NUMBER SYSTEMS BASED ON LOGICAL CALCULUS

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Abstract. The reference [1] denote number systems with a logical calculus, but the form of natural numbers are not consistently in these number systems. So we rewrite number systems to correct the defect.

1. Introduction

In [1], the logical calculus for number systems can produce new real numbers with new operations. However, the form of natural numbers are not consistently in these number systems. So we rewrite number systems to correct this defect.

The number systems in this paper can show typical features as follows:

1. The logical calculus provides a uniform frame for arithmetic axioms. Based on the same logical calculus, small number system can import new axioms to produce big number systems.

2. Consistent binary relation are the nature of number systems. The order relation and equivalence relation of each number system is consistent, so all numbers of each number system are layed in fixed positions of a number line.

3. The number systems unify the numbers and operations as a whole, and the only number 1 and various operations compose all numbers. So operation distinguishes different number systems and is the foremost component of number system.

4. The logical calculus for real number system produces new real numbers with new operations. While producing new real numbers in arithmetic, the new operations certainly produce new equations and inequalities in algebra. So the logical calculus not only extends real number system, but also extends equations and inequalities.

The paper is organized as follows. In Section 2, we construct a general logical calculus for number systems. In Section 3, we construct a logical calculus to denote natural number system. In Section 4, we construct a logical calculus to denote integral number system. In Section 5, we construct a logical calculus to denote rational number system. In Section 6, we construct a logical calculus to denote real number system.

2. Logical Calculus

In mathematical logic, logical calculus is a formal system to abstract and analyze the induction and deduction apart from specific meanings. In this section, however, we construct a logical calculus by virtue of formal language and deduce numbers to intuitively and logically denote number systems. The logical calculus not only denotes real numbers, but also allows them to join in algebraical operations.

The introduction of formal language aims to use computer fast execute real number operations. For clarity, we will explain the logical calculus with natural language.

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In [2], the producer "$\rightarrow$" substitutes the right permutations for the left permutations to produce new permutations. In [3], the connectives "$\neg$", "$\wedge$", "$\vee$", "$\Rightarrow$" and "$\Leftrightarrow$" stand for "not", "and", "or", "implies" and "if and only if" respectively. Here, the producer "$\rightarrow$" is considered as a predicate symbol and embedded into logical calculus.

Definition 2.1. $\{\Phi, \Psi\}$ is a logical calculus such that:

$$\Phi\{ \tag{2.1}$$

$$V\{\emptyset, a, b\cdots\}, \tag{2.2}$$

$$C\{\emptyset, 1, +\cdots\}, \tag{2.3}$$

$$P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \leq, \cdots\}, \tag{2.4}$$

$$V \circ C\{\emptyset, a, b\cdots, 1, +\cdots, aa, ab\cdots, a1, a+\cdots, ba, bb\cdots, b1, b+\cdots, \\ aaa, aab\cdots, aa1, aa+\cdots, baa, bab\cdots, ba1, ba+\cdots\}, \tag{2.5}$$

$$C \circ C\{\emptyset, 1, +\cdots, 11, 1+\cdots, 111, 11+\cdots\}, \tag{2.6}$$

$$V \circ C \circ P\{\emptyset, a, b\cdots, 1, +\cdots, \in, \subseteq\cdots, aa, ab\cdots, a1, a+\cdots, a\in, a\subseteq\cdots, \\ ba, bb\cdots, b1, b+\cdots, b\in, b\subseteq\cdots, aaa, aab\cdots, aa1, aa+\cdots, aa\in, aa\subseteq\cdots, \\ baa, bab\cdots, ba1, ba+\cdots, ba\in, ba\subseteq\cdots\}, \tag{2.7}$$

$$(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots), \tag{2.8}$$

$$(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots), \tag{2.9}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots \vee (\hat{a} \equiv 1)\cdots \vee (\hat{a} \equiv aa) \\ \vee(\hat{a} \equiv ab)\cdots \vee (\hat{a} \equiv a1)\cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab)\cdots \vee (\hat{a} \equiv aa1)\cdots), \tag{2.10}$$

$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+)\cdots \\ \vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+)\cdots), \tag{2.11}$$

$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots \vee (\hat{a} \equiv\in)\cdots \vee (\hat{a} \equiv aa) \\ \vee(\hat{a} \equiv ab)\cdots \vee (\hat{a} \equiv a\in)\cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab)\cdots \vee (\hat{a} \equiv aa\in)\cdots), \tag{2.12}$$

$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \\ \wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C)) \\ \cdots \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P))\cdots, \tag{2.13}$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee(\bar{a} \subseteq \bar{g}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \tag{2.14}$$

$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))) \cdots$

$\}$,

(2.15) $\Psi\{$

(2.16) $(\bar{a} \subseteq \bar{b}) \Leftrightarrow (\bar{b} = \bar{c}\bar{a}\bar{d})$,

(2.17) $(\bar{a} \to \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \Rightarrow (\bar{a} \to \bar{b}\bar{e}\bar{d})$,

(2.18) $(\bar{\bar{a}} \to \bar{\bar{b}}|\bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{b}}) \wedge (\bar{\bar{a}} \to \bar{\bar{c}}))$,

(2.19) $(\bar{\bar{a}}|\bar{\bar{b}} \to \bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{c}}) \wedge (\bar{\bar{b}} \to \bar{\bar{c}}))$,

(2.20) $(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{b} < \bar{a})$,

(2.21) $(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{a} = \bar{b})$,

(2.22) $(\bar{a} < \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow (\bar{a} < \bar{c})$,

(2.23) $(\bar{a} < \bar{b}) \wedge (\bar{a} \in (C \circ C)) \wedge (\bar{b} \in (C \circ C)) \Rightarrow (\bar{a} \wedge \bar{b})$,

(2.24) $(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d})$,

(2.25) $(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} = \bar{e}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d})$,

(2.26) $(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d})$,

(2.27) $(\bar{a} < \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \to \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} < \bar{b}\bar{f}\bar{d}\bar{f}\bar{e})$,

(2.28) $(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} \to \bar{e}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}\}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d})$,

(2.29) $(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} < \bar{d}\bar{f}\bar{e})$,

(2.30) $(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} < \bar{d}\bar{g}\bar{e}\bar{g}\bar{f})$,

(2.31) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c}\bar{f}\bar{d} < \bar{e})$,

(2.32) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c}\bar{g}\bar{d} < \bar{e}\bar{g}\bar{f})$,

(2.33) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \to \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} < \bar{e}\bar{h}\bar{f}\bar{h}\bar{g})$,

(2.34) $\bar{a} = \bar{a}$,

(2.35) $(\bar{a} = \bar{b}) \Rightarrow (\bar{b} = \bar{a})$,

(2.36) $(\bar{a} = \bar{b}) \Rightarrow \neg(\bar{a} < \bar{b})$,

(2.37) $(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d})$,

(2.38) $(\bar{a}\bar{b}\bar{c}) \wedge (\bar{b} = \bar{d}) \Rightarrow (\bar{a}\bar{b}\bar{c} = \bar{a}\bar{d}\bar{c})$,

(2.39) $(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d})$,

(2.40) $(\bar{a} = \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \to \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} = \bar{b}\bar{f}\bar{d}\bar{f}\bar{e})$,

(2.41) $(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} = \bar{d}\bar{f}\bar{e})$,

(2.42) $(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} = \bar{d}\bar{g}\bar{e}\bar{g}\bar{f})$,

(2.43) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} = \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \to \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} = \bar{e}\bar{h}\bar{f}\bar{h}\bar{g})$,

(2.44) $(\bar{a} \leq \bar{b}) \Leftrightarrow ((\bar{a} < \bar{b}) \vee (\bar{a} = \bar{b}))$,

(2.45) $(\bar{a} \leq \bar{b}) \wedge (\bar{b} \leq \bar{c}) \Rightarrow (\bar{a} \leq \bar{c})$

$\}$.

