The Secret Side of Reflexivity

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In dealing with calculations on a topological vector space it is very often convenient to take refuge to its dual space instead of doing so on the vector space itself.

The obstacle for many not to do so is in the fear for the existence of non-reflexive (topological) vector spaces.
However, the following may help:

Proposition: Every metrizable locally convex vectorspace is a dense topological subspace of a (complete) reflexive locally convex space, namely its bidual.

Since the topology of a metrizable locally convex space X is defined by a countable family of seminorms of X, it will be seen to follow from the proof of the following lemma:

Lemma: The closure of every normed space X is reflexive, i.e.: every Banachspace is reflexive.

Proof 1: Let ||·|| be the norm on X. Then the dual X^* i.e. the vectorspace of all continuous linear functionals on X is well-known to be a Banachspace under the (naturally induced) norm $\|\cdot\|^*: X^* \ni \mathrm{f} \mapsto \sup_{x\in X, \|x\|\le 1} |f(x)|$.
By this, each x in X, in turn is a continuous functional on X^*, hence X is a subspace of the Banachspace X^{**}, and the embedding is an isometry, since $\|x\| = \sup_{f\in X^*,\ \|f\|^*\le 1} |f(x)|$.

We are hence left to prove that X** is contained in the closure of X: Let there be $x \in X^{**}$ which is not in the closure of X, and dividing by its norm, we can assume that $\|x\|=1$. Then there is an $f \in X^{***}$ such that $f(x)=1$. Then the kernel, $\ker f$, is a subspace of X** which is closed, since $\ker f = f^{-1}(0)$ is the preimage of the closed set $\{0\}\subset K$ with K being the field over which X is defined. It follows that $\pi: X^{**} \ni y \mapsto f(y)x \in span(x) \subset X^{**}$ is a continuous projection. This allows the splitting of X^{**} into the direct sum of $span(x)$ and its complement $span(x)^C$ (which in turn is isomorphic to $X^{**}/span(x)$). Since $X \subset span(x)^C$, x is a zero-functional on X^*, and as an element of the dual of X^*: $x(f)=0\ \forall f \in X^* \Rightarrow x=0$.
Therefore, no such x can't exist, and X^{**} must be the closure of X. ■

Proof 2: A normed space is reflexive if and only if its unit ball is weakly compact. With the above notation, we may assume that X is complete and have to show that every $f \in X^*$ maps the closed unit ball $B=\{x\in X: \|x\|\le 1\}$ into a compact set. Since $|f(B)|\le\|f\|$, the image f(B) is bounded in K. To prove its compactness, it is left to prove that f(B) is closed. This is trivial for $f=0$, so let $f\ne 0$, and let $x\in X$ with $f(x)\ne 0$. There are two ways to prove this:
The 1st is by applying the open mapping theorem which states that given two metrizable, complete locally convex spaces E and F, every continuous, surjective linear mapping from E onto F is open (i.e.: maps open sets into open, and thus closed into closed ones). Since the field K is a 1-dimensional Banachspace and $f\ne 0$ is continuous, the condition holds, and f(B) is closed.
The 2nd is via quotient spaces: As in proof 1, the kernel $\ker f$ is a closed subspace of X, $X/\ker f$

isomorphic to a closed (1-dimensional) subspace of X, the canonical projection $\Phi : X \to X / \ker f$ continuous, open, and onto, $\Phi(B)$ therefore closed in $X / \ker f$, and $f = \Phi \circ g$ for some $g \in (X /(\ker f))^*$. Since g is a continuous, injective linear mapping of 1-dimensional Banachspaces, it is also open and onto. Hence, g maps $\Phi(B)$ into a closed subset of K. ■

Since every (continuous) seminorm p on a locally convex vectorspace X analogously induces a seminorm $p^* : X^* \ni f \mapsto \sup_{x \in X/N, p(x) \le 1} | f(x)|$ with N being the (closed) subspace of all $x \in X$ with $p(x) = 0$, and since by the assumed metrizability we can construct a well-ordered sequence of refining seminorms defining the topology on X (which in turn induces a refining sequence of seminorms on X*) the lemma generalizes to the above proposition.■

Let's examine a well-known example and dig out some surprises:

Let c_0 be the vectorspace of all null-converging sequences which is a separable Banachspace under the supremum norm. Its dual, l_1 is the space of all absolutely summable sequences, and have a look at its bidual c_0^{**}: It is currently hold that the dual of l_1 were l_∞, the space of all bounded sequences. Let us see what goes wrong, and take a look at an always neglected superspace of c_0, namely the space c_∞ of all converging sequences, which again is a Banachspace with supremum norm, and it is a closed subspace of l_∞: Let $f = (a_k)_{k>0}$ be a sequence converging to 1. Then for any $n > 1 : f = f_n + g_n$ where $f_n = (a_1,..., a_{n-1},0,..)$ and $g_n = (0,..., a_n, a_{n+1},..)$ .Now, as $n \to \infty$, f_n on c_0 weakly converges to some element of the weak closure of c_0 (which is c_0 again), and g_n on l_1 weakly converges to 0, however g_n does not weakly converge to zero, since $\lim_{k\to\infty} a_k$ defines a continuous linear form on c_∞ which does not vanish as $k \to \infty$. Now, as f_n even converges weakly on c_∞^*, $g_n = f - f_n$ converges weakly, and, since c_∞ is weakly closed, so this weak limit must be an element of c_∞. Since it is unequal zero and a zero linear form on l_1, it can't be an element of l_1^*. The point therefore is: Because all elements in l_1 vanish in infinity, so must all of l_1^*!
Since dual spaces of separable locally convex spaces are weakly (and strongly) closed, c_∞ is seen to be the direct sum of c_0 and its complement which is the space of limits of c_∞ in infinity, or, profanely a one dimensional Banachspace.
In fact, $f_n = (a_1,..., a_{n-1},0,..)$ weakly converges in c_0 for every $f = (a_k)_{k>0} \in c_\infty$, and therefore $\pi : c_\infty \ni (a_k)_{k>0} \mapsto \lim_{n\to\infty} (a_1,..., a_{n-1},0,..) \in c_0$ is a well-defined, continuous projection.
As a consequence, c_∞^* is the direct sum of l_1 and the one-dimensional Banachspace of all functionals on the infinite – and oops: we're into boundary values and functionals just by toplogical closure and completion.

Wait: we can do even better: We calculated the complement of c_0 in c_∞. Wouldn't you like to know what the complement of c_0 in l_∞ is?
First, let us prove that c_0 really has a toplological complement in l_∞:
For natural $n>0$ let $\pi_n : l_\infty \ni (a_k)_{k>0} \mapsto (a_1,.., a_n,0,0...) \in c_0$. With this, $(\pi_n)_{n>0}$ is a bounded sequence of projections which weakly converges in c_0 as $n \to \infty$. Since c_0 is weakly closed,

$\pi = \lim_{n\to\infty} \pi_n$ is a continuous projection from l_∞ into c_0. So, we can split l_∞ into the direct sum of c_0 and its complementary subspace $range(id_{l_\infty} - \pi)$ which in turn is isomorphic to l_∞ / c_0. (Alternatively, you may follow [Robertson/Robertson, Topological Vector Spaces, 2nd ed, Cambridge University Press, Ch. VI, Corollary 2 of Proposition 13], which states that for Fréchet spaces direct sums of two closed subspaces are topological direct sums.)
Next, it is straightforward to prove the following: Let $f = (a_k)_{k>0}$ be as above a sequence converging to 1. Then – with K being the field over which the Banachspace l_∞ is defined – for each $h \in l_\infty$ there is a triple $(h_0, k_1, k_2 \cdot g)$ with $h_0 \in c_0, k_1, k_2 \in K$ and g being a sequence with $\limsup_{k\to\infty} |g_k| = 1$, $\liminf_{k\to\infty} |g_k| = 0$, such that $h = h_0 + k_1 f + k_2 g$. That means: l_∞ differs from c_0 by the one-dimensional space spanned by f plus the Banachspace spanned by those alternating sequences g with the norm $g \mapsto \limsup_{k\to\infty} |g_k|$ modulo c_0. That subspace of alternating, bounded sequences modulo c_0 is known to have an overcountable dimension. Visually speaking, the complementary space of c_0 in l_∞ is the Banachspace of all countable, K-valued, bounded tuples ("sitting on the infinite" – ouch!) with supremum-norm.

Look at that: Given an open subset U of **R**^{n} equipped with the topology of compact subsets of U, and consider the space $C_c(U)$ of all continuous functions of compact support in U with the supremum norm. Its completion is the analogue to c_0. These functions all vanish on the boundary of U, and so do their dual functionals. However, the space $C_b(U)$ of all bounded, uniformly continuous functions on U does not, and so $C_b(U)^* = C_c(U)^* \oplus C_b(\Gamma(U))^*$ is the direct sum of $C_c(U)^*$ plus its complement $C_b(\Gamma(U))^*$, where $\Gamma(U) = \overline{U} \setminus U$ is the "boundary" of U. (Note that the metric on U and $\Gamma(U)$ are given by restriction of the metric of **R**^{n} which well-define the spaces $C_b(U)$ and $C_b(\Gamma(U))$ as the space of continuous functions on U and $\Gamma(U)$, and therefore their dual and bidual spaces are well-defined.)

Now, let U be even bounded. Then the Hilbertspace $L^2(U)$ of square Lebesgue-integrable functions on U is seen to be the closure of $C_c(U)$. But – contrary to what is often said – it does *not* embed $C_b(U)$, since $C_b(\Gamma(U))$ is projected to zero under the norm of $L^2(U)$, where $\Gamma(U)$ again denotes $\overline{U} \setminus U$. Of course, one can embed $L^2(U)$ into $L^2(R^n)$, but still the obstacle will be that often $\Gamma(U)$ is a set of Lebesgue measure zero. The commodity of self duality and the restriction of Borel measures to the Lebesgue measure thus have to be paid by the loss of continuity and a unified general approach.

It should now be straightforward to figure out the complementary subspace of the space $C_0(U)$ of uniformly continuous, bounded functions on R^n vanishing outside the open set $U \subset R^n$ in the space $C_\infty(U)$ of all continuous, bounded functions in U.