ON THE CONVERGENCE OF THE DIRICHLET SERIES WITH MOBIUS FUNCTION.

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ABSRTACT. In this paper, we introduced a method to modify the Dirichlet series over the Mobius function by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime p_r. The properties of the new series are analyzed as p_r approaches infinity and its relationship to the function $\exp[E_1((1-s)\log p_r)]$ and the partial Euler product is established and then used to examine the validity of the Riemann Hypothesis.

1- Introduction:

The Riemann Zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain

(1.1) $$\zeta(1-s) = 2(2\pi)^2 \cos(0.5\, s\pi)\Gamma(s)\zeta(s) ,$$

where $s=\sigma+it$ is a complex variable and $s\neq 0$ [1].

For $\sigma>1$ (or $\Re(s)>1$), $\zeta(s)$ can be expressed by the following series

(1.2) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} ,$$

and by the following product over the primes p_i 's

(1.3) $$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1-\frac{1}{p_i^s}\right) ,$$

where $p_1=2$, $\prod_{i=1}^{\infty}\left(1-1/p_i^s\right)$ is the Euler product and $\prod_{i=1}^{r}\left(1-1/p_i^s\right)$ is the partial Euler product. The series $\zeta(s)$ is absolutely convergent for $\sigma>1$.

The region of the convergence can be extended to $\Re(s)>0$ by using the alternating series $\eta(s)$ where

(1.4) $$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} ,$$

and

(1.5) $$\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s).$$

One may notice that the term $1-2^{1-s}$ is zero at $s = 1$. This zero cancels the simple pole that $\zeta(s)$ has at $s = 1$ enabling the extension of the zeta function series representation over the critical strip $0<\Re(s)<1$.

It is well known that all the non-trivial zeros of $\zeta(s)$ are located in the critical strip $0<\Re(s)<1$. Riemann stated that all the non-trivial zeros were very probably located on the critical line $\Re(s)=1/2$ [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function. The Mobius function $\mu(n)$ is define as follows

$$\mu(n)=1, \text{if } n=1.$$
$$\mu(n)=(-1)^k, \ n=\prod_{i=1}^{k} p_i, \ p_i's \text{ are distinct primes}$$
$$\mu(n)=0, \ p^2|n \text{ for some } p.$$

The Dirichlet series *Mu(s)* with the Mobius function is defined as

$$Mu(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}.$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s)>1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s)=1$. The Riemann hypothesis is equivalent to the statement that *Mu(s)* satisfies the following equation

(1.6) $$Mu(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s} = \frac{1}{\zeta(s)}$$

for $\Re(s)>0.5$.

In this paper, we have introduced a method to modify the Dirichlet series *Mu(s)* (defined by Equation (1.6)) by first eliminating the numbers that have the prime factor 2 to generate the series *Mu(s,2).* For the series *Mu(s,2)*, we then eliminate the numbers with prime factor 3 to generate the series *Mu(s,3)*, and so on, up to the prime number p_r. In essence, we have applied the sieving technique to modify the series *Mu(s)* to include only the numbers with prime factors greater than p_r. In the following sections, the properties of the new series will be analyzed and its use to compute the prime counting function and to examine the validity of the RH will be presented.

2- Applying the Sieving Method to the Dirichlet Series *Mu(s)*:

The Dirichlet series *Mu(s)* with the Mobius function $\mu(n)$ is defined as

$$Mu(s)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$Mu(s)=1-\frac{1}{2^s}-\frac{1}{3^s}+\frac{0}{4^s}-\frac{1}{5^s}+\frac{1}{6^s}-\ .....\ .$$

Now, we introduce the series *Mu(s,2)* by eliminating all the numbers that have a prime factor 2. Thus, *Mu(s,2)* can be written as

$$Mu(s,2)=1-\frac{1}{3^s}-\frac{1}{5^s}-\frac{1}{7^s}+\frac{0}{9^s}-\frac{1}{11^s}-\frac{1}{13^s}+\frac{1}{15^s}-\ .....\ .$$

To have the same index for both series *Mu(s)* and *Mu(s,2)* referring to the same term, the above series can be re-written as

$$Mu(s,2)=1+\frac{0}{2^s}-\frac{1}{3^s}+\frac{0}{4^s}-\frac{1}{5^s}+\frac{0}{6^s}-\frac{1}{7^s}+\frac{0}{8^s}+\frac{0}{9^s}+\frac{0}{10^s}\ .....\ ,$$

or

(2.1) $$Mu(s,2)=\sum_{n=1}^{\infty}\frac{\mu(n,2)}{n^s},$$

where
$\mu(n,2)=\mu(n)$, if n is an odd number.
$\mu(n,2)=0$, if n is an even number.

The above series *Mu(s,2)* can be further modified by eliminating all the numbers that have a prime factor 3 to get the series *Mu(s;3)* where

$$Mu(s,3)=1-\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{11^s}-\frac{1}{13^s}-\frac{1}{17^s}-\frac{1}{19^s}-\frac{1}{23^s}+\frac{0}{25^s}\ .....\ ,$$

or more conveniently

$$Mu(s,3)=1+\frac{0}{2^s}-\frac{0}{3^s}+\frac{0}{4^s}-\frac{1}{5^s}+\frac{0}{6^s}-\frac{1}{7^s}+\frac{0}{8^s}+\frac{0}{9^s}+\frac{0}{10^s}\ .....$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i > p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(2)\}$ is the set of square free odd numbers), then one may define the series $Mu(s, p_r)$ as

$$Mu(s, p_r) = \sum_{n \in \{1, I(p_r)\}} \frac{\mu(n)}{n^s},$$

or

(2.2) $$Mu(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s},$$

where
$\mu(n, p_r) = \mu(n)$, if $n \in \{1, I(p_r)\}$.
Otherwise, $\mu(n, p_r) = 0$

It can be easily shown that $Mu(s, p_r)$ converges absolutely for $\Re(s) > 1$ for every prime number p_r. Furthermore, it can be also shown that, for $\Re(s) > 1$, $Mu(s, p_r)$ satisfies the following equation

(2.3) $$Mu(s) = Mu(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right).$$

Since

$$Mu(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

one may then conclude that, for $\Re(s) > 1$, $Mu(s, p_r)$ approaches 1 as p_r approaches infinity.

3- Convergence of the series $Mu(s, p_r)$ within the strip $0.5 < \Re(s) \leq 1$:

In this section, we will first deal with the question of the conditional convergence of the series $Mu(s, p_r)$ over the strip $0.5 < \Re(s) \leq 1$. Toward this end, we need to examine the convergence of the series only along the real axis (or along the line $0.5 < \sigma \leq 1$). Theorem 1 establishes the relationship along the line $0.5 < \sigma \leq 1$ between the conditional convergence of the two series $Mu(\sigma, p_r)$ and $Mu(\sigma)$.

Theorem 1: For $s = \sigma + i0$, $0.5 < \sigma \leq 1$ and for every integer *r*, the series $Mu(\sigma)$ converges conditionally if and only if the series $Mu(\sigma, p_r)$ converges conditionally. Furthermore, $Mu(\sigma)$ and $Mu(\sigma, p_r)$ are related as follows

(3.1) $$Mu(\sigma) = Mu(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{\sigma}}\right).$$

The proof of Theorem 1 is outlined in Appendix 1.

Theorem 2: For $s = \sigma + it$, $0.5 < \sigma \leq 1$ and for every integer *r*, the series $Mu(s)$ converges conditionally if and only if $Mu(s, p_r)$ converges conditionally. Moreover, $Mu(s)$ and $Mu(s, p_r)$ are related as follows

(3.2) $$Mu(s) = Mu(s, p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{s}}\right).$$

The proof of the first part of Theorem 2 follows from the fact that $Mu(s, p_r)$ is a Dirichlet series and consequently this series is conditionally convergent if and only if the series $Mu(\sigma, p_r)$ is conditionally convergent.

The second part of the theorem can be proved by first defining $Mu(s, p_r; N_{1,} N_2)$ as the sum

(3.3) $$Mu(s, p_r; N_{1,} N_2) = \sum_{n=N_1}^{N_2}\frac{\mu(n, p_r)}{n^{s}}.$$

Then, one can write

(3.4) $$Mu(s, p_{r-1}; 1, Np_r) = Mu(s, p_r; 1, Np_r) - \frac{1}{p_r^{s}}Mu(s, p_r; 1, N).$$

If both series $Mu(s, p_{r-1})$ and $Mu(s, p_r)$ are convergent, then as N approaches infinity, we obtain

$$Mu(s, p_{r-1}) = \left(1-\frac{1}{p_r^{s}}\right)Mu(s, p_r).$$

By repeating this process r-1 times, we then obtain

$$Mu(s) = Mu(s, p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{s}}\right).$$

Note that if we multiply both sides of the above equation by $\prod_{i=1}^{r}\left(1+p_i^{-s}\right)$, then as p_r approaches infinity, one may obtain

(3.5) $$Mu(s, p_r) = \frac{\zeta(2s)}{\zeta(s)}\prod_{i=1}^{r}\left(1+\frac{1}{p_i^{s}}\right).$$

It should be pointed out that the sieving method applied to the Dirichlet series with Mobious function can be also applied to the Dirichlet series with Lioville function. The Dirichlet series *Lv(s)* with Lioville Function $\lambda(n)$ is defined as

(3.6) $$Lv(s) = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{s}},$$

where $\lambda(n)$ is the Liouville function defined as
$\lambda(n) = 1$, if $n = 1$.
$\lambda(n) = 1$, if n has an even number of prime factors including multiplicities.

$\lambda(n) = -1$, if n has an odd number of prime factors including multiplicities.

Following the same process, one may define the series $Lv(s, p_r)$ as

$$Lv(s, p_r) = \sum_{n \in \{1, I(p_r)\}} \frac{\lambda(n)}{n^s},$$

or

(3.7) $$Lv(s, p_r) = \sum_{n=1}^{\infty} \frac{\lambda(n, p_r)}{n^s},$$

where
$\mu(n, p_r) = \lambda(n)$, if $n \in \{1, I(p_r)\}$.
Otherwise, $\lambda(n, p_r) = 0$

It can be easily shown that $Lv(s, p_r)$ converges absolutely for $\Re(s)>1$ for every prime number p_r. Furthermore, it can also be shown that, for $\Re(s)>1$, $Lv(s, p_r)$ satisfies the following equation

(3.8) $$Lv(s, p_r) = Lv(s) \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

4- Relationship between the series $Mu(s, p_r)$ and the Exponential Integral $E_1((s-1)\log p_r)$:

In this section, we will derive a functional representation for the series $Mu(s, p_r)$ as p_r approaches infinity. This task can be easily achieved for $\sigma > 1$ by noting that

(4.1) $$1/\zeta(s) = Mu(s) = Mu(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right).$$

Since

$$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

therefore,

$$Mu(s, p_r) = \prod_{i=r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right).$$

For $s = \sigma + i0$, the above equation can be written as

$$Mu(\sigma, p_r) = \prod_{i=r+1}^{\infty} \left(1 - \frac{1}{p_i^{\sigma}}\right),$$

or

$$\log\left(Mu(\sigma, p_r)\right) = \sum_{i=r+1}^{\infty} \log\left(1 - \frac{1}{p_i^{\sigma}}\right).$$

Hence,

$$\log\left(Mu(\sigma, p_r)\right) = \sum_{i=r+1}^{\infty} \left(-\frac{1}{p_i^{\sigma}} - \frac{1}{2\,p_i^{2\sigma}} - \frac{1}{3\,p_i^{3\sigma}} - \ldots\right).$$

However, for $\sigma > 0.5$, the sum $\sum_{i=r1}^{\infty} \left(-\frac{1}{2\,p_i^{2\sigma}} - \frac{1}{3\,p_i^{3\sigma}} - \ldots\right)$ approaches zero as $r1$ approaches infinity.

Thus,

(4.2) $$\log\left(\prod_{i=r1}^{r2} \left(1 - \frac{1}{p_i^{\sigma}}\right)\right) = \sum_{i=r1}^{r2} \left(-\frac{1}{p_i^{\sigma}} - \frac{1}{2\,p_i^{2\sigma}} - \frac{1}{3\,p_i^{3\sigma}} - \ldots\right) = -\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} + \delta,$$

where $\delta = O\left(p_{r1}^{1-2\sigma}\right)$ is an arbitrary small number for sufficiently large $r1$.

On the other hand, using the Prime Number Theorem (PNT) with a suitable constant $a > 0$, the number of primes less than x is given by [4, page 43]

(4.3) $$\pi(x) = \mathrm{Li}(\mathrm{x}) + O\left(x\, e^{-a\sqrt{\log x}}\right),$$

or

(4.4) $$\pi(x) = \mathrm{Li}(x) + O\left(x/(\log x)^{k}\right),$$

where Li(x) is the Logarithmic Integral of x and k is a number greater than zero.

Using Stieltjes integral [5], one may write the sum $\sum_{i=r1}^{r2} 1/p_i^{\sigma}$ for $\sigma > 1$ as follows

(4.5) $$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{x=p_{r1}}^{p_{r2}} \frac{d\,\pi(x)}{x^{\sigma}}.$$

Using Equation (4.4) for the representation of $\pi(x)$, one may then write the integral in Equation (4.5) as [5, Theorem 2, page 57]

(4.6) $$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right),$$

where k is a number greater than zero. Thus, Equation (4.6) can be written as

(4.7) $$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{x=p_{r1}}^{\infty} \frac{1}{x^{\sigma}} \frac{1}{\log x} dx - \int_{x=p_{r2}}^{\infty} \frac{1}{x^{\sigma}} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right).$$

Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_{r}^{\infty} \frac{e^{-u}}{u} du,$$

and using the substitutions $u=(\sigma-1)\log x$, $du=(\sigma-1)dx/x$ and $x^{\sigma}/x=e^{u}$, then for $\sigma>1$, one may write Equation (4.7) as

(4.8) $$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + O\left(\frac{1}{(\log p_{r1})^k}\right).$$

Combining Equations (4.2) and (4.8) and noting that $E_1((\sigma-1)\log p_{r2})$ approaches zero as p_{r2} approaches infinity, one may write Equation (4.1) for $\sigma>1$ as

$$-\log\zeta(\sigma) = \sum_{i=1}^{r-1} \log\left(1-\frac{1}{p_i^{\sigma}}\right) - \sum_{i=r}^{\infty} \frac{1}{p_i^{\sigma}} + \delta,$$

or

$$\log\zeta(\sigma) + \sum_{i=1}^{r-1} \log\left(1-\frac{1}{p_i^{\sigma}}\right) - E_1((\sigma-1)\log p_r) = \Delta,$$

where $\Delta=O(1/(\log p_r)^k)$ is an arbitrarily small number attained by setting p_r sufficiently large. Therefore,

(4.9) $$\zeta(\sigma) \exp(-E_1((\sigma-1)\log p_r)) \prod_{i=1}^{r-1} \left(1-\frac{1}{p_i^{\sigma}}\right) = 1+\Delta.$$

As p_r approaches infinity, Δ approaches zero. Thus, the right side of the above equation approaches 1 as p_r approaches infinity. Similarly, for $\sigma>1$, it can be shown that

(4.10) $$\lim_{r\to\infty}\left\{\zeta(s) \prod_{i=1}^{r-1} \left(1-\frac{1}{p_i^{s}}\right) \exp(-E_1((s-1)\log p_r))\right\} = 1.$$

Now, define the function $G(s, p_r)$ as

(4.11) $$G(s, p_r) = \zeta(s) \exp\left(-E_1((s-1)\log p_r)\right) \prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right),$$

where $G(s, p_r)$ is a regular (analytic and single valued) function for $\Re(s)>1$. Referring to Equation (4.10), the function $G(s, p_r)$ approaches 1 as p_r approaches infinity. It should be noted that $G(s, p_r)$ can be considered as a sequence of analytic functions. Furthermore, as p_r (or *r*) approaches infinity, this sequence is uniformly convergent over the half plane $\sigma>1$. Thus, using Weiestrass theorem , the limit is also an analytic function [6] (Weiestrass theorem states that if the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then f is also analytic on Ω and f'_n converges uniformly to f' on Ω). If we define this limit as $G(s)$, where

(4.12) $$G(s)=\lim_{r\to\infty} G(s, p_r),$$

then, $G(s)$ is an analytic function over the half plane $\sigma>1$ and it is equal to 1 by the virtue of Equation (4.10).

The Prime Number Theorem (PNT) allows us to extend the above results to the line $s=1+it$. Moreover, we will show that if RH is valid, then for the strip $s=\sigma+it$ where $0.5<\sigma<1$, the above results will also be valid with the limit of $G(s, p_r)$ is 1 as p_r approaches infinity.