First, we will explain the primitive symbols of the logical calculus $\{\Phi, \Psi\}$ with natural language.

The symbols "{", "}", ",", "(", ")" are punctuation. The symbol "∅" indicates emptiness. The symbol "⋯" indicates an omission.

(2.1) denotes Φ as a set of notations and particular axioms between { and }. Different logical calculus correspond to different notations and particular axioms.

(2.2) denotes V as a set of variables between { and }.

(2.3) denotes C as a set of constants between { and }.

(2.4) denotes P as a set of predicate symbols between { and }.

(2.5) denotes $V \circ C$ as a set of concatenations between V and C.

(2.6) denotes $C \circ C$ as a set of concatenations between C and C.

(2.7) denotes $V \circ C \circ P$ as a set of concatenations among V, C and P.

(2.8) $\sim$ (2.12) define a set of axioms on the binary predicate symbol $\in$.

(2.13) defines an axiom on new variables ranging over $V \circ C$.

(2.14) defines an axiom on the binary predicate symbol $\subseteq$.

(2.15) denotes Ψ as a set of general axioms between { and }. Different logical calculus correspond to the same general axioms.

(2.16) defines an axiom on the binary predicate symbol $\subseteq$.

(2.17) defines an axiom on the binary predicate symbol $\rightarrow$.

(2.18) $\sim$ (2.19) define a set of axioms on the binary predicate symbol $|$.

(2.20) $\sim$ (2.33) define a set of axioms on the binary predicate symbol $<$.

(2.34) $\sim$ (2.43) define a set of axioms on the binary predicate symbol $=$.

(2.44) $\sim$ (2.45) define a set of axioms on the binary predicate symbol $\leq$.

Firstly, we define number with such a logical calculus.

Definition 2.2. In a logical calculus $\{\Phi, \Psi\}$, if $\bar{a} \equiv true$, then $\bar{a}$ is a number.

Then, we will prove that the logical calculus $\{\Phi, \Psi\}$ can deduce common number systems.

3. Natural Number System

Theorem 3.1.

If $\Phi\{$

(3.1) $V\{\emptyset, a, b\},$

(3.2) $C\{\emptyset, 1, +, [,]\},$

(3.3) $P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \leq\},$

(3.4) $V \circ C\{\emptyset, a, b \cdots, 1, + \cdots, aa, ab \cdots, a1, a+ \cdots, ba, bb \cdots, b1, b+ \cdots,$
$aaa, aab \cdots, aa1, aa+ \cdots, baa, bab \cdots, ba1, ba+ \cdots\},$

(3.5) $C \circ C\{\emptyset, 1, + \cdots, 11, 1+ \cdots, 111, 11+ \cdots\},$

(3.6) $V \circ C \circ P\{\emptyset, a, b \cdots, 1, + \cdots, \in, \subseteq \cdots, aa, ab \cdots, a1, a+ \cdots, a\in, a\subseteq \cdots,$
$ba, bb \cdots, b1, b+ \cdots, b\in, b\subseteq \cdots, aaa, aab \cdots, aa1, aa+ \cdots, aa\in, aa\subseteq \cdots,$
$baa, bab \cdots, ba1, ba+ \cdots, ba\in, ba\subseteq \cdots\},$

(3.7) $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots),$

(3.8) $(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots),$

(3.9) $(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa)$

$\vee(\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots)$,

(3.10) $(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots$
$\vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots)$,

(3.11) $(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa)$
$\vee(\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots)$,

(3.12) $(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C))$
$\wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C))$
$\wedge(\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P))$,

(3.13) $((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j})))$,

(3.14) $a \to 1|[a + a]$,

(3.15) $a < [1 + a]$,

(3.16) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}])$,

(3.17) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}])$
},

then $N\{\Phi, \Psi\}$ *denotes natural number system.*

Proof.

$(A1)$	$(a \to 1|[a + a]) \Rightarrow (a \to 1)$	$by(3.14), (2.18)$
$(A2)$	$\Rightarrow (a \to [a + a])$	$by(2.18)$
$(A3)$	$\Rightarrow (a \to [1 + a])$	$by(A2), (A1), (2.17)$
$(A4)$	$\Rightarrow (a \to [1 + 1])$	$by(A3), (A1), (2.17)$
$(A5)$	$(a < [1 + a]) \Rightarrow (1 < [1 + 1])$	$by(3.15), (A1), (2.29)$
$(A6)$	$\Rightarrow 1$	$by(2.23)$
$(A7)$	$\Rightarrow [1 + 1]$	$by(A5), (2.23)$
$(A8)$	$\Rightarrow ([1 + 1] < [1 + [1 + 1]])$	$by(3.15), (A4), (2.29)$
$(A9)$	$\Rightarrow [1 + 1]$	$by(2.23)$

$$\begin{array}{lll}
(A10) & \Rightarrow [1+[1+1]] & by (A8), (2.23) \\
(A11) & \Rightarrow (a \rightarrow [1+[a+a]]) & by (A3), (A2), (2.17) \\
(A12) & \Rightarrow (a \rightarrow [1+[1+a]]) & by (A11), (A1), (2.17) \\
(A13) & \Rightarrow (a \rightarrow [1+[1+1]]) & by (A12), (A1), (2.17) \\
(A14) & \Rightarrow ([1+[1+1]] < [1+[1+[1+1]]]) & by (3.15), (A13), (2.29) \\
(A15) & \Rightarrow [1+[1+[1+1]]] & by (2.23) \\
\vdots & \vdots & \vdots
\end{array}$$

Then according to Definition 2.2, we can deduce from $N\{\Phi, \Psi\}$ the numbers as follows:

$$\{1, [1+1], [1+[1+1]], [1+[1+[1+1]]] \cdots\}$$

$$\begin{array}{lll}
(B1) & [1+1] = [1+1] & by (A6), (3.16) \\
(B2) & [1+[1+1]] = [[1+1]+1] & by (A6), (A7), (3.16) \\
(B3) & [1+[1+[1+1]]] = [[1+[1+1]]+1] & by (A6), (A10), (3.16) \\
(B4) & [1+[1+[1+1]]] = [[1+1]+[1+1]] & by (A6), (A7), (3.17) \\
\vdots & \vdots & \vdots
\end{array}$$

Then we can deduce from $N\{\Phi, \Psi\}$ the equalities as follows:

$$\begin{array}{rcl}
[1+1] & = & [1+1], \\
[1+[1+1]] & = & [[1+1]+1], \\
[1+[1+[1+1]]] & = & [[1+[1+1]]+1], \\
[1+[1+[1+1]]] & = & [[1+1]+[1+1]], \\
\vdots & \vdots & \ddots
\end{array}$$

The deduced numbers correspond to natural numbers as follows:

$$\begin{array}{rcl}
1 & \equiv & 1, \\
[1+1] & \equiv & 2, \\
[1+[1+1]] & \equiv & 3, \\
[1+[1+[1+1]]] & \equiv & 4, \\
\vdots & \vdots & \ddots
\end{array}$$

The equalities on deduced numbers correspond to the addition in natural number system. So the claim follows. $\square$

4. Integral Number System

Theorem 4.1.

