Matrix Transformation and Solutions of Wave Equation of Free Electromagnetic Field

Xianzhao Zhong*

Meteorological College of Yunnan Province, Kunming, 650228, China

Abstract

In this paper, the generalized differential wave equation for free electromagnetic field is transformed and formulated by means of matrixes. Then Maxwell wave equation and the second form of wave equation are deduced by matrix transformation. The solutions of the wave equations are discussed . Finally, two differential equations of vibration are established and their solutions are discussed .

Key words: matrix transformation, wave equation.

PACS: 03.50.De

1 Introduction

By means of matrix establishment and transformation [1], the generalized wave equation for free electromagnetic field [2] is transformed and formulated in terms of a diagonal matrix. Then both wave equations, Maxwell wave

* *E-mail address*: zhxzh46@163.com

equation [3] and the second form of wave equation are deduced from the the generalized wave equation in terms of matrixes. Solutions of the two kinds of wave equations are discussed. For Maxwell wave equation, the wave amplitude $\mathbb{F}_0$ is independent of wave number k and wave frequency ω. For the generalized wave equation, it is a function of wave number k and wave frequency ω.

Solving the second form of wave equation, two solutions, $\mathbb{F}_1'$ and $\mathbb{F}_1''$, are obtained. Multiplying $\mathbb{F}_1'$ by $\mathbb{F}_1''$ results in a function $\mathbb{F}$ that is identical to the solution of the generalized wave equation. Finally, a pair of differential vibration equations are established , whose solution product is identical to the solution of the generalized wave equation .

2 Transforming the Generalized Wave Equation by Means of Matrix

The generalized wave equation of free electromagnetic field is

$$\nabla^2\mathbb{F} - \frac{1}{c^2}\mathbb{C}\cdot\nabla\frac{\partial\mathbb{F}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbb{F}}{\partial t^2} = 0, \tag{1}$$

where $\mathbb{F}$ is the electromagnetic field [2]. Now we transform Eq.(1) into diagonal matrix. Suppose

$$\mathbb{C}\cdot\nabla = p \tag{2}$$

and

$$\frac{\partial}{\partial t} = \varepsilon. \tag{3}$$

Thus Eq.(1) is rewritten as

$$(p^2 - p\varepsilon - \varepsilon^2)\mathbb{F} = 0. \tag{4}$$

It is required from Eq.(4) that , for arbitrary $\mathbb{F}$,

$$p^2 - p\varepsilon - \varepsilon^2 = 0. \tag{5}$$

Rewriting Eq.(5), we have

$$p^2 - \frac{1}{2}p\varepsilon - \frac{1}{2}p\varepsilon - \varepsilon^2 = 0, \tag{6}$$

which is then transformed into

$$\begin{pmatrix} p & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} p \\ \varepsilon \end{pmatrix} = 0. \tag{7}$$

Let

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \tag{8}$$

whose eigenvalue equation is

$$det(\lambda E - A) = 0. \tag{9}$$

Solving Eq.(9), two eigenvalues are obtained. They are

$$\lambda_1 = \frac{\sqrt{5}}{2} \tag{10}$$

and

$$\lambda_2 = -\frac{\sqrt{5}}{2}. \tag{11}$$

Solving the equation of eigenvector

$$(\lambda E - A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \tag{12}$$

we obtain the eigenvectors as

$$\alpha_1 = (-1 \quad \sqrt{5} - 2)^T \quad (for \lambda_1) \tag{13}$$

and

$$\alpha_2 = (1 \quad \sqrt{5}+2)^T \quad (for \lambda_2). \tag{14}$$

Then two matrixes

$$Y = \begin{pmatrix} -1 & 1 \\ \sqrt{5}-2 & \sqrt{5}+2 \end{pmatrix} \tag{15}$$

and

$$Y^T = \begin{pmatrix} -1 & \sqrt{5}-2 \\ 1 & \sqrt{5}+2 \end{pmatrix} \tag{16}$$

are built up in terms of the eigenvectors α_1 in Eq.(13) and α_2 in Eq.(14).