We will start this task by showing that although both $\zeta(s)$ and $E_1((s-1)\log p_r)$ have a singularity at *s*=1, the product $G(s, p_r)$ has a removable singularity at *s*=1, Furthermore, we will show that the function $G(s, p_r)$ converges for every p_r with $s=1+it$. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity $s=1$

(4.13) $$\zeta(s) = \frac{1}{s-1}+\gamma-\gamma_1(s-1)+\gamma_2\frac{(s-1)^2}{2!}-\gamma_3\frac{(s-1)^3}{3!}+ \ldots,$$

where γ is the Euler-Mascheroni constant and γ_i 's are the Stieltjes constants. For $s=1+\varepsilon$, where $\varepsilon=\epsilon+i\delta$, ϵ and δ are arbitrary small numbers, the above equation can be written as

$$\zeta(s) = \frac{1}{\varepsilon}+\gamma-\gamma_1\varepsilon+\gamma_2\frac{\varepsilon^2}{2!}-\gamma_3\frac{\varepsilon^3}{3!}+ \ldots .$$

Furthermore, for the right side complex plain, using the definition of the Exponential Integral $E_1(s)$, one may write $E_1(s)$ as

(4.14) $$E_1(s) = -\gamma-\log s+s-\frac{s^2}{2!2}+\frac{s^3}{3!3}-\frac{s^4}{4!4}+ \ldots .$$

Thus, for $s=1+\varepsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right) \;=\; e^{\gamma}\,\varepsilon\,\log p_r\,\exp\left(-\varepsilon\log p_r+\frac{(\varepsilon\log p_r)^2}{2\,2!}-\frac{(\varepsilon\log p_r)^3}{3\,3!}+....\right).$$

By taking the product $\zeta(s)\,\exp\{-E_1((s-1)\log p_r)\}$ and allowing ε to approach zero, one may then obtain at $s=1$ (in the same sense as computing (sin *x*)/*x* at *x*=0)

$$\zeta(s)\,\exp\left(-E_1((s-1)\log p_r)\right)=e^{\gamma}\,\log p_r\,. \tag{4.15}$$

However, it well known that the partial Euler product at $s=1$ can be written as [3]

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) \;=\; \frac{e^{-\gamma}}{\log p_r}+O\left(\frac{1}{(\log p_r)^2}\right). \tag{4.16}$$

Multiplying Equations (4.15) and (4.16), one may conclude that at $s=1$, $G(s\,,p_r)$ approaches 1 as p_r approaches infinity. Furthermore, for $s=1+it$ and $t\neq0$, the value of $\exp(-E_1(jx))$ approaches 1 as |*x*| (or p_r) approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{s}}\right)\right\}=1\,,$$

therefore, we conclude that, for $s=1+it$, we have the following

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{s}}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\} \;=\; 1\,.$$

To analyze the function $G(s\,,p_r)$ in the strip $0.5<\Re(s)<1$ assuming RH is valid, we first notice that the function $G(s\,,p_r)$ is an entire function for every p_r. In the following (assuming RH is valid), we will prove that, for $0.5<\sigma<1$, the function $G(\sigma\,,p_r)$ is convergent as p_r approaches infinity. In Appendix 3, we will extends these results for the strip $0.5<\sigma<1$. To show that the function $G(\sigma\,,p_r)$ is convergent as p_r approaches infinity, we first write the expressions for $G(\sigma\,,p_{r1})$ and $G(\sigma\,,p_{r2})$ (where $r1<r2$)

$$G(\sigma\,,p_{r1})=\zeta(\sigma)\,\exp\left(-E_1((\sigma-1)\log p_{r1})\right)\prod_{i=1}^{r1-1}\left(1-\frac{1}{p_i^{\sigma}}\right), \tag{4.17a}$$

$$G(\sigma\,,p_{r2})=\zeta(\sigma)\,\exp\left(-E_1((\sigma-1)\log p_{r2})\right)\prod_{i=1}^{r2-1}\left(1-\frac{1}{p_i^{\sigma}}\right). \tag{4.17b}$$

Dividing Equation (4.17b) by Equation (4.17a) and taking the logarithm

$$\log\left(\frac{G(\sigma\,,p_{r2})}{G(\sigma\,,p_{r1})}\right)=E_1\left((\sigma-1)\log p_{r1}\right)-E_1\left((\sigma-1)\log p_{r2}\right)+\log\left(\prod_{i=r1-1}^{r2-1}\left(1-\frac{1}{p_i^{\sigma}}\right)\right). \tag{4.18}$$

To compute the logarithm of the partial Euler product in the above equation, we recall Equation (4.2)

$$\log\left(\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^{\sigma}}\right)\right) = \sum_{i=r1}^{r2}\left(-\frac{1}{p_i^{\sigma}}-\frac{1}{2\,p_i^{2\sigma}}-\frac{1}{3\,p_i^{3\sigma}}-\dots\right) = -\sum_{i=r1}^{r2}\frac{1}{p_i^{\sigma}}+\delta\,,$$

where $\delta=O(p_{r_1}{}^{1-2\sigma})$. Also, since we are assuming that RH is valid, then

(4.19) $$\pi(x) = \mathrm{Li}(x)+O\left(\sqrt{x}\log x\right),$$

where Li(x) is the Logarithmic Integral of x. Using this representation for the prime counting function, one may then obtain (Appendix 2)

$$\sum_{i=r1-1}^{r2-1}\frac{1}{p_i^{\sigma}} = E_1((\sigma-1)\log p_{r1})-E_1((\sigma-1)\log p_{r2})+\varepsilon\,,$$

where $\varepsilon=O\left(\frac{t}{(\sigma-0.5)^2}\,p_{r1}{}^{1/2-\sigma}\log(p_{r1})\right)$ and Equation (4.18) can be written as

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = \varepsilon+\delta\,,$$

where, for $\sigma>0.5$, $\varepsilon+\delta$ approaches zero as p_{r1} approaches infinity. Hence,

$$G(\sigma, p_{r2}) = G(\sigma, p_{r1})\,e^{\varepsilon+\delta}\,.$$

Since $\varepsilon+\delta$ can be made arbitrary small by choosing p_{r1} arbitrary large, thus the limit of $G(\sigma, p_{r2})$ as p_{r2} approaches infinity exists and is given by

$$G(\sigma) = \lim_{p_{r2}\to\infty} G(\sigma, p_{r2})\,.$$

This proves that, assuming RH, $G(\sigma, p_r)$ is convergent as p_{r_1} approaches infinity and thus $G(\sigma)$ exists for $\sigma>0.5$. In Appendix 3, we have proved the same results for $s=\sigma+it$ where $\sigma>0.5$. In other words; we have proved that assuming RH is valid, then for $\sigma>0.5$, $G(s, p_r)$ is convergent as p_r approaches infinity and therefore $G(s)$ exists for $\sigma>0.5$.

It should be noted that while the function sequence $G(s, p_r)$ is not uniformly convergent when the region of convergence is extended to the line $\sigma=0.5$. It is however uniformly convergence for any strip $0.5+\epsilon<\sigma<1$ where ϵ is an arbitrary small number. The uniform convergence is a necessary requirement for the function sequence $G(s, p_r)$ to converge to an analytic function (Weiestrass theorem [6]). Since $G(s, p_r)$ is uniformly convergent in the strip $0.5+\epsilon<\sigma<1$, therefore $G(s)$ is an analytic function. Furthermore, since this strip ($0.5+\epsilon<\sigma<1$) is connected to the half plane ($\sigma\geq 1$) where the function $G(s)$ is equal to 1, thus the function $G(s)$ should also be equal to 1 for $0.5+\epsilon<\sigma<1$, where ϵ is arbitrary small. Hence, we have the following theorem,

Theorem 3 (Main Theorem 1): For $s=\sigma+it$ and $\sigma>0.5$, the followings hold if RH is valid

$$\text{(4.20)} \qquad \lim_{r\to\infty}\left\{\zeta(s)\,\exp\!\left(-E_1((s-1)\log p_r)\right)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{\,s}}\right)\right\}=1\ ,$$

and

$$\text{(4.21)} \qquad \lim_{r\to\infty}\left\{Mu(s\,,\,p_r)\;\;\exp\!\left(E_1((s-1)\log p_r)\right)\right\}=1\ .$$

Using Theorem 3, we can provide on RH an estimate of how well the partial Euler product represents $\zeta(s)$ in the strip $0.5<\sigma<1$. Referring to Appendix 3 and Equation (A3-3), where we set $p_{r1}=p_r$ and let p_{r2} approaches infinity, one may write Equation (4.20) for $\sigma>0.5$ as follows

$$\text{(4.22)} \qquad \frac{1}{\zeta(s)}\;=\;\exp\!\left(-E_1((s-1)\log p_r)\right)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{\,s}}\right)\;\exp\!\left(O\!\left(\frac{t}{\sigma-0.5}\,p_r^{\;1/2-\sigma}\log p_r\right)\right).$$

Equation (4.22) is obtained using Theorem 3 to have the following

$$\log\,\zeta(s)\;=\;E_1((s-1)\log p_{r2})-\sum_{i=1}^{r2}\log\left(1-\frac{1}{p_i^{\,s}}\right),$$

where the equality of both sides is attained as p_r approaches infinity. It should be pointed out that both functions $\log\zeta(s)$ and $E_1((s-1)\log p_{r2})$ have a branch cut along the real axis where $0.5<\sigma<1$. Alternatively, the above equation can be written as

$$\log\,\zeta(s)\;=\;E_1((s-1)\log p_{r2})-\sum_{i=1}^{r}\log\left(1-\frac{1}{p_i^{\,s}}\right)-\sum_{i=r}^{r2}\log\left(1-\frac{1}{p_i^{\,s}}\right).$$

Since

$$-\sum_{i=r}^{r2}\log\left(1-\frac{1}{p_i^{\,s}}\right)=-E_1((s-1)\log p_{r2})+E_1((s-1)\log p_r)+O\!\left(\frac{t}{(\sigma-0.5)^2}\,p_r^{\;1/2-\sigma}\log(p_r)\right),$$

therefore

$$\text{(4.23)} \qquad \log\,\zeta(s)\;=\;E_1((s-1)\log p_r)-\sum_{i=1}^{r}\log\left(1-\frac{1}{p_i^{\,s}}\right)+O\!\left(\frac{t}{(\sigma-0.5)^2}\,p_r^{\;1/2-\sigma}\log(p_r)\right).$$

From which Equation (4.22) follows. For sufficiently large p_r , Equation (4.22) can be written as

$$\frac{1}{\zeta(s)} \;=\; \prod_{i=1}^{r}\left(1-\frac{1}{p_i^{\,s}}\right)\;\exp\big(-E_1((s-1)\log p_r)\big)\;\left(1+O\left(\frac{t}{(\sigma-0.5)^2}\,p_r^{\,1/2-\sigma}\log p_r\right)\right).$$

For the cases where $E_1((s-1)\log p_r)=0$ (or, $\exp\big(-E_1((s-1)\log p_r)\big)=1$), one obtains

$$\zeta(s)\;\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{\,s}}\right) \;=\; 1+O\left(\frac{t}{(\sigma-0.5)^2}\,p_r^{\,(1/2-\sigma)^2}\log p_r\right).$$

For $0.5<\sigma<1$, one way to achieve this result (i.e. $E_1((s-1)\log p_r)=0$) is to choose *t* sufficiently large. In this case, we take advantage of the following asymptotic representation of the Exponential Integral

(4.24)
$$E_1(z)=\frac{e^{-z}}{z}\left(1+O\left(\frac{1}{z}\right)\right).$$

With $z=(\sigma+it-1)\log p_r$, one may then obtain

$$\left|E_1((s-1)\log p_r)\right| \;=\; \frac{p_r^{\,1-\sigma}}{t\log p_r}\left(1+O\left(\frac{1}{t\log p_r}\right)\right).$$

Thus,

(4.25)
$$\left|\exp\big(E_1((s-1)\log p_r)\big)\right| \;=\; 1+O\left(\frac{p_r^{\,1-\sigma}}{t\log p_r}\right).$$

By the virtue of Equation (4.22), we then have

(4.26)
$$\zeta(s)\;\prod_{p_i=2}^{r}\left(1-\frac{1}{p_i^{\,s}}\right) \;=\; 1+O\left(\frac{t}{(\sigma-0.5)^2}\,p_r^{\,1/2-\sigma}\log p_r\right)+O\left(\frac{p_r^{\,1-\sigma}}{t\log p_r}\right).$$

For $0.75<\sigma<1$ and assuming RH is valid, by choosing *t* and p_r such that $p_r^{1-\sigma}\ll t\ll p_r^{\sigma-0.5}$, we may use the partial Euler product as a good representation for $\zeta(s)$. In the next section, we will use the von Manglodt function to provide a better estimate for the first *O* term that allows the use of the partial Euler product over the range $0.5<\sigma<1$ with the proper choice of *t* and p_r .

The rest of this section will be devoted to the derivation of an expression for $\zeta'(s)$ using Equation (4.20). As mentioned earlier, assuming the validity of RH, the function sequence $G(s\,,p_r)$ (defined in Equation (4.11)) is analytic and uniformly convergent for any $\sigma>0.5+\epsilon$. Thus, by the Weiestrass theorem, the limit of $G(s\,,p_r)$ as *r* approaches infinity is also an analytic function. We denoted this limit as $G(s)$ and we showed that $G(s)=1$. Weiestrass theorem also states that as *r* approaches infinity, the derivative function sequence $G^{'}(s\,,p_r)$ should also converge uniformly to $G'(s)=0$.

Thus, assuming the validity of RH and for $\sigma > 0.5$, we obtain

$$(4.27) \qquad \zeta'(s) \lim_{r\to\infty}\left\{\exp\left(-E_1((s-1)\log p_r)\right) \prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\right\} = -\zeta(s)\frac{d}{ds}\left(\lim_{r\to\infty}\left\{\exp\left(-E_1((s-1)\log p_r)\right) \prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\right\}\right).$$

For $s=1+it$ where $t\neq 0$, we have $\lim_{r\to\infty} E_1((s-1)\log p_r)=0$. We also have for $s=1+it$ where $t\neq 0$

$$\zeta(s) = \lim_{r\to\infty}\left\{\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)^{-1}\right\}.$$

Thus, for $s=1+it$ where $t\neq 0$, one can write Equation (4.27) as

$$\zeta'(s) = -\zeta^2(s)\frac{d}{ds}\left(\lim_{r\to\infty}\left\{\exp\left(-E_1((s-1)\log p_r)\right) \prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\right\}\right).$$

Hence,

$$(4.28) \qquad \zeta'(s) = \zeta(s) \lim_{r\to\infty}\left\{\frac{e^{-(s-1)\log p_r}}{(s-1)} - \sum_{i=1}^{r}\frac{\log p_i}{p_i^s\left(1-p_i^{-s}\right)}\right\},$$

or

$$(4.29) \qquad \zeta'(s) = \zeta(s) \lim_{r\to\infty}\left\{\frac{e^{-it\log p_r}}{(it)} - \sum_{i=1}^{r}\frac{\log p_i}{p_i^s\left(1-p_i^{-s}\right)}\right\}.$$

Each term (of the the two terms) on the right side of Equation (4.28) doesn't converge as p_r approaches infinity. However, we can show that the sum of these two terms converges as p_r approaches infinity. If we denote $\zeta'(s,r)$, for $t\neq 0$, as

$$\zeta'(s,r) = \zeta(s)\left\{\frac{e^{-it\log p_r}}{(it)} - \sum_{i=1}^{r}\frac{\log p_i}{p_i^s\left(1-p_i^{-s}\right)}\right\},$$

then

$$\zeta'(s,r2) - \zeta'(s,r1) = \zeta(s)\left\{\frac{e^{-it\log p_{r2}}}{(it)} - \frac{e^{-it\log p_{r1}}}{(it)} - \sum_{i=r1}^{r2}\frac{\log p_i}{p_i^s\left(1-p_i^{-s}\right)}\right\}.$$

In Appendix 4, we have shown that using the prime number theorem, the above equation can be written as

$$\zeta'(s, r2) - \zeta'(s, r1) = \zeta(s) O(1/\log p_{r1}) .$$

Thus, the limit of the sum on the right side of Equation (4.29) exists (although each term does not converge). Therefore, Equation (4.29) can be used to compute $\zeta'(s)$ on the line $\sigma=1$ (where $t \neq 0$).