If $\Phi\{$

(4.1) $V\{\emptyset, a, b, c\},$

(4.2) $C\{\emptyset, 1, +, [,], -\},$

(4.3) $P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \leq\},$

(4.4) $V \circ C\{\emptyset, a, b \cdots, 1, + \cdots, aa, ab \cdots, a1, a + \cdots, ba, bb \cdots, b1, b + \cdots,$
$aaa, aab \cdots, aa1, aa + \cdots, baa, bab \cdots, ba1, ba + \cdots\},$

(4.5) $C \circ C\{\emptyset, 1, + \cdots, 11, 1 + \cdots, 111, 11 + \cdots\},$

(4.6) $V \circ C \circ P\{\emptyset, a, b \cdots, 1, + \cdots, \in, \subseteq \cdots, aa, ab \cdots, a1, a + \cdots, a \in, a \subseteq \cdots,$
$ba, bb \cdots, b1, b + \cdots, b \in, b \subseteq \cdots, aaa, aab \cdots, aa1, aa + \cdots, aa \in, aa \subseteq \cdots,$
$baa, bab \cdots, ba1, ba + \cdots, ba \in, ba \subseteq \cdots\},$

(4.7) $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots),$

(4.8) $(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots),$

(4.9) $(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa)$
$\vee(\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots),$

(4.10) $(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots$
$\vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots),$

(4.11) $(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa)$
$\vee(\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots),$

(4.12) $(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C))$
$\wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C))$
$\wedge(\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)),$

(4.13) $((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),$

(4.14) $a \rightarrow 1|[aba],$

(4.15) $b|c \to +|-$,
(4.16) $a < [1+a]$,
(4.17) $\bar{a} \Rightarrow ([\bar{a}-\bar{a}] = [1-1])$,
(4.18) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a}+\bar{b}] = [\bar{b}+\bar{a}])$,
(4.19) $\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a}-\bar{b}]+\bar{b}] = \bar{a})$,
(4.20) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([[\bar{a}-\bar{b}]+\bar{c}] = [\bar{a}+[\bar{c}-\bar{b}]])$,
(4.21) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}+[\bar{b}+\bar{c}]] = [[\bar{a}+\bar{b}]+\bar{c}])$,
(4.22) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}+[\bar{b}-\bar{c}]] = [[\bar{a}+\bar{b}]-\bar{c}])$,
(4.23) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}-[\bar{b}+\bar{c}]] = [[\bar{a}-\bar{b}]-\bar{c}])$,
(4.24) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}-[\bar{b}-\bar{c}]] = [[\bar{a}-\bar{b}]+\bar{c}])$
},

then $Z\{\Phi,\Psi\}$ denotes integral number system.

Proof.

$$
\begin{array}{lcl}
(A1) & (a \to 1|[aba]) \Rightarrow (a \to 1) & by(4.14),(2.18) \\
(A2) & \Rightarrow (a \to [aba]) & by(4.14),(2.18) \\
(A3) & \Rightarrow (a \to [1ba]) & by(A2),(A1),(2.17) \\
(A4) & \Rightarrow (a \to [1b1]) & by(A3),(A1),(2.17) \\
(A5) & (b|c \to +|-) \Rightarrow (b|c \to -) & by(4.15),(2.18) \\
(A6) & \Rightarrow (b \to -) & by(A5),(2.19) \\
(A7) & \Rightarrow (a \to [1-1]) & by(A4),(A6),(2.17) \\
(A8) & (a < [1+a]) \Rightarrow (1 < [1+1]) & by(4.16),(A1),(2.29) \\
(A9) & \Rightarrow 1 & by(2.23) \\
(A10) & \Rightarrow [1+1] & by(A8),(2.23) \\
(A11) & (a < [1+a]) \Rightarrow ([1-1] < [1+[1-1]]) & by(4.16),(A7),(2.29) \\
(A12) & \Rightarrow ([1-1] < [[1-1]+1]) & by(A11),(4.18),(2.24) \\
(A13) & \Rightarrow ([1-1] < 1) & by(A12),(4.19),(2.24) \\
(A14) & \Rightarrow [1-1] & by(2.23) \\
\vdots & \vdots & \vdots
\end{array}
$$

Then according to Definition 2.2, we can deduce from $Z\{\Phi,\Psi\}$ the numbers as follows:

$$\{1,[1+1],[1-1],[1+[1+1]],[1-[1+1]]\cdots\}$$

$$
\begin{array}{lcl}
(B1) & [1+[1+1]] = [[1+1]+1] & by(A9),(A10),(4.18) \\
(B2) & [1+[1-1]] = [[1-1]+1] & by(A9),(A14),(4.18) \\
(B3) & [1+[1-1]] = [[1+1]-1] & by(A9),(4.22) \\
(B4) & [[1+1]+[1-1]] = [[1-1]+[1+1]] & by(A10),(A14),(4.18) \\
(B5) & [[1+1]+[1-1]] = [1+[1+[1-1]]] & by(A10),(A14),(4.21)
\end{array}
$$

$$(B6)\quad [[1+1]+[1-1]] = [[[1+1]+1]-1] \quad by(A9),(A10),(4.22)$$
$$\vdots \qquad \vdots \qquad \vdots$$

Then we can deduce from $Z\{\Phi, \Psi\}$ the equalities as follows:

$$\begin{aligned}
[1+[1+1]] &= [[1+1]+1],\\
[1+[1-1]] &= [[1-1]+1],\\
[1+[1-1]] &= [[1+1]-1],\\
[[1+1]+[1-1]] &= [[1-1]+[1+1]],\\
[[1+1]+[1-1]] &= [1+[1+[1-1]]],\\
[[1+1]+[1-1]] &= [[[1+1]+1]-1],\\
\vdots \quad &\vdots \quad \vdots.
\end{aligned}$$

The deduced numbers correspond to integral numbers as follows:

$$\begin{aligned}
\vdots \quad &\vdots \quad \vdots,\\
[1-[1+[1+1]]] &\equiv -2,\\
[1-[1+1]] &\equiv -1,\\
[1-1] &\equiv 0,\\
1 &\equiv 1,\\
[1+1] &\equiv 2,\\
[1+[1+1]] &\equiv 3,\\
[1+[1+[1+1]]] &\equiv 4,\\
\vdots \quad &\vdots \quad \vdots.
\end{aligned}$$

The equalities on deduced numbers correspond to the addition and subtraction in integral number system. So the claim follows. □

5. Rational Number System

Theorem 5.1.

If $\Phi\{$

(5.1) $V\{\emptyset, a, b, c, d\}$,

(5.2) $C\{\emptyset, 1, +, [,], -\}$,

(5.3) $P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \leq\}$,

(5.4) $V \circ C\{\emptyset, a, b\cdots, 1, +\cdots, aa, ab\cdots, a1, a+\cdots, ba, bb\cdots, b1, b+\cdots,$
$aaa, aab\cdots, aa1, aa+\cdots, baa, bab\cdots, ba1, ba+\cdots\}$,

(5.5) $C \circ C\{\emptyset, 1, +\cdots, 11, 1+\cdots, 111, 11+\cdots\}$,

(5.6) $V \circ C \circ P\{\emptyset, a, b\cdots, 1, +\cdots, \in, \subseteq\cdots, aa, ab\cdots, a1, a+\cdots, a\in, a\subseteq\cdots,$
$ba, bb\cdots, b1, b+\cdots, b\in, b\subseteq\cdots, aaa, aab\cdots, aa1, aa+\cdots, aa\in, aa\subseteq\cdots,$
$baa, bab\cdots, ba1, ba+\cdots, ba\in, ba\subseteq\cdots\}$,

