The Concept of the Effective Mass Tensor in the General Relativity

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Abstract. It is well-known from the classical mechanics that there exists relation between the metric tensor and the effective mass tensor of the body (EMT). We have introduced the concept of the EMT in the General Relativity and we have found that there exists similar relation between the metric tensor and the EMT for the moving body in a weak gravitational field. We propose the an experimental verification of our considerations.

We compared a few physical features concerning of the space-time curvature with the physical features of the EMT and the result of this comparisons are presented in the form of a table in this paper.

keywords: *general theory of gravity; the effective mass tensor*

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1. Introduction

Typically, mass of the body (the inertial or gravitational) is a scalar. The concept of mass in the General Relativity (GR) is more complex than the concept of mass in the special relativity. In fact, general relativity does not offer a single definition for the term mass, but offers several different definitions which are applicable under different circumstances [1, 4].

It is well-known from the classical mechanics that there exists relation between the metric tensor and the effective mass tensor of the body (EMT). In the Section 3 we will analyze whether exists the relation between the metric tensor and the EMT of the body in a weak gravitational field in the framework of the GR. So we will not compare the concept of the EMT with the different concepts of the quasi-local masses of a collapsing structure in the black hole models [2].

In the Appendix A2 we compare a few physical features concerning of the space-time curvature with the physical features of the EMT. The result of this comparisons are presented in the table.

We propose also the an experimental verification of our considerations.

2. The concept of the EMT

The concept of the EMT of the body plays important role in the contemporary physics. The EMT is well-known in the solid-state physics. When an electron is moving inside a solid material, the force between other atoms will affect its movement and it will not be described by Newton's law [5].

The concept of the EMT is a very attractive because the equations of the motion includes full information about all fields (for example electromagnetic etc.) surrounding the body without their exact analysis. EMT can be isotropic or anisotropic, positive or negative. For the free body his effective mass tensor is equal to *the bare mass m*, where *m* is the scalar.

3. The concept of the EMT in the GR

Carl Gustav Jacob Jacobi was a first who studied the relation between curved geometry and particle dynamics. With the origin of tensor calculus, it became clear that there existed a map between the trajectories of the certain mechanical systems in configuration space and the geodesics of a curved manifold [6]. Nevertheless, despite the intense use of geometrical techniques in the context of dynamics, it seems that the relation between mechanics and geometry was not clearly appreciated in the literature of the GR in the aspect of the concept of the EMT.

It is well-known from the classical mechanics that there exists mathematical relation between the metric tensor g_{ij} and the position-dependent EMT of the body $m^{*}_{ij}(r)$ [7, 8, 9], components *i* and *j* are the Roman indices to denote spatial components (*i*, *j* = 1, 2, 3) and

$$g_{ij} = \frac{m^{*}_{ij}(r)}{m} \tag{1}$$

The components of the g_{ij} are identical with the components of the $m^{*}_{ij}(r)$.

By the analogy to the eq. (1) and for our considerations **we postulate** that in the particular case the EMT $m^{*}_{\mu\nu}$ can **mimics** the metric tensor $g_{\mu\nu}$ also in the GR and

$$\frac{m^{*}_{\mu\nu}}{m} = g_{\mu\nu} \tag{2}$$

where components μ, ν = 0, 1, 2, 3.

Now we would like to know when (under what conditions) such mimics is possible and the eq. (2) is correct.

Let's us consider the planet with mass *m* (e.g. Earth), which is moving on the elliptical orbit in the gravitational field. The source of this field is uncharged, non-rotating or a slowly rotating spherical star (e.g. Sun) with the mass *M* and the radius *R*.

For the further calculations we assume also that:

1. our considerations we will realize in the framework of the GR in the stationary, spherically symmetric and a weak gravitational field,
2. the planet is moving with the small velocity v around the star, where v << *c*, *c* is the speed of light in the vacuum,
3. the mass of the planet **depends on distance *r*** between the star and the planet and *m* = *m(r)*, where the distance *r* >> *R*,
4. the mass of the planet at perihelion is greater than the aphelion,
5. the mass of the planet is not a scalar but the EMT $m^{*}_{\mu\nu}(r)$,
6. the metric $ds^2(g_{\mu\nu}) = g_{\mu\nu}dx^{\mu}dx^{\nu}$,
7. the metric $ds^2(m_{\mu\nu}) = \frac{m_{\mu\nu}}{m}dx^{\mu}dx^{\nu}$,