5- Partial Euler Product Functional Representation of $\zeta(s)$ Using the von Mangoldt Function:

In the previous section, we have shown (Equation 4.23) that

$$\text{(5.1)} \quad \log \zeta(s) = -\sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) + E_1((s-1)\log p_r) + O\left(\frac{t}{(\sigma-0.5)^2} p_r^{1/2-\sigma} \log(p_r)\right) .$$

The *O* term is derived using the prime counting function (Equation (4.19)). Equation (5.1) represents well the singularity at *s*=1 and it allows analytic continuation for values of *s* with $\sigma<1$. This analytic continuation should extend all the way to the non-trivial zero('s) with the highest value of σ . Unfortunately, Equation (5.1) poorly represents the singularities in the critical strip (or non-trivial zeros) as the *O* term grows much faster than the growth of $\log \zeta(s)$ in the vicinity of the simple non-trivial zeros. In the following, we will use the von Mangoldt function to provide a better representation for $\log \zeta(s)$ in the vicinity of the no-trivial zeros.

The derivation of Equation (5.1) was based on computing the sum $\sum_{i=r1}^{r2} 1/p_i^s$ (see Appendix 3) as follows

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^s} d\pi(x) = \int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^s} d\, O(\sqrt{x} \log x) .$$

The above sum can be also computed using the von Mangoldt function $\Lambda(n)$ (where $\Lambda(n)=\log p$ if $n=p^k$ for some prime *p* and integer $k \geq 1$, otherwise $\Lambda(n)=0$) to obtain

$$\text{(5.2)} \qquad \sum_{i=r1}^{r2} \frac{1}{p_i^s} = \sum_{n=r1}^{r2} \frac{1}{n^s \log n} \Lambda(n) + \delta ,$$

where $\delta = O\left(p_{r1}^{(0.5-\sigma)}\right)$ is added to eliminate the contribution by the terms of the form m^{-s} , where $m=p^k$ and $2 \leq k < \log_2 p_{r2}$.

Since the Chebyshev function $\psi(x)$ is given by the following sum

$$\psi(x) = \sum_{n=1}^{x} \Lambda(n) ,$$

therefore, using the Stieltjes integral, one may write the sum of Equation (5.2) as the following integral

$$(5.3)\qquad \sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{r1}^{r2} \frac{1}{x^s \log x}\, d\,\psi(x) + \delta\,,$$

where $\psi(x)$ is also given by [1]

$$(5.4)\qquad \psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(o)}\,.$$

It should be pointed out that the first term *x* in Equation (5.4) is attributed to the pole of $\zeta(s)$ at *s*=1, the sum over ρ (or non-travail zeros) is attributed to the non-trivial zeros in the critical strip and the sum over *n* is attributed to the trivial zeros. Hence, Equation (5.3) can be written as

$$(5.5)\qquad \sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{P_{r1}}^{p_{r2}} \frac{1}{x^s \log x}\, d\,x - \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x}\, d\left(\sum_{\rho} \frac{x^{\rho}}{\rho}\right) + \delta\,\cdot$$

In Appendix 3, we have shown that

$$(5.6)\qquad \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x}\, d\,x = -E_1((s-1)\log p_{r2}) + E_1((s-1)\log p_{r1})\,\cdot$$

Similarly, for $\sigma > 0.5$, we can also show that

$$(5.7)\qquad \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x}\, d\left(\sum_{\rho} \frac{x^{\rho}}{\rho}\right) = \sum_{\rho} \left(-E_1((s-\rho)\log p_{r2}) + E_1((s-\rho)\log p_{r1})\right)\,\cdot$$

For the above integral, the interchange between the differentiation and summation is permissible by the virtue of the convergence of the sum $\sum_{\rho} (x^{\rho}/\rho)$ (alternatively, one may integrate by parts to get the same results where the sum becomes the integrand and the differentiation is applied to the term $1/(x^s \log x)$ instead of the sum). Also, the interchange between the integral and the sum is permissible by the virtue of the convergence of the sum on the right side of Equation (5.7) for values of *s* with σ higher than $\Re(\rho)$ for every ρ. The proof that this sum is convergent is outlined in Appendix 5.

Using Theorem (3), on RH and for $\sigma > 0.5$, we have shown that

$$\log \zeta(s) = E_1((s-1)\log p_{r2}) - \sum_{i=1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right)\,,$$

where the equality of both sides is attained as p_{r2} approaches infinity. Alternatively,

$$\log \zeta(s) = E_1((s-1)\log p_{r2}) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r}^{r2} \log\left(1 - \frac{1}{p_i^s}\right)\,,$$

or

$$(5.8) \qquad \log \zeta(s) = E_1((s-1)\log p_{r2}) - \sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^s}\right) + \sum_{i=r}^{r2}\left(\frac{1}{p_i^s}\right) + O(p_{r1}^{(1-2\sigma)}) .$$

Using Equations (5.5), (5.6) and (5.7) with $p_{r1}=p_r$ and noting that $E_1((s-\rho)\log p_{r2})$ in Equation (5.7) approaches zero when $\Re(s-\rho)\geq 0$ and $s\neq\rho$ (for every ρ and $\Re(s)>0.5$) as p_{r2} approaches infinity. we may then have the following theorem.

Theorem 4 (Main Theorem 2): If $\Re(s-\rho)\geq 0$ and $s\neq\rho$ for every non-trivial zero ρ where $\Re(s)>0.5$, then

$$(5.9) \quad \log \zeta(s) = -\sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^s}\right) + E_1((s-1)\log p_r) - \sum_{\rho} E_1((s-\rho)\log p_r) + \delta$$

where, $\delta=O(p_{r1}^{(0.5-\sigma)})$.

The differentiation of $\log\zeta(s)$ or $\zeta'(s)/\zeta(s)$ has been extensively used in the in the analysis of the Riemann zeta function. Using Equation (5.9), we may have the following expression for $\zeta'(s)/\zeta(s)$

Corollary 1: If $\Re(s-\rho)\geq 0$ and $s\neq\rho$ for every non-trivial zero ρ where $\Re(s)>0.5$, then

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\left(\log \prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\right) - \frac{p_r^{-(s-1)}}{s-1} + \sum_{\rho}\frac{p_r^{-(s-\rho)}}{s-\rho} + \delta$$

where, $\delta=O(p_{r1}^{(0.5-\sigma)})$.

Examining Equation (5.9), one may note that in the vicinity to the right of the non-trivial zero ρ_m , $\log\zeta(s)$ grows as $\log(s-\rho_m)$ and therefore $\log\zeta(s)$ approaches $-\infty$ as *s* approaches ρ_m . However, in the vicinity of ρ_m , the terms $-\sum_{i=1}^{r}\log(1-1/p_i^s)$ and $E_1((s-1)\log p_r)$ of Equation (5.9) are analytic. Therefore, in the vicinity of ρ_m , Equation (5.9) gives the value of $\log\zeta(s)$ as

$$(5.10) \quad \log \zeta(s) = -\sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^{\rho_m}}\right) + E_1((\rho_m-1)\log p_r) + O(|s-\rho_m|) - \sum_{\rho} E_1((s-\rho)\log p_r) + \delta .$$

Furthermore, examining Equation (5.10), one may notice that as *s* approaches ρ_m , $\log\zeta(s)$ can be written as $\log(s-\rho_m)+O(1)$. Moreover, as *s* approaches ρ_m and p_r approaches infinity, $\sum_{\rho} E_1((s-\rho)\log p_r)$ can be written as $E_1((s-\rho_m)\log p_r)+O(1)$. Hence, we may re-write Equation (5.10) as follows

$$\log(s-\rho_m)+O(1) = -\sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^{\rho_m}}\right) + E_1((\rho_m-1)\log p_r) + \log(s-\rho_m) + \log\log p_r$$

Thus, as s approaches ρ_m, we have

$$\sum_{i=1}^{r} \log\left(1-\frac{1}{p_i^{\rho_m}}\right) - E_1((\rho_m-1)\log p_r) = \log\log p_r + O(1) \tag{5.11}$$

5a- Using Theorem 4 for estimating the sums of the form $\sum_{i=1}^{r} (\log p_i)^m / p_i$:

The following equation has been used in the literature to to estimate the sums of the form $\sum_{i=1}^{r} (\log p_i)^m / p_i$.

$$\sum_{p\le x} f(p) = \int_2^x \frac{f(y)}{\log y} dx + f(2)\,\mathrm{li}(2) + f(x)(\pi(x)-\mathrm{li}(x)) - \int_2^x f'(y)(\pi(y)-\mathrm{li}(y))dy .$$

In the following, we will use corollary 1 of theorem 4 to compute these sums. We will start with task of computing the sum $\sum_{i=1}^{r} \log p_i / p_i$. Toward this end, we set $s=1+\epsilon$ (where ϵ is an arbitrary small positive number) to obtain

$$\frac{\zeta'(1+\epsilon)}{\zeta(1+\epsilon)} = -\sum_{i=1}^{r} \frac{\log p_i}{p_i^{1+\epsilon}(1-p_i^{-1-\epsilon})} - \frac{p_r^{-\epsilon}}{\epsilon} + \sum_{\rho} \frac{p_r^{-(1+\epsilon-\rho)}}{1+\epsilon-\rho} + O(p_r^{0.5-\sigma}) .$$

Since

$$\zeta(1+\epsilon) = \frac{1}{\epsilon} + \gamma - \gamma_1\epsilon + \gamma_2\frac{\epsilon^2}{2!} - \gamma_3\frac{\epsilon^3}{3!} + \ldots ,$$

and

$$\zeta'(1+\epsilon) = -\frac{1}{\epsilon^2} - \gamma_1 + \gamma_2\epsilon - \gamma_3\frac{\epsilon^2}{2!} + \ldots ,$$

thus

$$\frac{\zeta'(1+\epsilon)}{\zeta(1+\epsilon)} = -\frac{1}{\epsilon} + \gamma + O(\epsilon) .$$

Also, we have

$$\frac{p_r^{-\epsilon}}{\epsilon} = \frac{1}{\epsilon} + \log p_r + O(\epsilon) .$$

Therefore, as ϵ approaches zero, we obtain

(5.12) $$\sum_{i=1}^{r} \frac{\log p_i}{p_i(1-p_i^{-1})} = \log p_r - \gamma + \Delta ,$$

where

$$\Delta = \sum_{\rho} \frac{p_r^{-(1-\rho)}}{1-\rho} + O(p_r^{0.5-\sigma}) .$$

Furthermore, the above equation can be also written as

(5.13) $$\sum_{i=1}^{r} \frac{\log p_i}{p_i} = \log p_r - \gamma - \sum_{i=1}^{r} \frac{\log p_i}{p_i^2(1-p_i^{-1})} + \Delta .$$

Similarly, we can compute the sum $\sum_{i=1}^{r} (\log p_i)^2 / p_i$. Toward this end, we differentiate the ratio $\zeta'(s)/\zeta(s)$ with respect to s and then set $s=1+\epsilon$ to obtain

$$\frac{\zeta''(1+\epsilon)}{\zeta(1+\epsilon)} - \frac{(\zeta'(1+\epsilon))^2}{(\zeta(1+\epsilon))^2} = -\sum_{i=1}^{r} \frac{\log p_i}{p_i^{1+\epsilon}(1-p_i^{-1-\epsilon})^2} - \frac{p_r^{-\epsilon}}{\epsilon^2} + \frac{d\Delta}{d\epsilon}$$

Since

$$\frac{\zeta''(1+\epsilon)}{\zeta(1+\epsilon)} = \frac{2}{\epsilon^2} - \frac{2\gamma}{\epsilon} + 2\gamma_1 + O(\epsilon) ,$$

$$\frac{(\zeta'(1+\epsilon))^2}{(\zeta(1+\epsilon))^2} = \frac{1}{\epsilon^2} - \frac{2\gamma}{\epsilon} + \gamma^2 + O(\epsilon) ,$$

and

$$\frac{p_r^{-\epsilon}}{\epsilon^2} = \frac{1}{\epsilon^2} - \frac{\log p_r}{\epsilon} + \frac{(\log p_r)^2}{2} + O(\epsilon) .$$

therefore, as ϵ approaches zero, we obtain

$$\sum_{i=1}^{r} \frac{(\log p_i)^2}{p_i(1-p_i^{-1})^2} = \frac{(\log p_r)^2}{2} + 2\gamma_1 - \gamma^2 + \frac{d\Delta}{d\epsilon} .$$

The above equation can be also written as

(5.14) $$\sum_{i=1}^{r} \frac{(\log p_i)^2}{p_i} = \frac{(\log p_r)^2}{2} + 2\gamma_1 - \gamma^2 + \sum_{j=1}^{\infty} \frac{1}{j+1} \left(\sum_{i=1}^{r} \frac{(\log p_i)^2}{p_i^{j+1}} \right) + \frac{d\Delta}{d\epsilon} .$$

Similarly, sums of the form $\sum_{i=1}^{r} (\log p_i)^m / p_i$ for *m* greater than 2 can be computed.

5b- The sum $\sum_{i=r1}^{r2} 1/p_i^s$ and Riemann Hypothesis:

In the following, we will use theorem 4 to compute the sum $\sum_{i=r1}^{r2} 1/p_i^s$ and the role of the roots of the Riemann zeta function on its convergence. Toward this end, we assume that there are non-trivial zeros off the critical line and we assume that the right half plane with $\Re(s)>a$ (where $a>0.5$) is void of non-trivial zeros. In other words, we assume that non-trivial zeros exist with $\Re(s)=a$ or with $\Re(s)$ arbitrary close but not equal to *a*. We then define the function $J(s, p_{r1}, p_{r2})$ as

$$(5.15) \qquad J(s, p_{r1}, p_{r2}) = \sum_{i=r1}^{r2} \frac{1}{p_i^s} + E_1((s-1)\log p_{r2}) - E_1((s-1)\log p_{r1}).$$

The function $J(s, p_{r1}, p_{r2})$ is analytic for every p_{r1}, p_{r2} and *s*. This can be shown by noting that although the functions $E_1((s-1)\log p_{r1})$ and $E_1((s-1)\log p_{r2})$ have a branch cut on the negative real axis, the difference does not have a branch cut. Moreover, although the functions $E_1((s-1)\log p_{r1})$ and $E_1((s-1)\log p_{r2})$ have a singularity at $s\neq 1$, the difference has a removable singularity at $s\neq 1$. This follows from the fact the as *s* approaches zero, the difference can be written as

$$E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) = -\log((s-1)\log p_{r1}) - \gamma + E_1((s-1)\log p_{r2}) + \gamma$$

or,

$$E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) = -\log\log p_{r1} + \log\log p_{r2}$$

Hence, the function $J(s, p_{r1}, p_{r2})$ is analytic for every p_{r1}, p_{r2} and *s*.

Using Theorem 4, we then have

$$(5.16) \quad J(s, p_{r1}, p_{r2}) = \sum_{\rho} \left(-E_1((s-\rho)\log p_{r2}) + E_1((s-\rho)\log p_{r1})\right) + O\left(p_{r1}^{(0.5-\sigma)}\right),$$

and referring to Appendix 5, we then obtain (if $|s-\rho|\geq\epsilon>0$ for every ρ)

$$(5.17) \qquad J(s, p_{r1}, p_{r2}) = \frac{p_{r2}^{-s}}{\log p_{r2}} \sum_{\rho} \frac{p_{r2}^{\rho}}{s-\rho} - \frac{p_{r1}^{-s}}{\log p_{r1}} \sum_{\rho} \frac{p_{r1}^{\rho}}{s-\rho} + O(p_{r1}^{(0.5-\sigma)}).$$

Hence,

$$J(s, p_{r1}, p_{r2}) = O\left(\frac{p_{r1}^{a-\sigma}}{\log p_{r1}}\right) + O(p_{r1}^{(0.5-\sigma)}).$$

Consequently, on RH, we have

(5.18) $$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = -E_1((s-1)\log p_{r2}) + E_1((s-1)\log p_{r1}) + O(p_{r1}^{(0.5-\sigma)}).$$

Also, on RH and for $s=1$, we have

$$\sum_{i=r1}^{r2} \frac{1}{p_i} = \log\log p_{r2} - \log\log p_{r1} + O(p_{r1}^{(0.5-\sigma)}).$$

Thus, we conclude that the sum $\sum_{i=r1}^{r2} 1/p_i^{\sigma}$ is convergent for $0.5<\sigma<1$ if and only if the Riemann Hypothesis is true.

6- Representation of $Mu(s, p_r)$ using Mellin Transform:

In section 2, we showed that the series $Mu(s, p_r)$ converges absolutely or conditionally wherever the series $Mu(s)$ converges absolutely or conditionally. Furthermore, if series $Mu(s, p_r)$ is convergent, then it can written as

(6.1) $$Mu(s, p_r) = \frac{Mu(s)}{\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)} = \frac{1}{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)}.$$

The Mertens function *M(x)* is defined as

$$M(x) = Mu(0+i0; 1, x) = \sum_{n=1}^{x} u(n).$$

Similarly, we can define the function $M(x, p_r)$ as

(6.2) $$M(x, p_r) = Mu(0+i0, p_r; 1, x) = \sum_{n=1}^{x} u(n, p_r).$$

Using the Stieltjes integral, for $\Re(s)>1$ (or $\Re(s)>0.5$ on RH), Equation (6.1) can be written as

$$\frac{1}{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)} = \int_0^{\infty} x^{-s}\, dM(x, p_r).$$

Using integration by parts, one can write the above equation as a Mellin transform, (6.3) $$\frac{1}{s\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)} = \int_0^\infty x^{-s-1} M(x, p_r)\,dx .$$

where $\Re(s)>1$.