(5.7) $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots)$,

$$(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots), \tag{5.8}$$

$$(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa) \\ \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots), \tag{5.9}$$

$$(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots \\ \vee (\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots), \tag{5.10}$$

$$(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa) \\ \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots), \tag{5.11}$$

$$(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \\ \wedge (\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge \\ (\bar{j} \in (V \circ C)) \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge (\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)), \tag{5.12}$$

$$((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee (\bar{a} \subseteq \bar{g}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \\ \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge \\ ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee \\ (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))), \tag{5.13}$$

$$a \to 1|[aba], \tag{5.14}$$

$$b \to +|-, \tag{5.15}$$

$$c|d \to b| + +| - -, \tag{5.16}$$

$$a < [1 + a], \tag{5.17}$$

$$(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow ([\bar{a} + \bar{c}] < [\bar{b} + \bar{c}]), \tag{5.18}$$

$$(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow ([\bar{c} - \bar{b}] < [\bar{c} - \bar{a}]), \tag{5.19}$$

$$([1 - 1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{a} - -\bar{c}] < [\bar{b} - -\bar{c}]), \tag{5.20}$$

$$\bar{a} \Rightarrow ([\bar{a} - \bar{a}] = [1 - 1]), \tag{5.21}$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}]), \tag{5.22}$$

$$\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{b}] = \bar{a}), \tag{5.23}$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{c}] = [[\bar{a} + \bar{c}] - \bar{b}]), \tag{5.24}$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}]), \tag{5.25}$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} - \bar{c}]] = [[\bar{a} + \bar{b}] - \bar{c}]), \tag{5.26}$$

$$\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}]), \tag{5.27}$$

(5.28) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}])$,
(5.29) $\bar{a} \Rightarrow ([\bar{a} + +1] = \bar{a}) \wedge ([\bar{a} - -1] = \bar{a})$,
(5.30) $\neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} - -\bar{a}] = 1)$,
(5.31) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + +\bar{b}] = [\bar{b} + +\bar{a}])$,
(5.32) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + +\bar{c}]] = [[\bar{a} + +\bar{b}] + +\bar{c}])$,
(5.33) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + \bar{c}]] = [[\bar{a} + +\bar{b}] + [\bar{a} + +\bar{c}]])$,
(5.34) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} - \bar{c}]] = [[\bar{a} + +\bar{b}] - [\bar{a} + +\bar{c}]])$,
(5.35) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([[\bar{a} - -\bar{b}] + +\bar{b}] = \bar{a})$,
(5.36) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow ((([[\bar{a} - -\bar{b}] + +\bar{c}] = [[\bar{a} + +\bar{c}] - -\bar{b}]) \wedge$
$([[\bar{a} + \bar{c}] - -\bar{b}] = [[\bar{a} - -\bar{b}] + [\bar{c} - -\bar{b}]]) \wedge ([[\bar{a} - \bar{c}] - -\bar{b}] =$
$[[\bar{a} - -\bar{b}] - [\bar{c} - -\bar{b}]]))$,
(5.37) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([\bar{a} + +[\bar{b} - -\bar{c}]] = [[\bar{a} + +\bar{b}] - -\bar{c}]) \wedge$
$([\bar{a} - -[\bar{b} + +\bar{c}]] = [[\bar{a} - -\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} - -\bar{c}]] = [[\bar{a} - -\bar{b}] + +\bar{c}]))$
},

then $Q\{\Phi, \Psi\}$ denotes rational number system.

Proof.

$(A1)$	$(a \to 1 \vert [aba]) \Rightarrow (a \to 1)$	$by(5.14), (2.18)$
$(A2)$	$\Rightarrow (a \to [aba])$	$by(5.14), (2.18)$
$(A3)$	$\Rightarrow (a \to [1ba])$	$by(A2), (A1), (2.17)$
$(A4)$	$\Rightarrow (a \to [1b1])$	$by(A3), (A1), (2.17)$
$(A5)$	$(b \to + \vert -) \Rightarrow (b \to -)$	$by(5.15), (2.18)$
$(A6)$	$\Rightarrow (a \to [1 - 1])$	$by(A4), (A5), (2.17)$
$(A7)$	$(a < [1 + a]) \Rightarrow (1 < [1 + 1])$	$by(5.16), (A1), (2.29)$
$(A8)$	$\Rightarrow 1$	$by(2.23)$
$(A9)$	$\Rightarrow [1 + 1]$	$by(A7), (2.23)$
$(A10)$	$(a < [1 + a]) \Rightarrow ([1 - 1] < [1 + [1 - 1]])$	$by(5.16), (A6), (2.29)$
$(A11)$	$\Rightarrow ([1 - 1] < [[1 - 1] + 1])$	$by(A10), (5.18), (2.24)$
$(A12)$	$\Rightarrow ([1 - 1] < 1)$	$by(A11), (5.19), (2.24)$
$(A13)$	$\Rightarrow [1 - 1]$	$by(2.23)$
$(A14)$	$(b \to + \vert -) \Rightarrow (b \to +)$	$by(5.15), (2.18)$
$(A15)$	$\Rightarrow (a \to [1 + 1])$	$by(A4), (A14), (2.17)$
$(A16)$	$(a < [1 + a]) \Rightarrow ([1 + 1] < [1 + [1 + 1]])$	$by(5.16), (A15), (2.29)$
$(A17)$	$\Rightarrow ([1 - 1] < [1 + 1])$	$by(A12), (A7), (2.22)$
$(A18)$	$\Rightarrow ([1 - 1] < [1 + [1 + 1]])$	$by(A17), (A16), (2.22)$
$(A19)$	$\Rightarrow ([1 - -[1 + [1 + 1]]] < [[1 + 1] - -[1 + [1 + 1]]])$	$by(A12), (A7),$ $(A18), (5.20)$

$$
\begin{array}{lcl}
(A20) & \Rightarrow [1--[1+[1+1]]] & by(2.23)\\
(A21) & \Rightarrow [[1+1]--[1+[1+1]]] & by(A19),(2.23)\\
(A22) & \Rightarrow \neg([1+1]=[1-1]) & by(A17),(2.21)\\
\vdots & \vdots & \vdots
\end{array}
$$

Then according to Definition 2.2, we can deduce from $Q\{\Phi,\Psi\}$ the numbers as follows:

$$\{1,[1+1],[1-1],[1+[1+1]],[1-[1+1]],[1--[1+[1+1]]]\cdots\}$$

$$
\begin{array}{lcl}
(B1) & [1+[1+1]]=[[1+1]+1] & by(A8),(A9),(5.22)\\
(B2) & [1+[1-1]]=[[1-1]+1] & by(A8),(A13),(5.22)\\
(B3) & [1+[1-1]]=[[1+1]-1] & by(A8),(5.26)\\
(B4) & [[1+1]+[1-1]]=[[1-1]+[1+1]] & by(A9),(A13),(5.22)\\
(B5) & [[1+1]+[1-1]]=[1+[1+[1-1]]] & by(A9),(A13),(5.25)\\
(B6) & [[1+1]+[1-1]]=[[[1+1]+1]-1] & by(A8),(A9),(5.26)\\
(B7) & [1+[1--[1+[1+1]]]]=[[1--[1+[1+1]]]+1] & by(A8),(A20),(5.22)\\
(B8) & [[1--[1+1]]++[1+1]]=1 & by(A8),(A22),(5.35)\\
\vdots & \vdots & \vdots
\end{array}
$$

Then we can deduce from $Q\{\Phi,\Psi\}$ the equalities as follows:

$$
\begin{array}{rcl}
[1+[1+1]] & = & [[1+1]+1],\\
[1+[1-1]] & = & [[1-1]+1],\\
[1+[1-1]] & = & [[1+1]-1],\\
[[1+1]+[1-1]] & = & [[1-1]+[1+1]],\\
[[1+1]+[1-1]] & = & [1+[1+[1-1]]],\\
[[1+1]+[1-1]] & = & [[[1+1]+1]-1],\\
[1+[1--[1+[1+1]]]] & = & [[1--[1+[1+1]]]+1],\\
[[1--[1+1]]++[1+1]] & = & 1,\\
\vdots & \vdots & \vdots.
\end{array}
$$