8. for the distance $r \to \infty$ the $m^*_{\mu\nu}(r) \to m$ and EMT becomes a scalar,
9. for the distance $r \to \infty$ the $g_{\mu\nu}(r) \to \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski tensor,
10. in a weak gravitational field we can decompose of the EMT of the body to the simple form: $m^*_{\mu\nu}(r) = \phi_{\mu\nu} + \gamma^*_{\mu\nu}(r)$, where: $\phi_{\mu\nu} = \mathrm{m} \cdot \eta_{\mu\nu} = \mathrm{diag}(-\mathrm{m},+\mathrm{m},+\mathrm{m},+\mathrm{m})$, $\eta_{\mu\nu}$ is Minkowski tensor, $\gamma^*_{\mu\nu}(r) = m \cdot |h_{\mu\nu}(r)| << 1$ is a small EMT "perturbation" dependent on *r*.
11. this a small EMT "perturbation" should be measured e.g. in the *Solar System*.

Einstein's field equation has form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{3}$$

where: $R_{\mu\nu}$ is *the Ricci curvature tensor*, R is *the Ricci scalar*, G is *Newton's gravitational constant*, and $T_{\mu\nu}$ is *the stress-energy tensor*. The GR explains gravitation as a consequence of the curvature of space-time, while in turn space-time curvature is a consequence of the presence of matter. Space-time curvature affects the movement of matter, which reciprocally determines the geometric properties and evolution of space-time [10, 11].

In the first step we will solve Einstein's field equation for the fixed physical conditions and we get the metric $ds^2(g_{\mu\nu})$. In the second one we will find $m^*(\boldsymbol{r},t)$ for the moving body when $ds^2(g_{\mu\nu}) = ds^2(m_{\mu\nu})$ (see to end of the Section 3 and Appendix A1).

Considering the components of (00) in the equation (3), in the approximation of the stationary and the weak of the gravitational field, we can get the Poisson equation. We will use the alternative and covariant form of the equation (3):

$$R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu}\right) \tag{4}$$

where $T \equiv T^\mu_\mu$ .We are interested the expression in the form:

$$R_{00} = \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2} T g_{00}\right) \tag{5}$$

Our considerations we will realize in the weak gravitational field, which allows us to decompose the metric tensor into the flat Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6}$$

where: $|h_{\mu\nu}| << 1$ is a small perturbation. We will restrict ourselves to coordinates in which $\eta_{\mu\nu}$ takes its canonical form, $\eta_{\mu\nu}$ = diag(-1, +1, +1, +1). In the particular case for the weak and stationary gravitational field, for the small velocity v and for the perfect fluids $p/c^2 << \rho$, where p is the pressure, ρ is the mass density of the fluid element, the stress-energy tensor has simple form:

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{7}$$

what gives $T = \rho c^2$ and eq. (5) has form:

$$R_{00} = \frac{8\pi G}{c^4}\rho\left(u_0 u_0 - \frac{1}{2}c^2 g_{00}\right) \tag{8}$$

But $u_0 \approx c$ and $g_{00} \approx 1$ therefore

$$R_{00} \approx \frac{4\pi G}{c^4}\rho c^2 \tag{9}$$

From the second side of eq. (9) [10, 11] we have

$$R_{00} = -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} \tag{10}$$

where $\delta^{ij}\partial_i\partial_j = \nabla^2$.

In the GR we have to assume that

$$h_{00}(r) = -\frac{2V(r)}{c^2} = \frac{2GM}{c^2 r} \tag{11}$$

because Einstein sought dependencies between the metric tensor and Newton's potential *V(r)* in the non-relativistic limit.