And the matrix A in Eq.(8) is transformed into a diagonal matrix

$$Y^T A Y = \begin{pmatrix} -10+5\sqrt{5} & 0 \\ 0 & -10-5\sqrt{5} \end{pmatrix} = diag(\Lambda_1 \quad \Lambda_2) \tag{17}$$

in virtue of Eq.(15) and Eq.(16). Thus Eq.(4) is transformed into

$$\begin{pmatrix} p & \varepsilon \end{pmatrix} \begin{pmatrix} -10+5\sqrt{5} & 0 \\ 0 & -10-5\sqrt{5} \end{pmatrix} \begin{pmatrix} p \\ \varepsilon \end{pmatrix} \mathbb{F}_1 = 0. \tag{18}$$

After transformation, the electromagnetic field has changed from the field $\mathbb{F}$ to the field $\mathbb{F}_1$. Equation (18) is developed into

$$5\sqrt{5}(p^2 - \varepsilon^2)\mathbb{F}_1 - 10(p^2 + \varepsilon^2)\mathbb{F}_1 = 0. \tag{19}$$

Substituting Eq.(2) and Eq.(3) into Eq.(19) and taking its first bracket , we have

$$\nabla^2 \mathbb{F}_1 - \frac{1}{c^2}\frac{\partial^2 \mathbb{F}_1}{\partial t^2} = 0, \tag{20}$$

which is the well-known Maxwell wave equation. Taking the second bracket of Eq.(19) results in

$$(p^2 + \varepsilon^2)\mathbb{F}_1 = 0, \tag{21}$$

which leads to

$$\nabla^2 \mathbb{F}_1 + \frac{1}{c^2}\frac{\partial^2 \mathbb{F}_1}{\partial t^2} = 0, \tag{22}$$

the second form of wave equation.

3 Solutions of Generalized Wave Equation of Free Electromagnetic Field

The solution of the free electromagnetic field formulated by Eq.(1) is

$$\mathbb{F} = \mathbb{F}_0 exp(\lambda kr - \lambda\omega t) exp(i\lambda kr - i\lambda\omega t), \tag{23}$$

where k is the wave number and ω is the wave frequency.

In the last section, we transform the the generalized wave equation and obtain the well-known Maxwell wave equation [3] and the the second form of wave equation by matrix transformation. From Maxwell wave equation

$$\nabla^2 \mathbb{F}_1 - \frac{1}{c^2}\frac{\partial^2 \mathbb{F}_1}{\partial t^2} = 0 \tag{24}$$

we have the solution

$$\mathbb{F}_1 = \mathbb{F}_{10} exp(ikr - i\omega t), \tag{25}$$

where the wave amplitude $\mathbb{F}_{10}$ is a constant quantity, independent of the wave number k and wave frequency ω.

The other form of wave equation is

$$\nabla^2 \mathbb{F}_1 + \frac{1}{c^2}\frac{\partial^2 \mathbb{F}_1}{\partial t^2} = 0. \tag{26}$$

By separation of variables of the spatial r and the temporal t [4],

$$\mathbb{F}_1(r,t) = \mathbb{R}(r)T(t), \tag{27}$$

for the electromagnetic field $\mathbb{F}_1$ in Eq.(26) , there is the first form

$$\frac{c^2}{\mathbb{R}(r)}\frac{\partial^2\mathbb{R}(r)}{\partial r^2} = -\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \lambda^2\omega^2 \tag{28}$$

where ω is supposed to be a constant and the λ a variable. For Eq.(28), there is a solution

$$\mathbb{F}'_1 = \mathbb{R}_0 T_0 exp(\pm\lambda kr)exp(\pm i\lambda\omega t) \tag{29}$$

or rewritten

$$\mathbb{F}'_1 = \mathbb{F}'_0 exp(\pm\lambda kr)exp(\pm i\lambda\omega t). \tag{30}$$

For Eq.(26), there is the second form

$$-\frac{c^2}{\mathbb{R}(r)}\frac{\partial^2\mathbb{R}(r)}{\partial r^2} = \frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \lambda^2\omega^2. \tag{31}$$

For Eq.(31) , there is a solution

$$\mathbb{F}''_1 = \mathbb{R}_0 T_0 exp(\pm\lambda\omega t)exp(\pm i\lambda kr) \tag{32}$$

or rewritten

$$\mathbb{F}''{}_1 = \mathbb{F}''_0 exp(\pm\lambda\omega t)exp(\pm i\lambda kr). \tag{33}$$

Now multiplying Eq.(30) by Eq.(33) results in

$$\mathbb{F} = \mathbb{F}'{}_1\mathbb{F}''{}_1 = \mathbb{F}_0 exp(\lambda kr - \lambda\omega t)exp(i\lambda kr - i\lambda\omega t), \tag{34}$$

where k is the wave number and ω is the wave frequency ω. Equation (34) is identical to Eq.(23).