The deduced numbers correspond to rational numbers as follows:

$$
\begin{array}{rcl}
\vdots & \vdots & \vdots,\\
[1-[1+1]] & \equiv & -1,\\
\vdots & \vdots & \vdots,\\
[[1-1]-[1--[1+1]]] & \equiv & -\dfrac{1}{2},\\
\vdots & \vdots & \vdots,\\
[1-1] & \equiv & 0,\\
\vdots & \vdots & \vdots,
\end{array}
$$

$$\begin{aligned}
[1--[1+1]] &\equiv \frac{1}{2}, \\
\vdots\ &\vdots\ \vdots, \\
1 &\equiv 1, \\
\vdots\ &\vdots\ \vdots, \\
[1+[1--[1+1]]] &\equiv \frac{3}{2}, \\
\vdots\ &\vdots\ \vdots, \\
[1+1] &\equiv 2, \\
\vdots\ &\vdots\ \vdots.
\end{aligned}$$

The equalities on deduced numbers correspond to the addition, subtraction, multiplication, division in rational number system. So the claim follows. □

6. Real Number System

Definition 6.1. Real number system is a logical calculus $R\{\Phi, \Psi\}$ such that:

$\Phi\{$

(6.1) $V\{\emptyset, a, b, c, d, e, f, g, h, i, j, k, l\}$,

(6.2) $C\{\emptyset, 1, +, [,], -, /, \top, \bot, \_\}$,

(6.3) $P\{\emptyset, \in, \subseteq, \to, |, =, <, \leq, \|\}$,

(6.4) $V \circ C\{\emptyset, a, b\cdots, 1, +\cdots, aa, ab\cdots, a1, a+\cdots, ba, bb\cdots, b1, b+\cdots,$
$aaa, aab\cdots, aa1, aa+\cdots, baa, bab\cdots, ba1, ba+\cdots\}$,

(6.5) $C \circ C\{\emptyset, 1, +\cdots, 11, 1+\cdots, 111, 11+\cdots\}$,

(6.6) $V \circ C \circ P\{\emptyset, a, b\cdots, 1, +\cdots, \in, \subseteq\cdots, aa, ab\cdots, a1, a+\cdots, a\in, a\subseteq\cdots,$
$ba, bb\cdots, b1, b+\cdots, b\in, b\subseteq\cdots, aaa, aab\cdots, aa1, aa+\cdots, aa\in, aa\subseteq\cdots,$
$baa, bab\cdots, ba1, ba+\cdots, ba\in, ba\subseteq\cdots\}$,

(6.7) $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots)$,

(6.8) $(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots)$,

(6.9) $(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots \vee (\hat{a} \equiv 1)\cdots \vee (\hat{a} \equiv aa)$
$\vee(\hat{a} \equiv ab)\cdots \vee (\hat{a} \equiv a1)\cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab)\cdots \vee (\hat{a} \equiv aa1)\cdots)$,

(6.10) $(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+)\cdots$
$\vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+)\cdots)$,

(6.11) $(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b)\cdots \vee (\hat{a} \equiv \in)\cdots \vee (\hat{a} \equiv aa)$
$\vee(\hat{a} \equiv ab)\cdots \vee (\hat{a} \equiv a\in)\cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab)\cdots \vee (\hat{a} \equiv aa\in)\cdots)$,

(6.12) $(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C))$
$\wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge$
$(\bar{j} \in (V \circ C)) \wedge (\bar{k} \in (V \circ C)) \wedge (\bar{l} \in (V \circ C)) \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge$
$(\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P))$,

(6.13) $((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$
$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}, \bar{k}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$
$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}) \vee (\bar{a} \subseteq \bar{k}))),$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}, \bar{k}, \bar{l}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$
$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}) \vee (\bar{a} \subseteq \bar{k}) \vee (\bar{a} \subseteq \bar{l}))),$

(6.14) $(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{f}\bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \wedge \neg(\bar{f} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{g}\}) \wedge$
$((\bar{b} \to \bar{h}) \| (\bar{f} \to \bar{i})) \Rightarrow (\bar{a}\bar{h}\bar{c} = \bar{d}\bar{h}\bar{e}\bar{i}\bar{g}),$

(6.15) $(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e} = \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \wedge \neg(\bar{d} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{f}\}) \wedge ((\bar{b} \to \bar{g}) \| (\bar{d} \to \bar{h}))$
$\Rightarrow (\bar{a}\bar{g}\bar{c}\bar{h}\bar{e} = \bar{f}),$

(6.16) $a \to 1|[aba],$

(6.17) $b \to +|-,$

(6.18) $c|d \to e|f|g,$

(6.19) $e \to +|+e,$

(6.20) $f \to -|-f,$

(6.21) $g \to /|/g,$

(6.22) $(h \to +)\|(i \to -),$

(6.23) $(h \to +h)\|(i \to -i),$

(6.24) $(i \to -)\|(h \to +),$

(6.25) $(i \to -i)\|(h \to +h),$

(6.26) $(h \to +)\|(j \to /),$

(6.27) $(h \to +h)\|(j \to /j),$

(6.28) $k \to [1+1]|[1+k],$

(6.29) $l \to 1|[1+l],$

(6.30) $a < [1+a],$

(6.31) $(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a}+\bar{c}] < [\bar{b}+\bar{c}]) \wedge ([\bar{a}-\bar{c}] < [\bar{b}-\bar{c}]) \wedge ([\bar{c}-\bar{b}] < [\bar{c}-\bar{a}])),$

(6.32) $([1-1] \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ([\bar{a}--\bar{c}] < [\bar{b}--\bar{c}]),$

(6.33) $([1-1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ([\bar{c} - -\bar{b}] < [\bar{c} - -\bar{a}])$,

(6.34) $(1 < \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow (1 < [\bar{a} - -f\bar{b}])$,

(6.35) $(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([1-1] < [\bar{a}/g\bar{b}])$,

(6.36) $(1 < \bar{a}) \wedge (\bar{a} < \bar{b}) \Rightarrow (1 < [\bar{b}/g\bar{a}])$,

(6.37) $(1 \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}])$,

(6.38) $(1 \leq \bar{a}) \wedge (1 \leq \bar{b}) \wedge ([1-1] < \bar{c}) \wedge ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]) \Rightarrow (\bar{a} < \bar{b})$,

(6.39) $(1 < \bar{a}) \wedge ([1-1] \leq \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}])$,

(6.40) $(1 < \bar{a}) \wedge ([1-1] \leq \bar{b}) \wedge ([1-1] \leq \bar{c}) \wedge ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]) \Rightarrow (\bar{b} < \bar{c})$,

(6.41) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} - \bar{b}] = [\bar{a}/\bar{b}])$,

(6.42) $\bar{a} \wedge \neg(\bar{b} = [1-1]) \Rightarrow ([\bar{a} - -\bar{b}] = [\bar{a}//\bar{b}])$,

(6.43) $\bar{a} \Rightarrow ([\bar{a} - \bar{a}] = [1-1])$,

(6.44) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}])$,

(6.45) $\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{b}] = \bar{a})$,

(6.46) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{c}] = [[\bar{a} + \bar{c}] - \bar{b}])$,

(6.47) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}])$,

(6.48) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} - \bar{c}]] = [[\bar{a} + \bar{b}] - \bar{c}])$,

(6.49) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}])$,

(6.50) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}])$,

(6.51) $\bar{a} \Rightarrow ([\bar{a} + +1] = \bar{a}) \wedge ([\bar{a} - -1] = \bar{a})$,

(6.52) $\neg(\bar{a} = [1-1]) \Rightarrow ([\bar{a} - -\bar{a}] = 1)$,

(6.53) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + +\bar{b}] = [\bar{b} + +\bar{a}])$,

(6.54) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + +\bar{c}]] = [[\bar{a} + +\bar{b}] + +\bar{c}])$,

(6.55) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + \bar{c}]] = [[\bar{a} + +\bar{b}] + [\bar{a} + +\bar{c}]])$,