Referring to Equation (6.3), one may use the Mellin inversion theorem to compute $M(x, p_r)$ as the integral

(6.4) $$M(x, p_r) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)}\,ds ,$$

where the integral is valid for $\sigma>1$.

Theorem 5: For $\sigma>1$, the functions $M(x, p_r)=\sum_{n=1}^{x} u(n, p_r)$ and $1/\left(s\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\right)$ are Mellin transform pair where

(6.5a) $$\frac{1}{s\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)} = \int_0^\infty x^{-s-1} M(x, p_r)\,dx ,$$

(6.5b) $$M(x, p_r)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)}\,ds .$$

Similar results can be obtained using Perron's formula for $\Re(s)>1$. We first recall Lemma 3.12 in Reference [7] that states:

Let $$f(z)=\sum_{n=1}^{\infty}\frac{a_n}{n^z} \quad \text{for } \Re(z)>1 ,$$

where $a_n=O\{\psi(n)\},\ \psi(n)$ being non-decreasing and as $\sigma\to 1$

$$\sum_{n=1}^{\infty}\frac{a_n}{n^z}=O\left\{\frac{1}{(\sigma-1)^\alpha}\right\} .$$

Then, if $c>0,\ \sigma+c>1$ and *x* is an integer, we have

$$\sum_{n=1}^{x-1} \frac{a_n}{n^z} + \frac{a_x}{2\,x^z} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(z+s) \frac{x^s}{s} ds + O\left\{\frac{x^c}{T(\sigma+c-1)^{\alpha}}\right\} + O\left\{\frac{\psi(2x)\, x^{1-\sigma} \log x}{T}\right\} + O\left\{\frac{\psi(x)\, x^{-\sigma}}{T}\right\} .$$

Applying this lemma to the series $Mu(s, p_r)$, where $c>1$ and $\alpha=1$, one may obtain

$$\sum_{n=1}^{x-1} \frac{\mu(n, p_r)}{n^z} + \frac{\mu(x, p_r)}{2\,x^z} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\,\zeta(z+s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{z+s}}\right)} ds$$

$$+ O\left\{\frac{x^c}{T(\sigma+c-1)}\right\} + O\left\{\frac{x^{1-\sigma} \log x}{T}\right\} + O\left\{\frac{x^{-\sigma}}{T}\right\} .$$

Let x be an even number (thus, $\mu(x, p_r)=0$). Hence, as T approaches infinity, one can state the following theorem:

Theorem 6: For $\Re(z)+c>1$ and $c > 0$,

$$Mu(z, p_r; 1, x) = \sum_{n=1}^{x} \frac{\mu(n, p_r)}{n^z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s\,\zeta(z+s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{z+s}}\right)} ds .$$

Hence, for $z= 0+i0$ and $c > 1$

(6.6) $$M(x, p_r) = \sum_{n=1}^{x} \mu(n, p_r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s\,\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{s}}\right)} ds .$$

For $\sigma>1$, Equation (6.6) of Theorem 6 can be written as

(6.7) $$M(x, p_r) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s} \prod_{i=r}^{\infty} \left(1 - \frac{1}{p_i^{s}}\right) dt .$$

Equation (6.7) becomes evident if one denotes A as the integral

$$A = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(x\,y)^s}{s} dt .$$

then, $A=1$ if $x\,y>1$, $A=0$ if $x\,y<1$ and $A=0.5$ if $x\,y=1$ [1].

In the following section, we will use Theorems 5 and 6 to drive a formula for the prime counting function.

7- The Prime Counting Function $\pi(x)$:

The prime counting function $\pi(x)$ is a function that gives the number of primes less than or equal to *x*. In this section, we will use Theorem 5 to compute this function. We first recall that

$\mu(x, p_r)=0$ for $x<p_r$,
$\mu(x, p_r)=1$ if x is a prime number and $p_r<x<p_r^2$,
$\mu(x, p_r)=0$ if x is a composite number and $p_r<x<p_r^2$.

Thus, for $p_r<x<p_r^2$, $M(x, p_r)=\pi(x)-\pi(p_r)$. By the virtue of Theorem 5, one can state the following theorem.

Theorem 7: For $p_r<x<p_r^2$, the prime counting function $\pi(x)$ is given by

$$\pi(x) \;=\; \pi(p_r)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)}dt\,, \tag{7.1}$$

where $\sigma>1$.

Note that for $\sigma>1$ and $p_r<x<p_r^2$, Equation (7.1) can be written as

$$\pi(x) \;=\; \pi(p_r)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\prod_{i=r}^{\infty}\left(1-\frac{1}{p_i^s}\right)dt\,. \tag{7.2}$$

For sufficiently large *r* and for $p_r<x<p_r^2$, , one may also write Equation (7.2) as follows

$$\pi(x) \;=\; \pi(p_r)+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\exp\left(-\sum_{i=r}^{\infty}p_i^{-s}\right)dt\,.$$

The sum in the exponent can be written as the following integral

$$\sum_{i=r}^{\infty}p_i^{-s} \;=\; \int_{p_r}^{\infty}\frac{d\pi(x)}{x^s}\,.$$

Hence, for $p_r<x<p_r^2$, one may obtain the following

(7.3) $$\pi(x) \;=\; \pi(p_r) + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s}\exp\left(-\int_{p_r}^{\infty}\frac{d\,\pi(x)}{x^s}\right)dt\,.$$

In Appendix 6, we will use Theorem 3 to derive a formula for the growth of the difference between the actual prime numbers p_i and their ideal values of $\mathrm{Li}^{-1}(i)$.

In the following, we will apply Theorem 4 to Equation (7.2) to show that $\pi(x)=\mathrm{Li}(x)+O(\sqrt{x})$. This task is achieved by first writing Equation (7.2) as follows

$$\pi(x)-\pi(p_r) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\exp\left(\sum_{i=r}^{\infty}\log\left(1-\frac{1}{p_i^s}\right)\right)dt$$

where $\sigma=1+\epsilon$ and ϵ is arbitrary small. Referring to Equations (4.2), (5.2) and (5.5), one may write the above Equations as follows

(7.4) $$\pi(x)-\pi(p_r)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\exp(-A)\,dt+\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\exp(R)\,dt\,,$$

where A is given by

$$A=E_1((s-1)\log p_r)-\sum_{\rho}E_1((s-\rho)\log p_r),$$

and R is the sum of terms of the form $1/p^{-ns}$ where $n\geq 2$. Consequently, the order of the second integral (i.e. $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\exp(R)\,dt$) is given by $O(\sqrt{p_r})$.

The first integral in Equation (7.4) is computed by first expanding the term $\exp(-A)$ as

$$\exp(-A)=1-A+\frac{A^2}{2!}-\frac{A^3}{3!}+\ldots$$

Only the first and second terms (i.e. $1-A$) contribute to the first integral of Equation (7.4) when $1\leq x<p_r^2$. The third term ($A^2/2!$) contributes to the integral only if $x\geq p_r^2$, the fourth term contributes to the integral only if $x\geq p_r^3$ and so on. Thus, for $1\leq x<p_r^2$, we have

(7.5) $$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\exp(-A)\,dt=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\left(1-E_1((s-1)\log p_r)+\sum_{\rho}E_1((s-\rho)\log p_r)\right)dt.$$

The integral on the right side is the sum of integrals of the form $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{a^s}{s}E_i((s-Z)b)\,dt$ where, $a,b>0$, $Z=X+iY$. This integral is computed using Cauchy's residue theorem where the integral is performed over the line extending from $\sigma - iT$ to $\sigma + iT$, then the counter continues from $\sigma + iT$ over a semicircle until it reaches just above the horizontal line $i(Y+\epsilon)$. The counter then continues along this line from left to right until it reaches $Z + i\epsilon$. The counter continues along a semicircle around Z until it reaches $Z - i\epsilon$. The counter then continues along the line just below the line $i(Y-\epsilon)$ from right to left and then complete the semicircle at $\sigma - iT$. The only singularity within this counter is at $s=0$ and therefore, we may have the following,

$$\frac{1}{2\pi i}\left(\int_{\sigma-iT}^{\sigma+iT}\frac{a^s}{s}E_1((s-Z)b)\,dy+\int_{-T+i(Y+\epsilon)}^{X+i(Y+\epsilon)}\frac{a^s}{s}E_1((s-Z)b)\,dx+\int_{x+i(Y-\epsilon)}^{-T+i(Y-\epsilon)}\frac{a^s}{s}E_1((s-Z)b)\,dx\right)=E_1(-Zb),$$

where the integral along the semicircle approaches zero as T approaches infinity. Since, for $x>0$ and as ϵ approaches zero we have $E_1(-x \pm i\epsilon) = -\mathrm{Ei}(x) \mp \epsilon$, hence the second and third integrals on the right side can be written as,

$$\int_{-T+i(Y+\epsilon)}^{X+i(Y+\epsilon)}\frac{a^s}{s}E_1((s-Z)b)\,dx+\int_{X+i(Y-\epsilon)}^{-T+i(Y-\epsilon)}\frac{a^s}{s}E_1((s-Z)b)\,dx=$$

$$-i\pi\int_{-T}^{X}\frac{e^{x\log a+i(Y+\epsilon)\log a}}{x+iY}\,dx+i\pi\int_{X}^{-T}\frac{e^{x\log a+i(Y-\epsilon)\log a}}{x+iY}\,dx.$$

Hence as ϵ approaches zero, we have

$$\int_{-T+i(Y+\epsilon)}^{X+i(Y+\epsilon)}\frac{a^s}{s}E_1((s-Z)b)\,dx+\int_{X+i(Y-\epsilon)}^{-T+i(Y-\epsilon)}\frac{a^s}{s}E_1((s-Z)b)\,dx=-2\mathrm{i}\pi\int_{-T}^{X}\frac{e^{x\log a+i(Y)\log a}}{x+iY}\,dx.$$

Thus,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{a^s}{s}E_1((s-Z)b)\,dy \;=\; E_1(-Zb)+\int_{-\infty}^{X}\frac{e^{x\log a+i(Y)\log a}}{x+iY}\,dx.$$

Since the conjugate of every non-trivial zero is also a zero, hence

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{a^s}{s}\left(E_1((s-Z)b)+E_1(s-Z^*)\right)dy \;=\; E_1(-Zb)+E_1(-Z^*b)$$

$$+\,2\int_{-\infty}^{X}\frac{e^{x\log a}}{x^2+Y^2}(x\cos(Y\log a)+Y\sin(Y\log a))\,dx$$

Using the following two identities,

$$\int\frac{xe^{cx}}{ax^2+b}\,dx=\frac{1}{2\mathrm{a}}\exp\left(-i\frac{\sqrt{b}}{\sqrt{a}}c\right)\left(\mathrm{Ei}\left(c\left(i\frac{\sqrt{b}}{\sqrt{a}}+x\right)\right)+\exp\left(2i\frac{c\sqrt{b}}{\sqrt{a}}\right)\mathrm{Ei}\left(c\left(-i\frac{\sqrt{b}}{\sqrt{a}}+x\right)\right)\right)$$

$$\int \frac{e^{cx}}{ax^2+b}dx=\frac{-i}{2\sqrt{a}\sqrt{b}}\exp\left(-i\frac{\sqrt{b}}{\sqrt{a}}c\right)\left[\exp\left(2i\frac{c\sqrt{b}}{\sqrt{a}}\right)\mathrm{Ei}\left(c\left(-i\frac{\sqrt{b}}{\sqrt{a}}+x\right)\right)-\mathrm{Ei}\left(c\left(i\frac{\sqrt{b}}{\sqrt{a}}+x\right)\right)\right]$$

we may then obtain,

$$2\int_{-\infty}^{X}\frac{e^{x\log a}}{x^2+Y^2}\left(x\cos(Y\log a)+Y\sin(Y\log a)\right)dx=$$
$$\left(e^{-iY\log a}\mathrm{Li}(a^Z)+e^{iY\log a}\mathrm{Li}(a^{Z^*})\right)\cos(Y\log a)-i\left(e^{-iY\log a}\mathrm{Li}(a^Z)+e^{iY\log a}\mathrm{Li}(a^{Z^*})\right)\sin(Y\log a).$$

This can be simplified as follows

$$2\int_{-\infty}^{X}\frac{e^{x\log a}}{x^2+Y^2}\left(x\cos(Y\log a)+Y\sin(Y\log a)\right)dx=$$
$$\mathrm{Li}(a^Z)+\mathrm{Li}(a^{Z^*})+i\sin(Y\log a)\cos(Y\log a)(\mathrm{Li}(a^Z)\mathrm{Li}(a^{Z^*})).$$

Hence,

(7.6) $$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{a^s}{s}\left(E_1((s-Z)b)+E_1(s-Z^*)\right)dy=E_1(-Zb)+E_1(-Z^*b)+\mathrm{Li}(a^Z)+\mathrm{Li}(a^{Z^*})$$

Equation (7.6) can then be used to compute the integral of Equation (7.5) to obtain

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\left(1-E_1((s-1)\log p_r)+\sum_{\rho}E_1((s-\rho)\log p_r)\right)dt=$$
$$1-\mathrm{Li}(p_r)+\sum_{\rho}\left(\mathrm{Li}(p_r^{\rho})+\mathrm{Li}(x^{\rho^*})\right)+\mathrm{Li}(x)-\sum_{\rho}\left(\mathrm{Li}(x^{\rho})+\mathrm{Li}(x^{\rho^*})\right)$$

Consequently,

(7.7) $$\pi(x)=\mathrm{Li}(x)-\sum_{\rho}\left(\mathrm{Li}(x^{\rho})+\mathrm{Li}(x^{\rho^*})\right)+O(\sqrt{x})$$

8- Some Properties of the Function $M(x, p_r)$:

In this section, we will use Theorems 5 and 6 to examine the properties of $M(x, p_r)$ as *r* approaches infinity. Using $s=1+\epsilon+it_1$ in Equation (6.5b) and letting ϵ approaches zero, one may obtain

$$M(x, p_r) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{x x^{it_1}}{(1+it_1)\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right)} dt_1 ,$$

or

$$M(x, p_r) = \frac{x}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{x^{it_1}}{(1+it_1) \prod_{i=r}^{\infty} \left(1 - \frac{1}{p_i^s}\right)^{-1}} dt_1 .$$

However, by the virtue of Equation (4.4), one may have

$$\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) = \exp\{E_1((s-1)\log p_r)\} \left(1 + O(1/(\log p_r)^k)\right) .$$

Hence,

$$M(x, p_r) = \frac{x}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{x^{it_1}\left(1 + O(1/(\log p_r)^k)\right)}{(1+it_1)\exp\{E_1(it_1 \log p_r)\}} dt_1 .$$

Let $x = p_r^{\omega}$, then

$$M(x, p_r) = \frac{x}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\exp(i\omega t_1 \log p_r)\left(1 + O(1/(\log p_r)^k)\right)}{(1+it_1)\exp\{E_1(it_1 \log p_r)\}} dt_1 .$$

With the change of variables $t = t_1 \log p_r$, the above integral can be rewritten as

(8.1) $$M(x, p_r) = \frac{x}{2\pi \log p_r i} \int_{1-i\infty}^{1+i\infty} \frac{\exp(i\omega t)\left(1 + O(1/(\log p_r)^k)\right)}{\left(1 + \frac{it}{\log p_r}\right)\exp(E_1(it))} dt .$$

For sufficiently large *r*, the above integral can be further simplified to

(8.2) $$M(x, p_r) = \frac{x}{2\pi \log p_r i} \int_{1-i\infty}^{1+i\infty} \left(1 + O(1/(\log p_r)^k)\right) \exp(-E_1(it)) \exp(i\omega t)\, dt .$$

If define the above integral without the term $O(1/(\log p_r)^k)$ as $M_A(x, p_r)$, then

$$M_A(x, p_r) = \frac{x}{2\pi \log p_r i} \int_{1-i\infty}^{1+i\infty} \exp(-E_1(it)) \exp(i\omega t)\, dt,$$

or

(8.3) $$M_A(x, p_r) = \frac{p_r^{\omega}}{2\pi \log p_r i} \int_{1-i\infty}^{1+i\infty} \exp(-E_1(it)) \exp(i\omega t)\, dt.$$

Notice that the integral in Equation (8.3) is the Fourier transform of the function $\exp(-E_1(it))$ and it is independent of p_r. Since the function $\exp(-E_1(it))$ is an entire function with an infinite order, thus the decay of its Fourier transform has the form $O(\exp(-\omega \log(\omega)))$ and $M_A(x, p_r)$ may be written as [8]

(8.4) $$M_A(x, p_r) = \frac{p_r^{\omega}}{\log p_r} O(\exp - \omega \log(\omega)) = O\left(\exp\left(\omega \log\left(\frac{p_r}{\omega}\right)\right)\right) = O\left(\left(\frac{p_r}{\omega}\right)^{\omega}\right).$$

Thus, for $p_r < x < p_r^2$ (or $1 < \omega < 2$), $M_A(x, p_r)$ has the value of Li(*x*). $M_A(x, p_r)$ continues to grow for values of ω greater than 2 and reaches its maximum for values of ω at or around p_r. For $\omega \gg p_r$, $M_A(x, p_r)$ decays to zero as determined by Equation (8.4).