For the our consideration we assume that the relation between the $h_{00}(r)$ and the $\gamma^*_{00}(r)$ has the form (see to Appendix A1)

$$h_{00}(r) = \gamma^*_{00}(r) = \frac{m^*(r)}{m} \tag{12}$$

According to the eq. (12) the metric tensor g_{00} (or the EMT m^*_{00}) have form

$$g_{00}(r) = m^*_{00}(r) = -1 + \gamma^*_{00}(r) = -1 + \frac{m^*(r)}{m} \tag{13}$$

Now we will try to find the h_{ij} components. The eq. (3) is the nonlinear equation. We can use perturbation theory to compute the weak-field, non-relativistic perturbation to the metric (see eq. 6) and we get the wave equation

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = \frac{16\pi G}{c^4}T_{\mu\nu} \tag{14}$$

where: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and the gauge condition $\partial_\mu \bar{h}^\mu_\lambda = 0$ [11]. In the vacuum the eq. (14) has form:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0 \tag{15}$$

A few calculations [11] gives

$$h_{ij}(r) = \gamma^*_{ij}(r) = \frac{m^*(r)}{m}\delta_{ij} \tag{16}$$

For a star (or a planet) in a weak-field limit the metric has form

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) \tag{17}$$

For the EMT the metric (see the eq. 2) in a weak-field limit has form

$$ds^2 = -\left(1 - \frac{m^*(r)}{m}\right)c^2 dt^2 + \left(1 + \frac{m^*(r)}{m}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) \tag{18}$$

Equations (17) and (18) are equivalent if and only if we assume that the function $\frac{m^*(r)}{m}$ which we will call *the mass function* is equal

$$\frac{m^*(r)}{m} = \frac{2GM}{c^2 r} \tag{19}$$

4. Physical experiment

We propose the an experimental verification of our considerations. We make the calculations (see eq. (19)) for the Earth in the perihelion and aphelion and the difference is equal

$$\frac{\delta m^*(r)}{m} = \frac{1}{m}\left[m^*(r)_{perih} - m^*(r)_{aphel}\right] = \frac{2GM}{c^2}\left(\frac{1}{r_{perih}} - \frac{1}{r_{aphel}}\right) \approx 6{,}6 \cdot 10^{-10} \tag{20}$$

It seems that the phenomenon of the EMT should exist in *the Solar System* and although is a very small it may be measured.

Using the Post-Newtonian formalism [12] we can calculate the difference of the metric tensor $g_{00}(r)$ in perihelion and aphelion:

$$\left|\delta g_{00}(r)\right| = \left|\left(g_{00}(r)\right)_{perih} - \left(g_{00}(r)\right)_{aphel}\right| = \frac{\delta m^*(r)}{m} \approx 6{,}6 \cdot 10^{-10} \tag{21}$$

Please note that none of the above quantity (eq. (20) and (21)) never has been measured.

5. Conclusion

In the first step we solved Einstein's field equation for the fixed physical conditions and we get the metrics $ds^2(g_{\mu\nu})$. In the second one we found the mass function $m^*(r)$ (eq. (19), Appendix) for the moving body, when $ds^2(g_{\mu\nu}) = ds^2(m_{\mu\nu})$. Of course in the strong gravitational field the mass function can be more complicated than in eq. (19) [2, 13-15], but we suppose that the physical idea of the EMT which is presented in this paper should be correct.

At the moment, there is no experimental evidence for the concept of EMT in the GR. But if this concept will experimentally confirmed then we will gain the following benefits:

1. Under certain circumstances the EMT $m_{\mu\nu}$ **perfectly mimics** the metric tensor $g_{\mu\nu}$ – eq. (2).
2. Under certain circumstances there is a simple relation between the curved geometry and the particle dynamics.
3. The mass of the body is not a scalar but **under the influence of the gravitational field becomes the tensor (EMT)**.
4. In *the Solar System* the difference between the EMT in the perihelion and aphelion and the mass like a scalar is very small and for the planet Earth is of the order $6{,}6\cdot 10^{-10}$.
5. In a weak gravitational field everywhere, where we can determine the metric tensor we can quickly and simply determine the EMT of the body.
6. All components of the $g_{\mu\nu}$ are identical with the components of the $m_{\mu\nu}$.
7. The only price we have to pay for it is it that we have to find the mass function [2, 13-15].
8. The Lagrangian function and the equations of motion for the body with mass *m* moving in the space-time curvature with the metric tensor $g_{\mu\nu}$ **are the same** like the Lagrangian function and the equations of motion for the body moving with the EMT in the flat Minkowski space-time (see to table). Both descriptions are equivalent.

Is in a strong gravitational field the EMT also perfectly mimics the metric tensor? This will be the subject for the further researches.