(6.56) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} - \bar{c}]] = [[\bar{a} + +\bar{b}] - [\bar{a} + +\bar{c}]])$,

(6.57) $\bar{a} \wedge \neg(\bar{b} = [1-1]) \Rightarrow ([[\bar{a} - -\bar{b}] + +\bar{b}] = \bar{a})$,

(6.58) $\bar{a} \wedge \neg(\bar{b} = [1-1]) \wedge \bar{c} \Rightarrow (([[\bar{a} - -\bar{b}] + +\bar{c}] = [[\bar{a} + +\bar{c}] - -\bar{b}]) \wedge$
$([[\bar{a} + \bar{c}] - -\bar{b}] = [[\bar{a} - -\bar{b}] + [\bar{c} - -\bar{b}]]) \wedge ([[\bar{a} - \bar{c}] - -\bar{b}] =$
$[[\bar{a} - -\bar{b}] - [\bar{c} - -\bar{b}]]))$,

(6.59) $\bar{a} \wedge \neg(\bar{b} = [1-1]) \wedge \neg(\bar{c} = [1-1]) \Rightarrow (([\bar{a} + +[\bar{b} - -\bar{c}]] = [[\bar{a} + +\bar{b}] - -\bar{c}]) \wedge$
$([\bar{a} - -[\bar{b} + +\bar{c}]] = [[\bar{a} - -\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} - -\bar{c}]] = [[\bar{a} - -\bar{b}] + +\bar{c}]))$,

(6.60) $\bar{a} \Rightarrow ([\bar{a} + + + 1] = \bar{a}) \wedge ([\bar{a} - - - 1] = \bar{a})$,

(6.61) $\bar{a} \Rightarrow ([1 + + + \bar{a}] = 1)$,

(6.62) $\neg(\bar{a} = [1-1]) \Rightarrow ([\bar{a} + + + [1-1]] = 1)$,

(6.63) $([1-1] < \bar{a}) \Rightarrow ([[1-1] + + + \bar{a}] = [1-1])$,

(6.64) $([1-1] < \bar{a}) \wedge \neg(\bar{b} = [1-1]) \wedge \bar{c} \Rightarrow (([[\bar{a} - - - \bar{b}] + + + \bar{b}] = \bar{a}) \wedge$
$([[\bar{a} - - - \bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] - - - \bar{b}]) \wedge ([\bar{a} + + + [\bar{c} - -\bar{b}]] =$
$[[\bar{a} + + + \bar{c}] - - - \bar{b}]))$,

(6.65) $([1-1] < \bar{a}) \wedge ([1-1] < \bar{b}) \wedge \bar{c} \Rightarrow ((([\bar{a} +++ [\bar{b}///\bar{a}]] = \bar{b}) \wedge$
$(([[\bar{a} +++ \bar{c}]///\bar{b}] = [\bar{c} ++ [\bar{a}///\bar{b}]]) \wedge ([[\bar{a} -- \bar{b}] +++ \bar{c}] =$
$[[\bar{a} +++ \bar{c}] -- [\bar{b} +++ \bar{c}]])),$

(6.66) $([1-1] < \bar{a}) \wedge ([1-1] < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ((([[\bar{a}///\bar{c}] -- [\bar{b}///\bar{c}]] =$
$[\bar{a}///\bar{b}]) \wedge ([[\bar{a} ++ \bar{b}]///\bar{c}] = [[\bar{a}///\bar{c}] + [\bar{b}///\bar{c}]]) \wedge ([[\bar{a} -- \bar{b}]///\bar{c}] =$
$[[\bar{a}///\bar{c}] - [\bar{b}///\bar{c}]])),$

(6.67) $\neg(\bar{a} = [1-1]) \wedge \neg(\bar{b} = [1-1]) \wedge \neg(\bar{c} = [1-1]) \Rightarrow ((([[\bar{a} ++ \bar{b}] +++ \bar{c}] =$
$[[\bar{a} +++ \bar{c}] ++ [\bar{b} +++ \bar{c}]]) \wedge ([\bar{a} +++ [\bar{b} ++ \bar{c}]] = [[\bar{a} +++ \bar{b}] +++ \bar{c}])$
$\wedge([\bar{a} +++ [\bar{b} + \bar{c}]] = [[\bar{a} +++ \bar{b}] ++ [\bar{a} +++ \bar{c}]]) \wedge ([\bar{a} +++ [\bar{b} - \bar{c}]] =$
$[[\bar{a} +++ \bar{b}] -- [\bar{a} +++ \bar{c}]])),$

(6.68) $([1-1] < \bar{a}) \wedge \neg(\bar{b} = [1-1]) \wedge \neg(\bar{c} = [1-1]) \Rightarrow (([\bar{a} --- [\bar{b} ++ \bar{c}]] =$
$[[\bar{a} --- \bar{b}] --- \bar{c}]) \wedge ([\bar{a} --- [\bar{b} -- \bar{c}]] = [[\bar{a} --- \bar{b}] +++ \bar{c}])),$

(6.69) $([1-1] < \bar{a}) \wedge ([1-1] < \bar{b}) \wedge \neg(\bar{c} = [1-1]) \Rightarrow ((([[\bar{a} ++ \bar{b}] --- \bar{c}] =$
$[[\bar{a} --- \bar{c}] ++ [\bar{b} --- \bar{c}]]) \wedge ([[\bar{a} -- \bar{b}] --- \bar{c}] =$
$[[\bar{a} --- \bar{c}] -- [\bar{b} --- \bar{c}]])),$

(6.70) $(1 \leq \bar{a}) \Rightarrow ([\bar{a} + e1] = \bar{a}),$

(6.71) $(1 \leq \bar{a}) \Rightarrow ([\bar{a} ++ e[1-1]] = 1),$

(6.72) $(1 \leq \bar{a}) \Rightarrow ([\bar{a} - f1] = \bar{a}),$

(6.73) $([1-1] \leq \bar{a}) \Rightarrow ([1 ++ e\bar{a}] = 1),$

(6.74) $([1-1] < \bar{a}) \Rightarrow ([1 -- f\bar{a}] = 1),$

(6.75) $(1 < \bar{a}) \Rightarrow ([1//g\bar{a}] = [1-1]),$

(6.76) $(1 < \bar{a}) \Rightarrow ([\bar{a}/g\bar{a}] = 1),$

(6.77) $(1 \leq \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow ([[\bar{a}i\bar{b}]h\bar{b}] = \bar{a}),$

(6.78) $(1 \leq \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow ([[\bar{a}h\bar{b}]i\bar{b}] = \bar{a}),$

(6.79) $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{b}h[\bar{a}j\bar{b}]] = \bar{a}),$

(6.80) $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([[\bar{b}h\bar{a}]j\bar{b}] = \bar{a}),$

(6.81) $(1 \leq \bar{a}) \wedge (1 \leq \bar{b}) \Rightarrow ([\bar{a} + e\bar{b}] = [\bar{a}e[\bar{a} + e[\bar{b} - 1]]]),$

(6.82) $\top\_1\_1\_ = [1-1],$

(6.83) $\top\_1\_[1+1]\_ = 1,$

(6.84) $\perp\_1\_1\_ = 1,$

(6.85) $\perp\_1\_[1+1]\_ = 1,$

(6.86) $\top\_k\_[[[1+1] ++ l] - 1]\_ = \top\_[k-1]\_l\_,$

(6.87) $\perp\_k\_[[[1+1] ++ l] - 1]\_ = \perp\_[k-1]\_l\_,$

(6.88) $\top\_k\_[[1+1] ++ l]\_ = [\top\_[k-1]\_l\_ + \top\_[k-1]\_[l+1]\_],$

(6.89) $\perp\_k\_[[1+1] ++ l]\_ = [\perp\_[k-1]\_l\_ + \perp\_[k-1]\_[l+1]\_],$

(6.90) $(1 \leq \bar{a}) \wedge (\top\_\bar{b}\_\bar{c}\_) \wedge (\perp\_\bar{b}\_\bar{c}\_) \Rightarrow ([\bar{a} + h[\top\_\bar{b}\_\bar{c}\_ -- \perp\_\bar{b}\_\bar{c}\_]] =$
$[[\bar{a} + h\top\_\bar{b}\_\bar{c}\_] - i\perp\_\bar{b}\_\bar{c}\_])$

}.