9- The series $Mu(s, p_r)$ at s = 1:

In this section, we will show that if we define the function $f(a, p_r)$ as

(9.1) $$f(a, p_r) = Mu(1, p_r; 1, p_r^a) = \sum_{n=1}^{p_r^a} \frac{\mu(n, p_r)}{n},$$

then, as p_r approaches infinity, the function $f(a, p_r)$ approaches a deterministic function $F(a)$ that is independent of p_r and is dependent on only a. In other words; if we plot $Mu(1, p_r; 1, N)$ (where $N = p_r^a$)as a function $\log(N)/\log(p_r)$ (or a), then for each value for a, as p_r approaches infinity, $F(a)$ approaches a unique value that is independent of p_r.

This can be shown by first dividing the prime numbers that are in the range $p_r \le x < p_r^2$ into *N* sections. The first section comprises of all the prime numbers that are in the range $p_r \le x < p_r^{1+\epsilon}$. The second section comprises of all the prime numbers that are in the range $p_r^{1+\epsilon} < x < p_r^{(1+\epsilon)^2}$ and so on (where the *i*-th section comprises of all the prime numbers that are in the range $p_r^{(1+\epsilon)^i} < x < p_r^{(1+\epsilon)^{i+1}}$). Hence,

$$(1+\epsilon)^N = 2,$$

or

(9.2) $$\epsilon = \frac{\log 2}{N}.$$

The process of dividing the prime numbers into sections continues for primes greater than $p_r^{\,2}$. Thus, the total number of sections *M* over the range $p_r \le x < p_r^a$ is given by

(9.3) $$M = \frac{\log a}{\epsilon}.$$

If we define K_i as the sum of the reciprocal of the prime numbers in section *i,* then by Mertens' Theorem K_i is given by

$$K_i \;=\; \log\log p_r^{(1+\epsilon)^{i+1}} - \log\log p_r^{(1+\epsilon)^i} + O(1/\log p_r).$$

Hence, for sufficiently small ϵ and for sufficiently large p_r, one may then obtain

(9.4) $$K_i \;=\; \epsilon + O(1/\log p_r),$$

where $O(1/\log p_r)$ can be made arbitrary small by selecting p_r arbitrary large. Thus, K_i will have the same value ϵ for each of the M sections as p_r approaches infinity.

In the following, we will device an algorithm to construct a series that is equivalent to the series $Mu(1, p_r; 1, p_r^{\,\omega})$ from these *M* sections (that are comprised of the prime numbers) and the products of these sections (with the appropriate signs). This series starts with the number 1. Then, instead of subtracting the terms $1/p_i$ (in the order based on their values, where $p_r \le p_i < p_r^2$), we subtract the values of K_i's of the first *N* sections. These sections are ordered based on the value of the largest member within each section. It should pointed out that, the value of $Mu(1, p_r; 1, p_r^{\,2})$ constructed by this method is given $1-\log 2$ (plus a factor that is determined by the sum of *N* terms of the form $O(1/\log p_r)$ and, as mentioned earlier, this factor can be made arbitrary small by selecting p_r arbitrary large).

The terms of the series $Mu(1, p_r; 1, p_r^{\,a})$ in the range $p_r^2 \le p_i < p_r^3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. To reconstruct these terms, we subtract the sum of K_i's for the sections of primes in the range $p_r^2 \le p_i < p_r^3$ and then add it to the sum of the terms that are the product of K_i's and K_j's for any two sections of the prime numbers where the maximum value of members within the product is less than p_r^3.

Similarly, we reconstruct the terms of $Mu(1, p_r; 1, p_r^{\,a})$ in the range $p_r^3 \le p_i < p_r^4$ by subtracting the sum of K_i's for the sections of primes in the range $p_r^3 \le p_i < p_r^4$ and then adding it to the sum of the terms that are the product of K_i's and K_j's for any two sections of the prime numbers where the maximum value of the members within the product is less than p_r^4. We then subtract the result from the sum of the terms that are the product of K_i's, K_j's and K_l's for any three sections of the prime numbers where the maximum value of the members within the product is less than p_r^4.

We repeat this process $a-1$ times to reconstruct all the terms of $Mu(1, p_r; 1, p_r^a)$. Thus, one may conclude that except for the terms of the form $O(1/\log p_r)$ (that can made arbitrarily small by choosing p_r arbitrarily large), $Mu(1, p_r; 1, p_r^a)$ is only dependent on $M = \log a/\epsilon$ (i.e. the number of sections used to construct $Mu(1, p_r; 1, p_r^a)$) and ϵ (the value associated with the sum of the reciprocal of the primes within each section). Hence, $Mu(1, p_r; 1, p_r^a)$ is independent of p_r. However, it is well known through elementary methods that for every integer N>1, we have have the following inequality [3];

$$\left|\sum_{j=1}^{J} \frac{\mu(j)}{j}\right| = |Mu(1; 1, J)| \leqslant 1 .$$

Thus, using Theorem 1, both $Mu(1; p_r^a)$ and $Mu(1, p_r; 1, p_r^a)$ have a limit as a approaches infinity (i.e. as J approaches infinity). Therefore, one may conclude that as ϵ approaches zero (and at the same time, p_r approaches infinity since we have to keep the terms of the form $O(1/\log p_r)$ much smaller than ϵ), $Mu(1, p_r; 1, p_r^a)$ becomes dependent on only a. This implies that for sufficiently large p_r, the value of $Mu(1, p_r)$ is the same for all p_r's. However, Theorem 2 states that for every p_r we have

$$Mu(1) = Mu(1, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) .$$

This can be achieved only if

$$Mu(1) = Mu(1, p_r) = 0 .$$

It should be pointed out that the series generated by this algorithm includes both square-free terms (that forms $Mu(1, p_r; 1, p_r^a)$) as well as the non square-free terms. Therefore, the series generated by this algorithm is in fact $Lv(1, p_r; 1, p_r^a)$ instead of $Mu(1, p_r; 1, p_r^a)$. In the following, we will show that as p_r approaches infinity, the contribution by the non square-free terms approaches zero as well. Thus, our analysis in this sections holds for both $Lv(1, p_r)$ and $Mu(1, p_r)$. Toward this end, let S_0 be the sum of the terms associated with the square of the prime p_r. Let S_{r+1} be the sum of the remaining terms that are associated with the square of the prime p_{r+1}, and so on. Let T be sum of all the terms associated with non square-free terms of $Lv(1, p_r; 1, p_r^a)$. Thus, T is given by

$$T = \frac{1}{p_r^2} S_0 + \frac{1}{p_{r+1}^2} S_1 + .. + \frac{1}{p_{r+L}^2} S_{L-1} ,$$

where p_{r+L}^2 is the largest prime square that is less than p_r^a. However,

$$|S_0| , \ |S_1| , \ .... , \ |S_{L-1}| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p_r^a} .$$

Thus,

$$|S_0| , \ |S_1| , \ .... , \ |S_{L-1}| < \log p_r .$$

Hence,

$$T < \left(\frac{1}{p_r^2} + \frac{1}{p_{r+1}^2} + .. + \frac{1}{p_{r+L}^2} \right) a \log p_r ,$$

or

$$T < \frac{a \log p_r}{p_r} .$$

Thus, for any a , *T* approaches zero as p_r approaches zero. Therefore, both $Lv(1, p_r)$ and $Mu(1, p_r)$ attain the same value of zero for all p_r 's.

10- The series $Mu(s, p_r)$ for $\Re(s) \leq 1$ and the validity of the Riemann Hypothesis:

In this section, we modify the method described in the previous section to provide an estimate for the decay of the function $Mu(1, p_r; 1, p_r^a)$ as *a* and p_r approach infinity. We then establish the relationship between $Mu(1, p_r; 1, p_r^a)$ and $Mu(\sigma, p_r; 1, p_r^a)$ where $\sigma < 1$ as *a* and p_r approach infinity. Our analysis shows that the series $Mu(1, p_r)$ fails to converge for $\sigma < 1$ and this is the basis for our claim that RH is invalid. In fact, our analysis points to the presence of non-trival zeros arbitrary close to 1.

The main idea behind our approach to examine the validity of the Riemann Hypothesis is to determine the rate at which the function $Mu(s=1, p_r; 1, p_r^a)$ decays as *a* and p_r approach infinity. It is well known that the series $Mu(1)$ is equal to zero and Riemann Hypothesis is valid if and only if $Mu(\sigma)$ is convergent and is equal to $1/\zeta(\sigma)$ for $\sigma > 0.5$. However, it is unknown how the function $Mu(s=1; 1, n) = \sum_{i=1}^{n} \frac{\mu(i)}{i}$ approaches zero as *n* approaches infinity. This follows from the random nature of the Mobius function $\mu(n)$. Figure (1A) shows the value of the function $Mu(s=1; 1, n)$ for $1 \leq n \leq 64$ which evidently shows the random behavior of $Mu(1; 1, n)$ as it approaches zero. Figure (1B) shows the function $Mu(s=1,3; 1,3^A)$ for $1 \leq A \leq 6$. It is clear from this figure how our approach has effectively smoothed the random variation in $Mu(1; 1, n)$. Figure (1C) shows the function $Mu(s=1,17; 1,17^A)$ for $1 \leq A \leq 6$ and Figure (1D) shows an expanded view of $Mu(17; 1,17^A)$ for $4 \leq A \leq 6$ which demonstrates the smooth approach of the function $Mu(17; 1,17^A)$ to zero. As p_r approaches infinity, the function $Mu(s=1, p_r; 1, p_r^A)$ approaches a deterministic function that we have shown to be dependent on only *A*. It is the rate of decay (at which this function $Mu(1, p_r; 1, p_r^A)$ approaches zero) that we will analyze in this section. This rate is then used to examine the validity of RH.

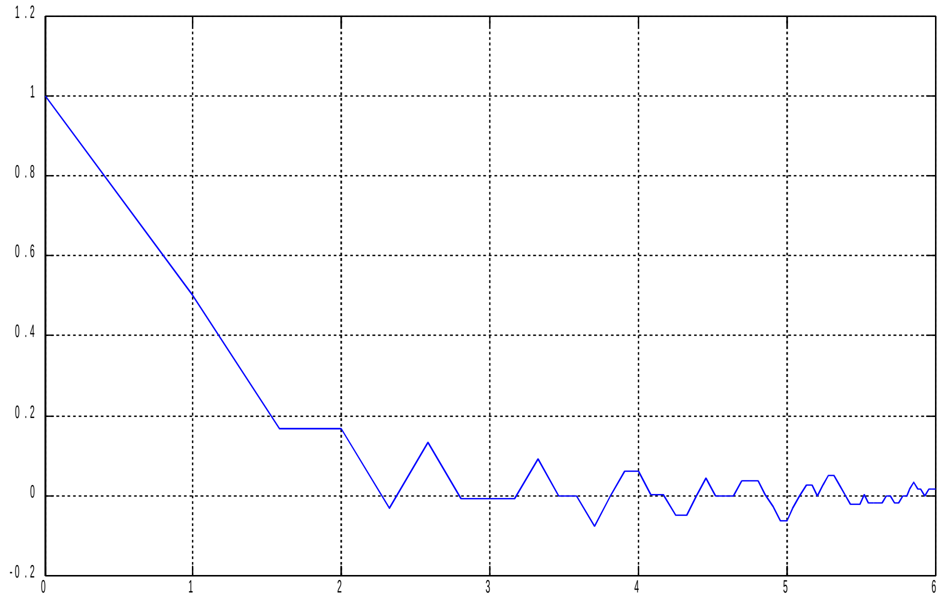

Figure (1A): The function $Mu(1;1,n)$ or $Mu(s=1;1,n)=\sum_{i=1}^{n}\mu(i)/i$ vs. $\log_2 n$ which shows the random variation of $Mu(1;1,n)$ as it approaches zero.

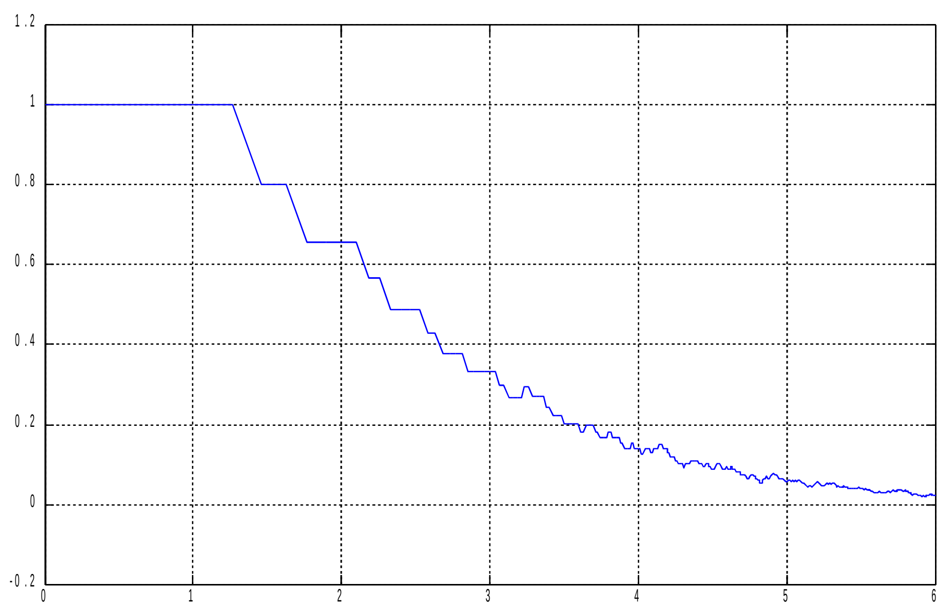

Figure (1B): The function $Mu(s=1;1,3^A)=Mu(1;1,3^A)$ vs. A (where $A=\log_3 n$). Notice the smoother approach of $Mu(1;1,3^A)$ to zero as A (or n) approaches infinity.

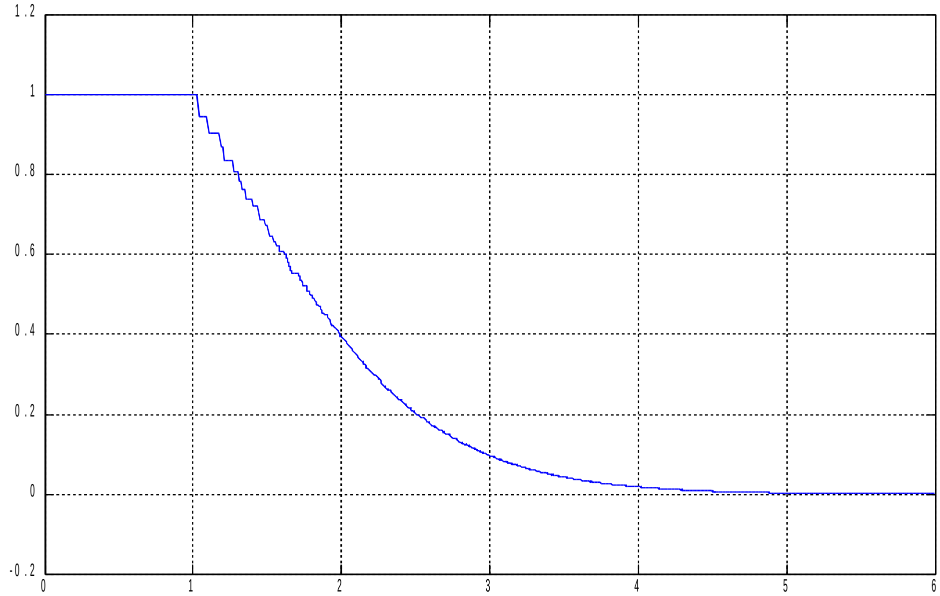

Figure (1C): The function $Mu(1;1,17^{A})$ vs. A (where $A=\log_{17} n$). Notice a much smoother approach of $Mu(1;1,17^{A})$ to zero as A (or n) approaches infinity.

