbf{r}_i) \tag{15}$$

For a volume charge density ρ, (15) becomes

$$W = \int \rho\phi \, dV \tag{16}$$

where dV is the volume element. Using Poisson's equation $\nabla^2\phi = -\rho/\epsilon_0$ we can write (16) as

$$W = -\epsilon_0 \int \phi\nabla^2\phi \, dV \tag{17}$$

Integration by parts then leads to

$$W = \epsilon_0 \int |\nabla\phi|^2 \, dV \tag{18}$$

and, since $E = |\nabla\phi|$, we get

$$W = \epsilon_0 \int E^2 dV \tag{19}$$

This represents the energy stored in the electric field, therefore, we can interpret $\epsilon_0 E^2$ as the energy density u, or

$$u = \epsilon_0 E^2 \tag{20}$$

Clearly, this is in contradiction to the conventional physics claim that the energy density is $u = (\epsilon_0/2)E^2$. The extra energy density is due to the extra work that must be done on the charges already assembled in order to keep them in place against the field of each new charge as we add it to the configuration.