It should be noted that (6.82) $\sim$ (6.90) restrict $[\top\_\bar{b}\_\bar{c}\_--\bot\_\bar{b}\_\bar{c}\_]$ to be a Farey fraction. In the following, we will deduce some numbers and equalities as examples.

$(A1)$	$(a \to 1\|[aba]) \Rightarrow (a \to 1)$	$by(6.16),(2.18)$
$(A2)$	$\Rightarrow (a \to [aba])$	$by(6.16),(2.18)$
$(A3)$	$\Rightarrow (a \to [1ba])$	$by(A2),(A1),(2.17)$
$(A4)$	$\Rightarrow (a \to [1b1])$	$by(A3),(A1),(2.17)$
$(A5)$	$(b \to +\|-) \Rightarrow (b \to -)$	$by(6.17),(2.18)$
$(A6)$	$\Rightarrow (a \to [1-1])$	$by(A4),(A5),(2.17)$
$(A7)$	$(a < [1+a]) \Rightarrow (1 < [1+1])$	$by(6.30),(A1),(2.29)$
$(A8)$	$\Rightarrow 1$	$by(2.23)$
$(A9)$	$\Rightarrow [1+1]$	$by(A7),(2.23)$
$(A10)$	$(a < [1+a]) \Rightarrow ([1-1] < [1+[1-1]])$	$by(6.30),(A6),(2.29)$
$(A11)$	$\Rightarrow ([1-1] < [[1-1]+1])$	$by(A10),(6.44),(2.24)$
$(A12)$	$\Rightarrow ([1-1] < 1)$	$by(A11),(6.45),(2.24)$
$(A13)$	$\Rightarrow [1-1]$	$by(2.23)$
$(A14)$	$(b \to +\|-) \Rightarrow (b \to +)$	$by(6.17),(2.18)$
$(A15)$	$\Rightarrow (a \to [1+1])$	$by(A4),(A14),(2.17)$
$(A16)$	$(a < [1+a]) \Rightarrow ([1+1] < [1+[1+1]])$	$by(6.30),(A15),(2.29)$
$(A17)$	$\Rightarrow ([1-1] < [1+1])$	$by(A12),(A7),(2.22)$
$(A18)$	$\Rightarrow ([1-1] < [1+[1+1]])$	$by(A17),(A16),(2.22)$
$(A19)$	$\Rightarrow ([1--[1+[1+1]]] < [[1+1]--[1+[1+1]]])$	$by(A12),(A7),$ $(A18),(6.32)$
$(A20)$	$\Rightarrow [1--[1+[1+1]]]$	$by(2.23)$
$(A21)$	$\Rightarrow 1 < [1+[1+1]]$	$by(A7),(A16),(2.22)$
$(A22)$	$\Rightarrow \neg([1+1] = [1-1])$	$by(A17),(2.21)$
$(A23)$	$\Rightarrow (1 < [[1+[1+1]]--f[1+1]])$	$by(A21),(A17),(6.34)$
$(A24)$	$(f \to -\|-f) \Rightarrow (f \to -)$	$by(6.20),(2.18)$
$(A25)$	$\Rightarrow (f \to -f)$	$by(6.20),(2.18)$
$(A26)$	$\Rightarrow (f \to --)$	$by(A25),(A24),(2.17)$
$(A27)$	$\Rightarrow (1 < [[1+[1+1]]----[1+1]])$	$by(A23),(A26),(2.26)$
$(A28)$	$\Rightarrow [[1+[1+1]]----[1+1]]$	$by(2.23)$
$(A29)$	$\Rightarrow ([[1+1]--[1+1]] < [[1+[1+1]]--[1+1]])$	$by(A17),(A16),(6.32)$
$(A30)$	$\Rightarrow (1 < [[1+[1+1]]--[1+1]])$	$by(A22),(6.52),(2.25)$
$(A31)$	$\Rightarrow [[1+[1+1]]--[1+1]]$	$by(2.23)$
$\vdots$	$\vdots$	$\vdots$

Then according to Definition 2.2, we can deduce from $R\{\Phi, \Psi\}$ the numbers as follows:

$$\{1, [1+1], [1-1], [1+[1+1]], [1-[1+1]], [1--[1+[1+1]]], \\ [[1+[1+1]]----[1+1]], [[1+[1+1]]--[1+1]]\cdots\}$$

(B1) $(\top\_k\_[[1+1]++l]\_ = [\top\_[k-1]\_l\_ + \top\_[k-1]\_[l+1]\_])$ by(6.88)

(B2) $\Rightarrow (\top\_[1+1]\_[[1+1]++l]\_ = [\top\_[[1+1]-1]\_l\_+ \top\_[[1+1]-1]\_[l+1]\_])$ by(6.28), (2.18), (2.42)

(B3) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_[[1+1]-1]\_1\_+ \top\_[[1+1]-1]\_[1+1]\_])$ by(6.29), (2.18), (2.42)

(B4) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_[[1-1]+1]\_1\_+ \top\_[[1-1]+1]\_[1+1]\_])$ by(6.46), (2.37)

(B5) $\Rightarrow (\top\_[1+1]\_[[1+1]++1]\_ = [\top\_1\_1\_ + \top\_1\_[1+1]\_])$ by(6.45), (2.37)

(B6) $\Rightarrow (\top\_[1+1]\_[1+1]\_ = [\top\_1\_1\_ + \top\_1\_[1+1]\_])$ by(6.51), (2.35), (2.37)

(B7) $\Rightarrow (\top\_[1+1]\_[1+1]\_ = [[1-1]+1])$ by(6.82), (6.83), (2.37)

(B8) $\Rightarrow (\top\_[1+1]\_[1+1]\_ = 1)$ by(6.45), (2.37)

(B9) $\Rightarrow ([1-1] < \top\_[1+1]\_[1+1]\_)$ by(A12), (2.24)

(B10) $\Rightarrow \top\_[1+1]\_[1+1]\_$ by(2.23)

(B11) $(\bot\_k\_[[1+1]++l]\_ = [\bot\_[k-1]\_l\_ + \bot\_[k-1]\_[l+1]\_])$ by(6.89)

(B12) $\Rightarrow (\bot\_[1+1]\_[[1+1]++l]\_ = [\bot\_[[1+1]-1]\_l\_+ \bot\_[[1+1]-1]\_[l+1]\_])$ by(6.28), (2.18), (2.42)

(B13) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_[[1+1]-1]\_1\_+ \bot\_[[1+1]-1]\_[1+1]\_])$ by(6.29), (2.18), (2.42)

(B14) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_[[1-1]+1]\_1\_+ \bot\_[[1-1]+1]\_[1+1]\_])$ by(6.46), (2.37)

(B15) $\Rightarrow (\bot\_[1+1]\_[[1+1]++1]\_ = [\bot\_1\_1\_ + \bot\_1\_[1+1]\_])$ by(6.45), (2.37)

(B16) $\Rightarrow (\bot\_[1+1]\_[1+1]\_ = [\bot\_1\_1\_ + \bot\_1\_[1+1]\_])$ by(6.51), (2.35), (2.37)

(B17) $\Rightarrow (\bot\_[1+1]\_[1+1]\_ = [1+1])$ by(6.84), (6.85), (2.37)

(B18) $\Rightarrow ([1-1] < \bot\_[1+1]\_[1+1]\_)$ by(A17), (2.24)