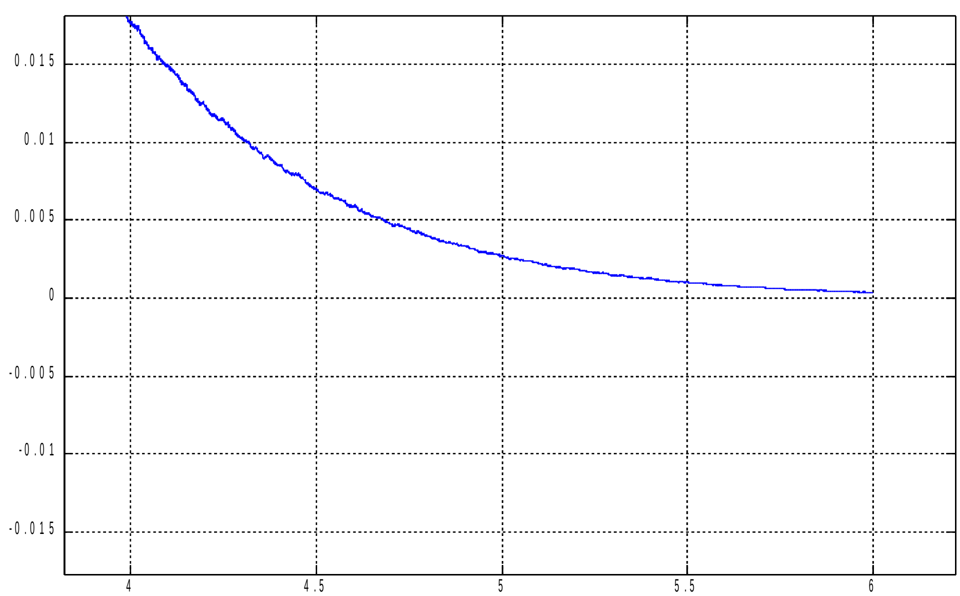

Figure (1D): An expanded view of $Mu(17;1,17^{A})$ for $4 \leq A \leq 6$ which demonstrates the smooth approach of the function $Mu(17;1,17^{A})$ to zero.

In the previous section, we constructed the series $Mu(1, p_r; 1, p_r^{\,a})$ by first dividing the prime numbers that are in the range $p_r \le x < p_r^2$ into *N* sections. The first section comprises of all the prime numbers that are in the range $p_r \le x < p_r^{1+\epsilon}$. The second section comprises of all the prime numbers that are in the range $p_r^{1+\epsilon} < x < p_r^{(1+\epsilon)^2}$ and so on. In this section we will also divide the prime numbers that are in the range $p_r \le x < p_r^2$ into *N* sections. However, the first section comprises of all the prime numbers that are in the range $p_r \le x < p_r^{1+\delta}$. the second section comprises of all the prime numbers that are in the range $p_r^{\delta} \le x < p_r^{\,1+2\delta}$ and so on (where the *j*-th section comprises of all the prime numbers that are in the range $p_r^{1+(j-1)\delta} \le x < p_r^{\,1+j\delta}$). Hence,

(10.1) $$N\delta = 1\ .$$

The process of dividing the prime numbers into sections continues for primes greater than $p_r^{\,2}$. Thus, the total number of sections *M* over the range $p_r \le x < p_r^a$ is given by

(10.2) $$M\delta = a\ .$$

If we define K_i as the sum of the reciprocals of the prime numbers in section j (where *i*=*j*+*N*) then by Mertens' Theorem, K_i is given by

(10.3) $$K_i \;=\; \log\log p_r^{(i+1)\delta} - \log\log p_r^{i\delta} + \frac{O(1/\log p_r)}{i\delta}\ ,$$

where $1 \le i\delta \le a$. Hence, for sufficiently small δ and for sufficiently large p_r, we may then obtain

(10.4) $$K_i \;=\; \frac{1}{i} + \frac{1}{i\delta}\,O(1/\log p_r) + O\left(1/i^2\right),$$

where $O(1/\log p_r)$ can be made arbitrary small by selecting p_r arbitrary large. As p_r and *N* approach infinity, we may then write

(10.5) $$K_i \;=\; \frac{1}{i}\ .$$

Similar to what we did in the section 9, in the following, we will device an algorithm to construct a series that is equivalent to the series $Mu(1, p_r; 1, p_r^{\,A})$ from these *M* sections (that are comprised of the prime numbers) and the products of these sections (with the appropriate signs). This series starts with the number 1. Then, instead of subtracting the terms $1/p_i$ (in the order based on their values, where $p_r \le p_i < p_r^2$), we subtract the values of K_i's for the first *N* sections. These sections are ordered based on the value of the largest member within each section. It can be easily shown that the value of $Mu(1, p_r; 1, p_r^{\,2})$ constructed by this method is given $1-\log 2$ (plus a factor that is determined by the sum of *N* terms of the form $(1/i\delta)O(1/\log p_r)$ and this factor (as mentioned earlier) can be made arbitrary small by selecting p_r arbitrary large).

Moreover, we will compute the sum of terms of the form $1/p_i$ over the range $p_r \le p_i < p_r^A$ and we will also compute the sum of the terms of the form $1/p_i$ over $p_r^A \le p_i < p_r^{A+\Delta A}$ (where $\Delta A = 1/N$). It can be easily shown the sum over the range $p_r \le p_i < p_r^A$ as p_r approaches infinity is given by $M_1(A) = \log A$ while the sum over the range $p_r^A \le p_i < p_r^{A+\Delta A}$ as p_r approaches infinity is given by $\Delta M_1(A) = \Delta A / A$ (one may notice that $\Delta M_1(A)$ is also given by $\Delta A \; dM_1(A)/dA$).

The terms of the series $Mu(1, p_r; 1, p_r^A)$ in the range $p_r^2 \le p_i < p_r^3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. To reconstruct these terms, we subtract the sum of K_i's for the sections of primes in the range $p_r^2 \le p_i < p_r^3$ and then add to it the sum of the terms that are the product of K_i's and K_j's for any two sections of the prime numbers where the maximum value of members within the product is less than p_r^3.

In the following, we will compute the sum of terms of the form $1/(p_i p_j)$ over the range $p_r \le p_i < p_r^A$. We will also compute the sum of the terms of the form $1/(p_i p_j)$ over $p_r^A \le p_i < p_r^{A+\Delta A}$. The sum $M_2(A)$ of the terms of the form $1/(p_i p_j)$ over the range $p_r \le p_i < p_r^A$ as p_r approaches infinity is given by:

$$M_2(A) = \frac{1}{2N} \sum_{i=N}^{M-N} \frac{1}{A - i/N} \log\left(\frac{i}{N}\right) = \frac{1}{2} \int_1^{A-1} \frac{\log a}{A-a} \, da \cdot$$

The factor of 1/2 was added since each term of the form $1/(p_i p_j)$ is accounted twice in the above sum. Computing the above integral, we then obtain

$$M_2(A) = \frac{1}{2} \log^2(A) + \mathrm{Li}_2\left(\frac{1}{A}\right) - \frac{\pi^2}{12},$$

where $\mathrm{Li}_2(x)$ is the dilogarithm of x and is given by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} dt \cdot$$

Similarly, the sum $\Delta M_2(A)$ of the terms of the form $1/(p_i p_j)$ over the range $p_r^A \le p_i < p_r^{A+\Delta A}$ as p_r approaches infinity is given by:

$$\Delta M_2(A) = = \frac{\Delta A}{2} \int_1^{A-1} \frac{1}{(A-a)a} \, da,$$

or

$$\Delta M_2(A) = \frac{\Delta A}{A} \log(A-1),$$

or

$$\Delta M_2(A) = \frac{\Delta A}{A} M_1(A-1).$$

Once again, we notice that $\Delta M_2(A)$ is also given by $\Delta A \, dM_2(A)/dA$.

Similarly, we reconstruct the terms of $Mu(1,p_r;1,p_r^A)$ in the range $p_r^3 \le p_i < p_r^4$ by subtracting the sum of K_i's for the sections of primes in the range $p_r^3 \le p_i < p_r^4$ and then adding it to the sum of the terms that are the product of K_i's and K_j's for any two sections of the prime numbers where the maximum value of the members within the product is less than p_r^4. We then subtract the result from the sum of the terms that are the product of K_i's, K_j's and K_l's for any three sections of the prime numbers where the maximum value of the members within the product is less than p_r^4. Similarly, the sum $M_3(A)$ of terms of the form $1/(p_i p_j p_k)$ over the range $p_r \le p_i < p_r^A$ as p_r approaches infinity can be computed as follows

$$M_3(A) = \frac{1}{3N} \sum_{i=2N}^{M-N} \frac{1}{A-i/N} \; M_2\left(\frac{i}{N}\right) = \frac{1}{3} \int_2^{A-1} \frac{M_2(a)}{A-a} \, da.$$

Furthermore, the sum $\Delta M_3(A)$ of the terms of the $1/(p_i p_j p_k)$ over the range $p_r^A \le p_i < p_r^{A+\Delta A}$ as p_r approaches infinity is given by:

$$\Delta M_3(A) = \frac{\Delta A}{3} \int_2^{A-1} \frac{1}{(A-a)} \frac{\log(a-1)}{a} \, da,$$

or

$$\Delta M_3(A) = = \frac{\Delta A}{3A} \int_2^{A-1} \frac{\log(a-1)}{A-a} \, da + \frac{\Delta A}{3A} \int_2^{A-1} \frac{\log(a-1)}{a} \, da.$$

Hence,

$$\Delta M_3(A) = \frac{\Delta}{A} \left(\frac{1}{2} \log^2(A-1) + \mathrm{Li}_2\left(\frac{1}{A-1}\right) - \frac{\pi^2}{12} \right)$$

or,

$$\Delta M_3(A) = \frac{\Delta A}{A} M_2(A-1),$$

We repeat this process $A-1$ times to reconstruct all the terms of $Mu(1,p_r;1,p_r^A)$. For each step j, we compute both $M_j(A)$ and $\Delta M_j(A)$ as p_r approaches infinity to obtain

(10.6) $$M_j(A) = \frac{1}{jN}\sum_{i=jN}^{(A-1)N}\frac{1}{A-i/N}\ M_{j-1}\left(\frac{i}{N}\right) = \frac{1}{j}\int_j^{A-1}\frac{M_{j-1}(a)}{A-a}\,da\,,$$

and

(10.7) $$\Delta M_j(A) = \frac{\Delta A^2}{j\ N}\sum_{i=jN}^{M-N}\frac{1}{A-i/N}\frac{1}{i/N}\Delta M_{j-1}\left(\frac{i}{N}\right) = \frac{\Delta A}{j}\int_j^{A-1}\frac{1}{(A-a)}\frac{\Delta M_{j-1}(a)}{a}\,da\,,$$

or

(10.8) $$\Delta M_j(A) = \frac{\Delta A}{jA}\int_j^{A-1}\frac{\Delta M_{j-1}(a)}{A-a}\,da + \frac{\Delta A}{jA}\int_j^{A-1}\frac{\Delta M_{j-1}(a)}{a}\,da\,,$$

and

(10.9) $$\Delta M_j(A) = \Delta A\,\frac{d\,M_j(A)}{dA}\,.$$

Thus, we may conclude that

(10.10) $$\frac{d\,M_j(A)}{dA} = \frac{M_{j-1}(A-1)}{A}\,.$$

Since $Mu(1, p_r; 1, p_r^A)$ is the superposition of the terms $M_j(A)$'s, hence

(10.11) $$\frac{d\,Mu(1, p_r; 1, p_r^A)}{dA} = -\frac{Mu(1, p_r; 1, p_r^{(A-1)})}{A}\,.$$

Thus, for sufficiently large *A*, $Mu(1, p_r; 1, p_r^A)$ is given by

(10.12) $$Mu(1, p_r; 1, p_r^A) = O(1/A)\,.$$

In other words; the Dirichlet series $Mu(s, p_r; 1, p_r^A)$ at $s=1$ approaches zero as approaches *A* infinity and for sufficiently large *A*, $Mu(1, p_r; 1, p_r^A)$ decays at a rate given by 1/*A*. In the following we will use this result to examine the validity of the Riemann Hypothesis. This task will be achieved by using the same approach described in this section to analyze the behavior of $Mu(\sigma, p_r; 1, p_r^A)$ for $0.5<\sigma<1$ as *A* approaches infinity.

In the following, we will use the same approach to reconstruct $Mu(\sigma, p_r; 1, p_r^a)$ where we divide the prime numbers that are in the range $p_r \le x < p_r^2$ into *N* sections. The first section comprises of all the prime numbers that are in the range $p_r \le x < p_r^{1+\delta}$. The second section comprises of all the prime numbers that are in the range $p_r^{1+\delta} < x < p_r^{1+2\delta}$ and so on (where the *j*-th section comprises of all the prime numbers that are in the range $p_r^{1+(j-1)\delta} < x < p_r^{1+j\delta}$). Hence, $N\delta = 1$. The process of dividing the prime numbers into sections continues for primes greater than $p_r^{\,2}$. Thus, the total number of sections *M* over the range $p_r \le x < p_r^a$ is given by $M\delta = a$.

If we define K_i as the sum of the reciprocal of the prime numbers in section *j* (where *i=j+N*), then on RH and using Theorem 4, we have (refer to Equation 5.5)

$$K_i = \sum_{p_r^{i\delta} \le p_j < p_r^{(i+1)\delta}} \frac{1}{p_j^{\sigma}} = \int_{p_r^{i\delta}}^{p_r^{(i+1)\delta}} \frac{1}{x^{\sigma}\log x} dx - \int_{p_r^{i\delta}}^{p_r^{(i+1)\delta}} \frac{1}{x^{\sigma}\log x} d\left(\sum_{\rho} \frac{x^{\rho}}{\rho}\right) + \delta ,$$

or

$$K_i = -E_1\left((\sigma-1)\log p_r^{(i+1)\delta}\right) + E_1\left((\sigma-1)\log p_r^{i\delta}\right) + O\left(\frac{1}{(\sigma-0.5)^2} p_r^{\,1/2-\sigma} \log(p_r)\right).$$

Using the following asymptotic representation of the Exponential Integral

$$E_1(z) = \frac{e^{-z}}{z}\left(1 + O\left(\frac{1}{z}\right)\right),$$

we may then have

$$-E_1\left((\sigma-1)\log p_r^{(i+1)\delta}\right) = \frac{e^{(1-\sigma)(i+1)\delta\log p_r}}{(1-\sigma)\log p_r^{(i+1)\delta}}\left(1 + O\left(\frac{1}{N\log p_r}\right)\right) = \frac{e^{(1-\sigma)(i+1)\delta\log p_r}}{(1-\sigma)i\delta\log p_r}\left(1 + O\left(\frac{1}{N\log p_r}\right)\right),$$

and

$$E_1\left((\sigma-1)\log p_r^{(i+1)\delta}\right) = -\frac{e^{(1-\sigma)i\delta\log p_r}}{(1-\sigma)i\delta\log p_r}\left(1 + O\left(\frac{1}{N\log p_r}\right)\right).$$

Hence

$$K_i = C\,\frac{p_r^{(1-\sigma)\delta i}}{i}\left(1 + O\left(\frac{1}{N\log p_r}\right)\right) + O\left(\frac{1}{(\sigma-0.5)^2} p_r^{\,1/2-\sigma} \log(p_r)\right),$$

where $C = (p_r^{(1-\sigma)\delta} - 1)/((1-\sigma)\delta\log p_r)$. Hence, as p_r and *N* approach infinity, we may then have

(10.13) $$K_i = p_r^{(1-\sigma)\delta i}\,\frac{C}{i}$$

Similar to the computation of $M_1(A) = \log A$ and $\Delta M_1(A) = \Delta A / A$, in the following we will compute the sum of terms of the form $1/p_i^{\sigma}$ in $Mu(\sigma, p_r)$ over the range over the range $p_r^A \le p_i < p_r^{A+\Delta A}$ (where $\Delta A = 1/N$). We will also introduce the series $Mu(1, p_r; C)$. This series is similar to the series $Mu(1, p_r; 1, p_r^{\,a})$ expect that the terms of the form $1/p_{il}$ are multiplied by the factor C, the terms of the form $1/p_{i1}p_{i2}$ are multiplied by the factor C^2 and the terms of the form $1/p_{i1}p_{i2}\cdots p_{¿}$ are multiplied by the factor C^n. If we denote the sum of terms of the form $1/p_i$ in this series over the range $p_r \le p_i < p_r^A$ as $M_1(A;C)$ and the sum of the terms of the form $1/p_i$ over $p_r^A \le p_i < p_r^{A+\Delta A}$ as $\Delta M_1(A;C)$, then we have

$$M_1(A;C) = C \log A\,,$$

$$\Delta M_1(A;C) = C\,\Delta A/A\,,$$

and the sum of the terms of the form $1/p_i^\sigma$ in $Mu(\sigma, p_r)$ over the range $p_r^A \le p_i < p_r^{A+\Delta A}$ is given by

$$\Delta M_1(A,\sigma) = C\, p_r^{(1-\sigma)A}\Delta A/A = \Delta M_1(A;C)\, p_r^{(1-\sigma)A}\,.$$