By matrix transformation, both wave equations, Eq.(24) and Eq.(26), are derived from the generalized wave equation Eq.(1), thus the general quadratic form becomes a standard quadratic one. The generalized wave equation conforms perfectly with Maxwell wave equation and the second form of wave equation.

4 Two Vibration Equations and Their Solutions

Now let us establish a pair of differential vibration equations for the electromagnetic field. The first vibration equation in terms of the temporal variable t is

$$\frac{d^2\mathbb{F}_2}{dt^2} + 2(\lambda\omega)\frac{d\mathbb{F}_2}{dt} + 2\lambda^2\omega^2\mathbb{F}_2 = 0. \tag{35}$$

In terms of the spatial variable, the second wave equation is

$$\nabla^2\mathbb{F}_2 - 2\lambda k\nabla\cdot\mathbb{F}_2 + 2\lambda^2k^2\mathbb{F}_2 = 0. \tag{36}$$

The solution of Eq.(35) is

$$\mathbb{F}_2 = \mathbb{F}_{20}exp(-\lambda\omega t)exp(-i\lambda\omega t). \tag{37}$$

Rewriting Eq.(37) as a function of t, we have

$$T(t) = \mathbb{F}_{20}exp(-\lambda\omega t)exp(-i\lambda\omega t). \tag{38}$$

In the spheroidal coordinates, Eq.(36) of spherically symmetric becomes

$$\frac{d^2\mathbb{F}_2}{dr^2} - 2(\lambda k)\frac{d\mathbb{F}_2}{dr} + 2\lambda^2k^2\mathbb{F}_2 = 0. \tag{39}$$

For Eq.(39) there is the solution

$$\mathbb{F}_2 = \mathbb{F}_{20}exp(\lambda kr)exp(i\lambda kr) \tag{40}$$

or rewritten

$$\mathbb{R}(r) = \mathbb{F}_{20}exp(\lambda kr)exp(i\lambda kr). \tag{41}$$

Evidently, for the free electromagnetic field $\mathbb{F}_2$, both equations, Eq.(35) and Eq.(36), are vibration equations. Equation (35) is one of the function of t and Eq.(36) is of the function of r . Multiplying Eq.(38) by Eq.(41), we arrive at

$$\mathbb{F} = \mathbb{R}(r)T(t) = \mathbb{F}_0 exp(\lambda kr - \lambda\omega t)exp(i\lambda kr - i\lambda\omega t). \quad (42)$$

Equation (42) is identical to Eq.(23) .

Subtracting Eq.(35) from Eq.(36) , we get

$$c^2\nabla^2\mathbb{F}_2 - 2\lambda\omega c\nabla\cdot\mathbb{F}_2 + 2\lambda^2\omega^2\mathbb{F}_2 - (\frac{d^2\mathbb{F}_2}{dt^2} + 2\lambda\omega\frac{d\mathbb{F}_2}{dt} + 2\lambda^2\omega^2\mathbb{F}_2) = 0. \quad (43)$$

Equation (43) needs to be rearranged because there are two variables, t and r , included in the equation. Then Eq.(43) is changed to

$$c^2\nabla^2\mathbb{F}_2 - \frac{\partial^2\mathbb{F}_2}{\partial t^2} - 2\lambda\omega(c\nabla\cdot\mathbb{F}_2 + \frac{\partial\mathbb{F}_2}{\partial t}) + 2\lambda^2\omega^2(\mathbb{F}_2 - \mathbb{F}_2) = 0. \quad (44)$$

Considering the continuity equation

$$\nabla\cdot c\mathbb{F}_2 + \frac{\partial\mathbb{F}_2}{\partial t} = 0 \quad (45)$$

for passive field, Eq.(44) becomes

$$\nabla^2\mathbb{F}_2 - \frac{1}{c^2}\frac{\partial^2\mathbb{F}_2}{\partial t^2} = 0, \quad (46)$$

which is Maxwell wave equation.

References

[1] Gene. H. Golub, Charles F.Ven loan. *MATRIX COMPUTATIONS.* ISBN 0-8018-5414-8. 3rd ed © 1996. Hopkins University press.

[2] X.Z. Zhong. *Electromagnetic Field Equation and Field Wave Equation.* vixra. org: [84] 1101.0057., 2011.

[3] J.D. Jackson. *CLASSICAL ELECTRODYNAMICS*, 3rd ed. John-Wiley & Sons, Inc., 1999.

[4] M.D. Weir, R.L. Finney and F.R. Giordano. *THOMAS' CALCULUS*. 10th ed. Publishing as Pearson Education. Inc., 2004.