(B19) $\Rightarrow \bot\_[1+1]\_[1+1]\_$ by(2.23)

$$(B20)\quad ([[1+1]+e[[1+[1+1]]--[1+1]]] = [[1+1]e[[1+1]+e[[[1+[1+1]]--[1+1]]-1]]]) \quad by (A7), (A30), (6.81)$$

$$(B21)\quad \Rightarrow ([[1+1]++e[[1+[1+1]]--[1+1]]] = [[1+1]+e[[1+1]++e[[[1+[1+1]]--[1+1]]-1]]]) \quad by (6.19), (2.42)$$

$$(B22)\quad \Rightarrow ([[1+1]+++e[[1+[1+1]]--[1+1]]] = [[1+1]++e[[1+1]+++e[[[1+[1+1]]--[1+1]]-1]]]) \quad by (6.19), (2.42)$$

$$(B23)\quad \Rightarrow ([[1+1]++++[[1+[1+1]]--[1+1]]] = [[1+1]+++[[1+1]++++[[[1+[1+1]]--[1+1]]-1]]]) \quad by (6.19), (2.42)$$

$$(B24)\quad \Rightarrow ([[1+1]++++[[1+[1+1]]--[1+1]]] = [[1+1]+++[[1+1]++++ [[[1--[1+1]]+[[1+1]--[1+1]]]-1]]]) \quad by (6.58), (2.37)$$

$$(B25)\quad \Rightarrow ([[1+1]++++[[1+[1+1]]--[1+1]]] = [[1+1]+++[[1+1]++++[[[1--[1+1]]+1]-1]]]) \quad by (A22), (6.52), (2.37)$$

$$(B26)\quad \Rightarrow ([[1+1]++++[[1+[1+1]]--[1+1]]] = [[1+1]+++[[1+1]++++[[[1--[1+1]]-1]+1]]]) \quad by (6.46), (2.37)$$

$$(B27)\quad \Rightarrow ([[1+1]++++[[1+[1+1]]--[1+1]]] = [[1+1]+++[[1+1]++++[1--[1+1]]]]) \quad by (6.45), (2.37)$$

$$(B28)\quad ([[1+1]+h[\top\_[1+1]\_[1+1]\_--\bot\_[1+1]\_[1+1]\_]] = [[[1+1]+h\top\_[1+1]\_[1+1]\_]-i\bot\_[1+1]\_[1+1]\_]) \quad by (A7), (B10), (B19), (6.90)$$

$$(B29)\quad \Rightarrow ([[1+1]+h[1--[1+1]]] = [[[1+1]+h1]-i[1+1]]) \quad by (B8), (B17), (2.35), (2.37)$$

$$(B30)\quad \Rightarrow ([[1+1]++h[1--[1+1]]] = [[[1+1]++h1]--i[1+1]]) \quad by (6.23), (6.14)$$

$$(B31)\quad \Rightarrow ([[1+1]+++h[1--[1+1]]] = [[[1+1]+++h1]---i[1+1]]) \quad by (6.23), (6.14)$$

$$(B32)\quad \Rightarrow ([[1+1]++++[1--[1+1]]] = [[[1+1]++++1]----[1+1]]) \quad by (6.22), (6.14)$$

$$(B33)\quad ([[1+1]+e1] = [1+1]) \quad by (A7), (6.70)$$

$$(B34)\quad \Rightarrow ([[1+1]++e1] = [1+1]) \quad by (6.19), (2.35), (2.39)$$

$$(B35)\quad \Rightarrow ([[1+1]+++e1] = [1+1]) \quad by (6.19), (2.35), (2.39)$$

$$(B36)\quad \Rightarrow ([[1+1]++++1] = [1+1]) \quad by (6.19), (2.35), (2.39)$$

$$(B37)\quad \Rightarrow ([[1+1]++++[1--[1+1]]] = [[1+1]----[1+1]]) \quad by(B32),(B36),(2.37)$$

$$(B38)\quad \Rightarrow ([[1+1]++++[[1+[1+1]]--[1+1]]] = [[1+1]+++[[1+1]----[1+1]]]) \quad by(B27),(B37),(2.37)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then we can deduce from $R\{\Phi, \Psi\}$ the equalities as follows:

$$\begin{aligned}
[[1+1]++[[1+1]---[1+1]]] &= [[[1+1]---[1+1]]++[1+1]], \\
[[1+1]++++[[1+[1+1]]--[1+1]]] &= [[1+1]+++[[1+1]----[1+1]]], \\
\vdots \quad \vdots \quad &\vdots.
\end{aligned}$$

The deduced numbers correspond to real numbers as follows:

$$\begin{aligned}
\vdots \quad &\vdots \quad \vdots, \\
[[1-1]-[[1+[1+1]]----[1+1]]] &\equiv \\
\vdots \quad &\vdots \quad \vdots, \\
[[1-1]-[[1+1]---[1+1]]] &\equiv -\sqrt[2]{2}, \\
\vdots \quad &\vdots \quad \vdots, \\
[[1-1]-[[1+1]---[1+[1+1]]]] &\equiv -\sqrt[3]{2}, \\
\vdots \quad &\vdots \quad \vdots, \\
[[1-1]-1] &\equiv -1, \\
\vdots \quad &\vdots \quad \vdots, \\
[[1-1]-[1--[1+1]]] &\equiv -\frac{1}{2}, \\
\vdots \quad &\vdots \quad \vdots, \\
[1-1] &\equiv 0, \\
\vdots \quad &\vdots \quad \vdots, \\
[1--[1+1]] &\equiv \frac{1}{2}, \\
\vdots \quad &\vdots \quad \vdots, \\
[[1+1]///[1+[1+1]]] &\equiv \log_3 2, \\
\vdots \quad &\vdots \quad \vdots, \\
1 &\equiv 1, \\
\vdots \quad &\vdots \quad \vdots, \\
[[1+1]---[1+[1+1]]] &\equiv \sqrt[3]{2},
\end{aligned}$$

$$
\begin{array}{rcl}
\vdots & \vdots & \vdots, \\
[[1+1]---[1+1]] & \equiv & \sqrt[2]{2}, \\
\vdots & \vdots & \vdots, \\
[1+[1--[1+1]]] & \equiv & \frac{3}{2}, \\
\vdots & \vdots & \vdots, \\
[[1+[1+1]]///[1+1]] & \equiv & \log_2 3, \\
\vdots & \vdots & \vdots, \\
[[1+[1+1]]----[1+1]] & \equiv & \\
\vdots & \vdots & \vdots, \\
[1+1] & \equiv & 2, \\
\vdots & \vdots & \vdots, \\
[[1+1]+[1--[1+1]]] & \equiv & \frac{5}{2}, \\
\vdots & \vdots & \vdots, \\
[1+[1+1]] & \equiv & 3, \\
\vdots & \vdots & \vdots, \\
[[[1+[1+1]]----[1+1]]++[1+1]] & \equiv & \\
\vdots & \vdots & \vdots.
\end{array}
$$

The equalities on deduced numbers correspond to addition, subtraction, multiplication, division, exponentiation operation, root-extraction operation, logarithm operation and more other operations in real number system. Note that in the correspondence above, some irrational numbers such as $[[1-1]-[[1+[1+1]]----[1+1]]]$, $[[1+[1+1]]----[1+1]]$ and $[[[1+[1+1]]----[1+1]]++[1+1]]$ do not correspond to any irrational number based on traditional operations. So the logical calculus $R\{\Phi,\Psi\}$ can deduce more irrational numbers than before.

In fact, the logical calculus $R\{\Phi,\Psi\}$ not only deduces more irrational numbers than before, but also makes its deduced numbers join in algebraical operations. So the logical calculus $R\{\Phi,\Psi\}$ intuitively and logically denote real number system.

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