Similarly, if we denote the sum of terms of the form $1/p_i p_j$ in the series $Mu(1, p_r; C)$ over the range $p_r \le p_i < p_r^A$ as $M_2(A;C)$ and the sum of the terms of the form $1/p_i p_j$ over $p_r^A \le p_i < p_r^{A+\Delta A}$ as $\Delta M_2(A;C)$ then we have

$$M_2(A;C) \;= C^2\left(\frac{1}{2}\log^2(A) + \mathrm{Li}_2\left(\frac{1}{A}\right) - \frac{\pi^2}{12}\right),$$

and

$$\Delta M_2(A;C) \;=\; C^2\,\frac{\Delta A}{A}\,\log(A-1)\,,$$

and the sum of the terms of the form $1/p_i^\sigma p_j^\sigma$ in $Mu(\sigma, p_r)$ over the range $p_r^A \le p_i < p_r^{A+\Delta A}$ is given by

$$\Delta M_2(A,\sigma) \;=\; C^2\,\frac{\Delta A}{A}\,\log(A-1)\, p_r^{(1-\sigma)A} = \Delta M_2(A;C)\; p_r^{(1-\sigma)A}\,,$$

and so on for all the terms of the two series $Mu(1, p_r; C)$ and $Mu(\sigma, p_r)$. Therefore, if the series $Mu(1, p_r; C)$ is convergent, then as it is the case with $Mu(1, p_r; 1, p_r^{\,A})$, $Mu(1, p_r; 1, p_r^{\,A}; C)$ should also decay at a rate given by $1/A$. Furthermore, we have

(10.1 $$\Delta M_i(A,\sigma) \;=\; \Delta M_i(A;C)\; p_r^{(1-\sigma)A}\,.$$

Since for $\sigma < 1$, the factor $p_r^{\,(1-\sigma)A}$ approaches infinity as A approaches infinity, therefore $Mu(\sigma, p_r; p_r^{\,A}, p_r^{\,A+\Delta A})$ also approaches infinity as A approaches infinity (Notice, that $Mu(\sigma, p_r; p_r^{\,A}, p_r^{\,A+\Delta A}) = \sum_{i=1}^{A}\Delta M_i(A,\sigma)$). Therefore, the series $Mu(\sigma, p_r)$ diverges and consequently the series $Mu(\sigma)$ also diverges for $\sigma < 1$ and the divergence of the the series $Mu(\sigma)$ is the basis for our claim that the Riemann Hypothesis is invalid.

Appendix 1:

To prove the first part of theorem 1 (i.e. for $s=\sigma+i0$ and $0.5<\sigma\leq 1$, the series $Mu(\sigma, p_r)$ converges conditionally if $Mu(\sigma)$ converges conditionally), we first start with proving that $Mu(\sigma, 2)$ is convergent if $Mu(\sigma)$ is convergent. Since $Mu(\sigma)$ is convergent, then for every arbitrary small number δ, there exists an integer N_0 such that for every integer $N > N_0$

$$Mu(\sigma; N, \infty) = \sum_{n=N}^{\infty} \frac{\mu(n)}{n^{\sigma}} < \delta .$$

Let the sums $Mu(\sigma; 1, N)$, $Mu(\sigma; N+1, 2N)$, $Mu(\sigma; 2\mathrm{N}+1, 2^2 N)$, $Mu(\sigma; 2^2 N+1, 2^3 N)$,…., $Mu(\sigma; 2^{L-1} N+1, 2^L N)$ be defined as;

$$\begin{aligned}
Mu(\sigma; 1, N) &= \sum_{n=1}^{N} \frac{\mu(n)}{n^{\sigma}} = A_1 , \\
Mu(\sigma; N+1, 2N) &= \sum_{n=N+1}^{2\mathrm{N}} \frac{\mu(n)}{n^{\sigma}} = \delta_1 , \\
Mu(\sigma; 2\mathrm{N}+1, 2^2 N) &= \sum_{n=2\mathrm{N}+1}^{2^2 N} \frac{\mu(n)}{n^{\sigma}} = \delta_2 , \\
Mu(\sigma; 2^2 N+1, 2^3 N) &= \sum_{n=2^2 N+1}^{2^3 N} \frac{\mu(n)}{n^{\sigma}} = \delta_3 , \qquad \text{(A1.1)} \\
Mu(\sigma; 2^{L-1} N+1, 2^L N) &= \sum_{n=2^{L-1} N+1}^{2^L N} \frac{\mu(n)}{n^{\sigma}} = \delta_{L-1} ,
\end{aligned}$$

where by the virtue of the convergence of $Mu(\sigma)$

$$|\delta_1|, |\delta_2|, |\delta_3|, \ldots, |\delta_{L-1}| < \delta \quad . \qquad \text{(A1.2)}$$

Furthermore, let the sums $Mu(\sigma, 2; 1, N)$, $Mu(\sigma, 2; N+1, 2N)$, $Mu(\sigma, 2; 2\mathrm{N}+1, 2^2 N)$, $Mu(\sigma, 2; 2^2 N+1, 2^3 N)$,…., $Mu(\sigma, 2; 2^{L-1} N+1, 2^L N)$ be defined as

$$\begin{aligned}
Mu(\sigma, 2; 1, N) &= \sum_{n=1}^{N} \frac{\mu(n, 2)}{n^{\sigma}} = B_1 , \\
Mu(\sigma, 2; N+1, 2N) &= \sum_{n=N+1}^{2\mathrm{N}} \frac{\mu(n, 2)}{n^{\sigma}} = \epsilon_1 , \\
Mu(\sigma, 2; 2\mathrm{N}+1, 2^2 N) &= \sum_{n=2\mathrm{N}+1}^{2^2 N} \frac{\mu(n, 2)}{n^{\sigma}} = \epsilon_2 , \\
Mu(\sigma, 2; 2^2 N+1, 2^3 N) &= \sum_{n=2^2 N+1}^{2^3 N} \frac{\mu(n, 2)}{n^{\sigma}} = \epsilon_3 , \qquad \text{(A1.3)}
\end{aligned}$$

$$Mu(\sigma,2;2^{L-1}N+1,2^{L}N) \;=\; \sum_{n=2^{L-1}N+1}^{2^{L}N} \frac{\mu(n,2)}{n^{\sigma}} \;=\; \epsilon_{L-1}\,.$$

Since

$$\sum_{n=1}^{2N} \frac{\mu(n)}{n^{\sigma}} \;=\; \sum_{n=1}^{2N} \frac{\mu(n,2)}{n^{\sigma}} - \sum_{n=1}^{N} \frac{\mu(n,2)}{(2n)^{\sigma}} \;=\; \sum_{n=1}^{2N} \frac{\mu(n,2)}{n^{\sigma}} - \frac{1}{2^{\sigma}} \sum_{n=1}^{N} \frac{\mu(n,2)}{n^{\sigma}}\,,$$

thus

(A1.4) $$Mu(\sigma;1{,}2N) \;=\; Mu(\sigma,2;1{,}2N) - \frac{1}{2^{\sigma}} Mu(\sigma,2;1,N)\,.$$

Similarly, we can write

$$\sum_{n=2^{l}N+1}^{2^{l+1}N} \frac{\mu(n)}{n^{\sigma}} \;=\; \sum_{n=2^{l}N+1}^{2^{l+1}N} \frac{\mu(n,2)}{n^{\sigma}} - \sum_{n=2^{l-1}N+1}^{2^{l}N} \frac{\mu(n,2)}{(2n)^{\sigma}} \;=\; \sum_{n=2^{l}N+1}^{2^{l+1}N} \frac{\mu(n,2)}{n^{\sigma}} - \frac{1}{2^{\sigma}} \sum_{n=2^{l-1}N+1}^{2^{l}N} \frac{\mu(n,2)}{n^{\sigma}}\,.$$

Thus,

(A1.5) $$Mu(\sigma;2^{l}N+1,\,2^{l+1}N) = Mu(\sigma,2;2^{l}N+1,\,2^{l+1}N) - \frac{1}{2^{\sigma}} Mu(\sigma,2;2^{l-1}N+1,\,2^{l}N)\,.$$

By the virtue of Equations (A1.2), (A1.3), (A1.4) and (A1.5)

(A1.6) $$A_1+\delta_1 = B_1+\epsilon_1-\frac{1}{2^{\sigma}}B_1\;,$$

(A1.7a) $$\delta_2 = \epsilon_2-\frac{1}{2^{\sigma}}\epsilon_1\;,$$

(A1.7b) $$\delta_3 = \epsilon_3-\frac{1}{2^{\sigma}}\epsilon_2\;,$$

(A1.7c) $$\delta_{L-1} = \epsilon_{L-1}-\frac{1}{2^{\sigma}}\epsilon_{L-2}\;.$$

where $|\delta_1|,|\delta_2|,|\delta_3|,\dots,|\delta_{L-1}| \;<\; \delta$, $|\delta_1+\delta_2|<\delta$, $|\delta_1+\delta_2+\delta_3|<\delta$,.., $|\delta_1+\delta_2+\delta_3+\dots+\delta_{L-1}|<\delta$ and δ is arbitrary small.

The above equations can be written as

$$\epsilon_2 = \frac{1}{2^{\sigma}}\epsilon_1+\delta_2\;,$$

(A1.8) $$\epsilon_3 = \frac{1}{2^\sigma}\epsilon_2+\delta_3=\frac{1}{2^{2\sigma}}\epsilon_1+\frac{1}{2^\sigma}\delta_2+\delta_3 ,$$

$$\epsilon_{L-1} = \frac{1}{2^\sigma}\epsilon_{L-2}+\delta_{L-1}=\frac{1}{2^{(L-2)\sigma}}\epsilon_1+\frac{1}{2^{(L-3)\sigma}}\delta_2+\frac{1}{2^{(L-4)\sigma}}\delta_3+..+\delta_{L-1} .$$

Thus,

(A1.9) $$\epsilon_1+\epsilon_2+\epsilon_3+..+\epsilon_{L-1}=(1+\frac{1}{2^\sigma}+\frac{1}{2^{2\sigma}}+..+\frac{1}{2^{(L-2)\sigma}})\epsilon_1+(\delta_2+\delta_3+..+\delta_{L-1}) + \frac{1}{2^\sigma}(\delta_2+\delta_3+..+\delta_{L-2})+\frac{1}{2^{2\sigma}}(\delta_2+\delta_3+..+\delta_{L-3})+..+\frac{1}{2^{(L-3)\sigma}}\delta_2 .$$

Since $|\delta_2|<|\delta|$, $|\delta_2+\delta_3|<|\delta|$,.., $|\delta_2+\delta_3+...+\delta_{L-1}|<|\delta|$,

$$|\delta_2+\delta_3+..+\delta_{L-1}|+\frac{1}{2^\sigma}|\delta_2+\delta_3+..+\delta_{L-1}|+..+\frac{1}{2^{(L-2)\sigma}}|\delta_2| < |\delta+\frac{1}{2^\sigma}\delta+\frac{1}{2^{2\sigma}}\delta+..+\frac{1}{2^{(L-2)\sigma}}\delta| ,$$

or

(A1.10) $$|\delta_2+\delta_3+..+\delta_{L-1}|+\frac{1}{2^\sigma}|\delta_2+\delta_3+..+\delta_{L-1}|+..+\frac{1}{2^{(L-2)\sigma}}|\delta_2| < |\delta|\frac{2^\sigma}{2^\sigma-1} ,$$

where $|\delta| = \delta$. Thus, Equation (A1.9) can be written as

(A1.11) $$\epsilon_1+\epsilon_2+\epsilon_3+..+\epsilon_{L-1} = \epsilon_1(1+\frac{1}{2^\sigma}+\frac{1}{2^{2\sigma}}+..+\frac{1}{2^{(L-2)\sigma}})+\gamma_1 ,$$

where γ_1 is a small number of the same order as δ . Since δ is arbitrary small that tends to zero as *N* approaches infinity, thus, γ_1 is also a small number that tends to zero as *N* approaches infinity. As *L* in Equation (A1.11) approaches infinity, one may then obtain

(A1.12) $$\sum_{i=1}^{\infty}\epsilon_i = \frac{2^\sigma}{2^\sigma-1}\epsilon_1+\gamma_1 .$$

Therefore, the sum $Mu(\sigma,2;N+1,\infty)$ (which is equal to $\epsilon_1+\epsilon_2+\epsilon_3+...$) is bounded by the sum $Mu(\sigma,2;N+1,2N)$ (which is equal to ϵ_1).

The process of Equations (A1.6) through (A1.12) can be repeated with 2*N* is substituted for *N* and Equation (A1.6) becomes

$$A_2+\delta_2=B_2+\epsilon_2-\frac{1}{2^\sigma}B_2 ,$$

where $A_2=Mu(\sigma;1,2N)$ and $B_2=Mu(\sigma,2;1,2N)$

Thus (refer to Equation (A1.7a))

$$A_2 = B_2 - \frac{1}{2^\sigma} B_2 + \frac{1}{2^\sigma} \epsilon_1 \ ,$$

and following the same process, it can be shown that the sum $Mu(\sigma, 2; 2\mathrm{N}+1, \infty)$ is given by

$$\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{2^\sigma - 1} \epsilon_1 + \gamma_2 \ ,$$

where γ_2 is a small number of the same order as γ_1. Since γ_1 tends to zero as N approaches infinity, thus, γ_2 also tends to zero as N approaches infinity

If we repeat the process l times, we get

$$A_l = B_l - \frac{1}{2^\sigma} B_l + \frac{1}{2^{(l-1)\sigma}} \epsilon_1 \ ,$$

where $A_l = Mu(\sigma; 1, 2^l N)$ and $B_l = Mu(\sigma, 2; 1, 2^l N)$. Furthermore, the sum $Mu(\sigma, 2; 2^l N+1, \infty)$ is given by

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{2^{(l-2)\sigma}} \frac{1}{2^\sigma - 1} \epsilon_1 + \gamma_l \ ,$$

where γ_l is a small number that tends to zero as N approaches infinity.

It is clear that $Mu(\sigma, 2; 2^l N+1, \infty)$ approaches zero as l approaches infinity. Furthermore, as l approaches infinity, B approaches its limit given by

$$\left(1 - \frac{1}{2^\sigma}\right) Mu(\sigma, 2; 1, \infty) = Mu(\sigma; 1, \infty) \ .$$

Following the same steps, it can be also shown that

(A1.13) $$\left(1 - \frac{1}{3^\sigma}\right) Mu(\sigma, 3; 1, \infty) = Mu(\sigma, 2; 1, \infty) \ ,$$

or

(A1.14) $$\left(1 - \frac{1}{2^\sigma}\right)\left(1 - \frac{1}{3^\sigma}\right) Mu(\sigma, 3; 1, \infty) = Mu(\sigma; 1, \infty) \ .$$

Equations (A1.13) and (A1.14) can be proved by first defining

$$Mu(\sigma, 2; 1, N) \ = \ \sum_{n=1}^{N} \frac{\mu(n, 2)}{n^\sigma} \ = \ A_1 \ ,$$

$$Mu(\sigma,2;N+1,3N) = \sum_{n=N+1}^{3N} \frac{\mu(n,2)}{n^\sigma} = \delta_1 ,$$
$$Mu(\sigma,2;3N+1,3^2 N) = \sum_{n=3N+1}^{3^2 N} \frac{\mu(n,2)}{n^\sigma} = \delta_2 ,$$

$$Mu(\sigma,2;3^{L-1}N+1,3^L N) = \sum_{n=3^{L-1}N+1}^{3^L N} \frac{\mu(n,2)}{n^\sigma} = \delta_{L-1} ,$$

and

$$Mu(\sigma,3;1,N) = \sum_{n=1}^{N} \frac{\mu(n,3)}{n^\sigma} = B_1 ,$$
$$Mu(\sigma,3;N+1,3N) = \sum_{n=N+1}^{3N} \frac{\mu(n,3)}{n^\sigma} = \epsilon_1 ,$$
$$Mu(\sigma,3;3N+1,3^2 N) = \sum_{n=3N+1}^{3^2 N} \frac{\mu(n,3)}{n^\sigma} = \epsilon_2 ,$$

$$Mu(\sigma,3;3^{L-1}N+1,3^L N) = \sum_{n=3^{L-1}N+1}^{3^L N} \frac{\mu(n,3)}{n^\sigma} = \epsilon_{L-1} .$$

Since

$$\sum_{n=1}^{3N} \frac{\mu(n,2)}{n^\sigma} = \sum_{n=1}^{3N} \frac{\mu(n,3)}{n^\sigma} - \sum_{n=1}^{N} \frac{\mu(n,3)}{(3n)^\sigma} = \sum_{n=1}^{3N} \frac{\mu(n,3)}{n^\sigma} - \frac{1}{3^\sigma}\sum_{n=1}^{N} \frac{\mu(n,3)}{n^\sigma} .$$

thus

$$Mu(\sigma,2;1,3N) = Mu(\sigma,3;1,3N) - \frac{1}{3^\sigma} Mu(\sigma,3;1,N) .$$

Similarly

$$Mu(\sigma,2;3^l N+1,3^{l+1}N) = Mu(\sigma,3;3^l N+1,3^{l+1}N) - \frac{1}{3^\sigma} Mu(\sigma,3;3^{l-1}N+1,3^l N) .$$

Following the same steps (A1.1) through (A1.12), we can show that

$$\sum_{i=1}^{\infty} \epsilon_i = \frac{3^\sigma}{3^\sigma - 1}\epsilon_1 + \gamma_1 ,$$

where γ_1 is an arbitrary small number.