Appendix A1. Now we compare the eq. (9) and (10)

$$-\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} \approx \frac{4\pi G}{c^4}\rho c^2 \tag{A1.1}$$

Assume that

$$h_{00}(r) = \gamma^*_{00}(r) = \frac{m^*(r)}{m} \tag{A1.2}$$

because we are looking for dependencies between EMT $\gamma^*_{00}(r)$ and the metric tensor $h_{00}(r)$ in the non-relativistic limit. For the $\delta^{ij}\partial_i\partial_j = \nabla^2$ we have

$$-\frac{1}{2m}\nabla^2 m^*(r) \approx \frac{4\pi G}{c^2}\rho \tag{A1.3}$$

and finally for $r >> R$ we have the mass function $m^*(r)$ for the body

$$\frac{m^*(r)}{m} \approx \frac{2G}{c^2}\int_0^r \frac{\rho dV}{r} = \frac{2GM}{c^2 r} \tag{A1.4}$$

The Schwarzschild metric has form [11]

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{A1.5}$$

The EMT in the Schwarzschild metric has form

$$ds^2 = -\left(1 - \frac{m^*(r)}{m}\right)c^2 dt^2 + \left(1 - \frac{m^*(r)}{m}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{A1.6}$$

The mass function $\frac{m^*(r)}{m}$ for the moving body in the Schwarzschild metric for the distance $r >> R$ is very similar to the eq. (A1.4).

Appendix A2. The space-time curvature vs. the EMT

Let's compare a few physical features concerning of the space-time curvature with the physical features of the EMT. The results of this comparison are presented in table below.

Table. The space-time curvature vs. the EMT.

Space-time curvature	**The EMT**
The metric tensor $g_{\mu\nu}$	*The effective mass tensor* $m^*_{\mu\nu} = m \cdot g_{\mu\nu}$
The mass The mass of the body is a scalar.	*The EMT* The mass of the body is not a scalar. Under the influence of the gravitational field the mass of the body becomes the EMT.
The weak spherical field $g_{\mu\nu}(r) = \eta_{\mu\nu} + h_{\mu\nu}(r)$ The weak of the gravitational field is expressed as ability to decompose the metric tensor into the flat Minkowski metric tensor plus a small perturbation tensor, $\left\|h_{\mu\nu}(r)\right\| << 1$.	*The weak spherical field* $m^*_{\mu\nu}(r) = \phi_{\mu\nu} + \gamma^*_{\mu\nu}(r)$, where: $\phi_{\mu\nu} = \mathrm{m} \cdot \eta_{\mu\nu} = \mathrm{diag}(-\mathrm{m},+\mathrm{m},+\mathrm{m},+\mathrm{m})$, $\eta_{\mu\nu}$ is Minkowski tensor, $\gamma^*_{\mu\nu}(r) = m \cdot \left\|h_{\mu\nu}(r)\right\| << 1$ is a small EMT "perturbation" dependent on r.
Lagrangian: $L = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$	*Lagrangian:* $L = \frac{1}{2} m^*_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$
The equation of motion (m = 1):	*The equation of motion:*

$g_{\alpha\mu}\frac{d^2x^\mu}{dt^2}+\frac{\partial g_{\alpha\mu}}{\partial x^\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$ $-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0$	$m^*_{\alpha\mu}\frac{d^2x^\mu}{dt^2}+\frac{\partial m^*_{\alpha\mu}}{\partial x^\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$ $-\frac{1}{2}\frac{\partial m^*_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0$
Field equation $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^4}T_{\mu\nu}$	*Field equation* Known but only in the weak gravitational field
Physical interpretation All classical tests of the GR are satisfied and they are generated by the curvature of space-time.	*Physical interpretation* All classical tests of the GR are satisfied and they are generated by the EMT.

Appendix A3. For the Schwarzschild metric the EMT in the matrix representation has form

$$m^*_{\mu\nu}(r)=m\begin{bmatrix} -\left(1-\frac{m^*(r)}{m}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{m^*(r)}{m}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{A3.1}$$

When the $m^*(r)=0$ then the $m^*_{\mu\nu}(r)\to m$ and EMT becomes a scalar. When the $m^*(r)>0$ then the time component of the $m^*_{tt}(r)<m$ and the radial component $m^*_{rr}(r)>m$. When we approach to the star the $m^*_{tt}(r)$ is getting smaller, while $m^*_{rr}(r)$ becomes bigger.

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