Similarly, if we define $A_2 = Mu(\sigma,2;1,3N)$ and $B_2 = Mu(\sigma,3;1,3N)$, then

$$A_2 = B_2 - \frac{1}{3^\sigma} B_2 + \frac{1}{3^\sigma} \epsilon_1 \ .$$

Therefore

$$\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{3^\sigma - 1} \epsilon_1 + \gamma_2 \ ,$$

where γ_2 is an arbitrary small number.

Repeating the steps *l* times, one may obtain

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{3^{(l-2)\sigma}} \frac{1}{3^\sigma - 1} \epsilon_1 + \gamma_l \ ,$$

thus, γ_l is a small number that tends to zero as *N* approaches infinity.

Hence, one may conclude that $Mu(\sigma, 3; 3^l N + 1, \infty)$ approaches zero as *l* approaches infinity. Furthermore, as *l* approaches infinity, *B* approaches its limit given by

$$(1 - \frac{1}{3^\sigma}) Mu(\sigma, 3; 1, \infty) = Mu(\sigma, 2; 1, \infty) \ .$$

Repeating the process *r* times, one may conclude

$$Mu(\sigma; 1, \infty) \ = \ Mu(\sigma, p_r; 1, \infty) \prod_{i=1}^{r} (1 - \frac{1}{p_i^\sigma}) \ ,$$

or

(A1.15)
$$Mu(\sigma) \ = \ Mu(\sigma, p_r) \prod_{i=1}^{r} (1 - \frac{1}{p_i^\sigma}) \ .$$

The second part of the theorem can be proved by recalling

$$Mu(\sigma, p_{r-1}; 1, Np_r) \ = \ Mu(\sigma, p_r; 1, Np_r) - \frac{1}{p_r^\sigma} Mu(\sigma, p_r; 1, N) \ .$$

If both series $Mu(\sigma, p_{r-1})$ and $Mu(\sigma, p_r)$ are convergent then as *N* approaches infinity, we obtain

$$Mu(\sigma, p_{r-1}) \ = \ \left(1 - \frac{1}{p_r^\sigma}\right) Mu(\sigma, p_r) \ .$$

By repeating this process *r* times, one then obtains

$$Mu(\sigma) = Mu(\sigma, p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^{\sigma}}\right).$$

Appendix 2:

Assuming RH is valid and for $0.5<\sigma<1$, to show that

$$\sum_{i=r1}^{r2}\frac{1}{p_i^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon ,$$

where $\varepsilon = O\left(\frac{1}{(\sigma-0.5)^2}\, p_{r1}^{1/2-\sigma}\log(p_{r1})\right)$, we first recall that

(A2.1) $$\sum_{i=r1}^{r2}\frac{1}{p_i^{\sigma}} = \int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}}d\pi(x) = \int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}\log x}dx + \int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}}dO(\sqrt{x}\log x).$$

We will first show that

$$\int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}\log x}dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}).$$

Using the substation $\log x = y$, one may then write the above integral as follows

$$\int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}\log x}dx = \int_{\log p_{r1}}^{\log p_{r2}}\frac{e^{(1-\sigma)y}}{y}dy = \int_{\epsilon}^{\log p_{r2}}\frac{e^{(1-\sigma)y}}{y}dy - \int_{\epsilon}^{\log p_{r1}}\frac{e^{(1-\sigma)y}}{y}dy.$$

With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, one then obtains

$$\int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}\log x}dx = \int_{\epsilon/\log p_{r2}}^{1}\frac{e^{(1-\sigma)\log p_{r2} z_2}}{z_2}dz_2 - \int_{\epsilon/\log p_{r1}}^{1}\frac{e^{(1-\sigma)\log p_{r1} z_1}}{z_1}dz_1.$$

With the variable substantiations $w_1 = (1-\sigma)\log p_{r1} z_1$ and $w_2 = (1-\sigma)\log p_{r2} z_2$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_2}}\frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_1}}\frac{dw_1}{w_1}$, one may then write

$$\int_{x=p_{r1}}^{p_{r2}}\frac{1}{x^{\sigma}\log x}dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}}\frac{e^{w_2}-1}{w_2}dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}}\frac{e^{w_1}-1}{w_1}dw_1 + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}}\frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}}\frac{dw_1}{w_1}$$

Since [9, page 230], for $x > 0$

$$\int_0^x \frac{e^t - 1}{t} dt = \mathrm{Ei}(x) - \log(x) - \gamma = -E_1(-x) - \log(x) - \gamma ,$$

one may conclude

$$\text{(A2.2)} \qquad \int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma} \log x} d\,x = E_1\big((\sigma-1)\log p_{r1}\big) - E_1\big((\sigma-1)\log p_{r2}\big) .$$

The integral with the O notation in Equation (A2.1) can be computed by integration by parts. Thus,

$$\int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} d\,O(\sqrt{x}\log x) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{p_{r2}^{\sigma}} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{p_{r1}^{\sigma}} - \int_{x=p_{r1}}^{p_{r2}} O(\sqrt{x}\log x)\, d\left(\frac{1}{x^{\sigma}}\right) .$$

Since $x > 0$, thus

$$\int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} d\,O(\sqrt{x}\log x) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{p_{r2}^{\sigma}} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{p_{r1}^{\sigma}} - O\left(\int_{x=p_{r1}}^{p_{r2}} \sqrt{x}\log x\, d\left(\frac{1}{x^{\sigma}}\right)\right) .$$

With the substitution of variables $\log x = y$, one may then obtain

$$\int_{x=p_{r1}}^{p_{r2}} \sqrt{x}\log x\; d\left(\frac{1}{x^{\sigma}}\right) = -\int_{y=\log p_{r1}}^{\log p_{r2}} \sigma\, y\, e^{(\frac{1}{2}-\sigma)y}\, dy .$$

Since

$$\int x\, e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} ,$$

therefore

$$\int_{x=p_{r1}}^{p_{r2}} \sqrt{x}\log x\, d\left(\frac{1}{x^{\sigma}}\right) = -\sigma\left(\frac{\log p_{r2}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) p_{r2}^{0.5-\sigma} + \sigma\left(\frac{\log p_{r1}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) p_{r1}^{0.5-\sigma} .$$

Hence, for $\sigma > 0.5$, one may conclude

$$\text{(A2.3)} \qquad \int_{x=p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} d\,O(\sqrt{x}\log x) = O\left(\frac{p_{r1}^{\;1/2-\sigma}\log(p_{r1})}{(\sigma-0.5)^2}\right) .$$

Appendix 3:

Assuming the validity of RH, to prove that,

$$-\sum_{i=r1}^{r2}\frac{1}{p_i^{\sigma+it}} = E_1((s-1)\log p_{r2}) - E_1((s-1)\log p_{r1}) + O\left(\frac{t}{(\sigma-0.5)^2}\, p_{r1}^{1/2-\sigma}\log(p_{r1})\right),$$

for $\sigma>0.5$, we first recall Equation (4.2) with $s=\sigma+it$

$$\log\left(\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^{\sigma+it}}\right)\right) = \sum_{i=r1}^{r2}\left(-\frac{1}{p_i^{\sigma+it}}-\frac{1}{2\,p_i^{2(\sigma+it)}}-\frac{1}{3\,p_i^{3(\sigma+it)}}-\ldots\right) = -\sum_{i=r1}^{r2}\frac{1}{p_i^{\sigma+it}}+\delta\,,$$

where $\delta=O\left(p_{r1}^{1-2\sigma}\right)$ is an arbitrary small number for sufficiently large $r1$ (it should be pointed out that the sum $\sum_{i=r1}^{r2}\frac{1}{2\,p_i^{2(\sigma+it)}}$ is finite for $\sigma\geq 0.5$ and $t\neq 0$ by the virtue of the prime number theorem or the absence of zeros on the line $\sigma=1$). We also have

$$-\sum_{i=r1}^{r2}\frac{1}{p_i^{\sigma+it}} = -\sum_{i=r1}^{r2}\frac{\cos(t\log p_i)}{p_i^{\sigma}} + i\sum_{i=r1}^{r2}\frac{\sin(t\log p_i)}{p_i^{\sigma}}\,.$$

We will first compute the first term using Equation (4.19) as the prime counting function to obtain

$$-\sum_{i=r1}^{r2}\frac{\cos(t\log p_i)}{p_i^{\sigma}} = -\int_{p_{r1}}^{p_{r2}}\frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx + \int_{x=p_{r1}}^{p_{r2}}\frac{\cos(t\log x)}{x^{\sigma}}d\,O(\sqrt{x}\log x)\,.$$

The integral with the *O* notation can be computed by integration by parts as shown in Appendix 2 to obtain

$$\int_{x=p_{r1}}^{p_{r2}}\frac{\cos(t\log x)}{x^{\sigma}}d\,O(\sqrt{x}\log x) = O\left(\frac{t}{(\sigma-0.5)^2}\, p_{r1}^{1/2-\sigma}\log(p_{r1})\right).$$

Hence,

$$-\sum_{i=r1}^{r2}\frac{\cos(t\log p_i)}{p_i^{\sigma}} = -\int_{p_{r1}}^{p_{r2}}\frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx + \left(\frac{t}{(\sigma-0.5)^2}\, p_{r1}^{1/2-\sigma}\log(p_{r1})\right).$$

Using the substation $\log x=y$, one may then write the above Equation as follows

$$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = -\int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}\cos(t\,y)}{y}\,dy + \Delta\,,$$

where $\Delta = O\left(\frac{t}{(\sigma-0.5)^2}\; p_{r1}^{\;1/2-\sigma}\log(p_{r1})\right)$. The above integral can be computed if it is modified as follows

$$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = -\int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}\cos(t\,y)}{y}\,dy + \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}\,dy - \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}\,dy + \Delta\,,$$

or

$$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}(1-\cos(t\,y))}{y}\,dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}(1-\cos(t\,y))}{y}\,dy -$$

$$\int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}\,dy + \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y}\,dy + \Delta\,.$$

With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, one then obtains

$$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)\log p_{r2} z2}(1-\cos(t\log p_{r2} z_2))}{z_2}\,dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)\log p_{r1} z_1}(1-\cos(t\log p_{r1} z_1))}{z_1}\,dz_1$$

$$-\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)\log p_{r2} z_2}}{z_2}\,dz_2 + \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)\log p_{r1} z_1}}{z_1}\,dz_1 + \Delta\,.$$

By the virtue of the following identity (Reference [9], page 230)

$$\int_0^1 \frac{e^{at}(1-\cos(bt))}{t}\,dt = \frac{1}{2}\log(1+b^2/a^2) + \mathrm{Ei}(a) + \Re\left[E_1(-a+ib)\right],$$

where $a>0$, one may then have the following

$$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = \Re\left[E_1((s-1)\log p_{r2})\right] + \mathrm{Ei}((1-\sigma)\log p_{r2}) - \Re\left[E_1((s-1)\log p_{r1})\right] - \mathrm{Ei}((1-\sigma)\log p_{r1})$$

$$-\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)\log p_{r2} z_2}}{z_2} dz_2 + \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)\log p_{r1} z_1}}{z_1} dz_1 + \Delta .$$

With the variable substantiations $w_1=(1-\sigma)\log p_{r1} z_1$ and $w_2=(1-\sigma)\log p_{r2} z_2$ and by adding and subtracting the terms $\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r_1}} \frac{d\, w_1}{w_1}$, one may then obtain

$$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = \Re\left[E_1((s-1)\log p_{r2})\right] - \mathrm{Ei}((1-\sigma)\log p_{r2}) -$$
$$\Re\left[E_1((s-1)\log p_{r1})\right] + \mathrm{Ei}((1-\sigma)\log p_{r1}) -$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 -$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_1}{w_1} + \Delta .$$

Since [9, page 230], for $x>0$

$$\int_0^x \frac{e^t-1}{t} dt = \mathrm{Ei}(x) - \log(x) - \gamma ,$$

hence, one may conclude

(A3-1) $$-\sum_{i=r1}^{r2} \frac{\cos(t\log p_i)}{p_i^{\sigma}} = \Re\left[E_1((s-1)\log p_{r2})\right] - \Re\left[E_1((s-1)\log p_{r1})\right] + \Delta .$$

Similarly, using the identity [9, page 230]

$$\int_0^1 \frac{e^{at}\sin(bt)}{t} dt = \pi - \arctan(b/a) + \Im\left[E_1(a+ib)\right] ,$$

one may show that

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\sin(t\log x)}{\log x} dx = \Im\left[E_1((s-1)\log p_{r2})\right] - \Im\left[E_1((s-1)\log p_{r1})\right] ,$$

from which one may obtain

(A3-2) $$-\sum_{i=r1}^{r2} \frac{\sin(t \log p_i)}{p_i^{\sigma}} = \Im\left[E_1((s-1)\log p_{r2})\right] - \Im\left[E_1((s-1)\log p_{r1})\right] + \Delta .$$

Combining Equations A3-1 and A3-2, one may conclude

(A3-3) $$-\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma+it}} = E_1((s-1)\log p_{r2}) - E_1((s-1)\log p_{r1}) + \left(\frac{t}{(\sigma-0.5)^2}\, p_{r1}{}^{1/2-\sigma} \log(p_{r1})\right).$$

It should be pointed out that (A3-3) can be also obtained directly by noting that

$$-\sum_{i=r1}^{r2} \frac{1}{p_i^{s}} = -\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x}\, dx + O\left(\frac{t}{(\sigma-0.5)^2}\, p_{r1}{}^{1/2-\sigma} \log(p_{r1})\right).$$

Using the definition of the Exponential integral as we did in Appendix 2, one may then be able to show

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x}\, dx = -E_1((s-1)\log p_{r2}) + E_1((s-1)\log p_{r1}) .$$

Appendix 4:

In the following, using the Prime Number Theorem (PNT) and for $s=1+it$ where $t\neq 0$, we will show that

(A4.1) $$-\sum_{i=r1}^{r2} \frac{\log p_i}{p_i^s\left(1-p_i^{-s}\right)} = \frac{e^{-(s-1)\log p_{r1}}}{(s-1)} - \frac{e^{-(s-1)\log p_{r2}}}{(s-1)} + O(1/\log p_{r1}{}^k) .$$

We first note that

(A4.2) $$\sum_{i=r1}^{r2} \frac{\log p_i}{p_i^s\left(1-p_i^{-s}\right)} = \sum_{i=r1}^{r2} \frac{\log p_i}{p_i^s}\left(1+\frac{1}{p_i^s}+\frac{1}{p_i^{2s}}+\frac{1}{p_i^{3s}}+\ldots\right).$$

The first sum on the right side of Equation (A4.2) can be written as the following integral

$$\sum_{i=r1}^{r2} \frac{\log p_i}{p_i^s} = \int_{p_{r1}}^{p_{r2}} \frac{\log x}{x^s}\, d\,\pi(x) ,$$

where using PNT, $\pi(x)$ is given by

$$\pi(x) = \mathrm{Li}(x) + O\left(x/(\log x)^k\right).$$

Thus, for $s=1+it$, we may then obtain

$$-\sum_{i=r1}^{r2} \frac{\log p_i}{p_i^{s}} = \frac{e^{-(s-1)\log p_{r1}}}{(s-1)} - \frac{e^{-(s-1)\log p_{r2}}}{(s-1)} + O(1/\log {p_{r1}}^{k}) .$$

Similarly, for $s=1+it$, we can also show that

$$\sum_{i=r1}^{r2} \frac{\log p_i}{p_i^{2s}} \left(1+\frac{1}{p_i^{s}}+\frac{1}{p_i^{2s}}+\frac{1}{p_i^{3s}}+... \right) = O(1/p_{r1}) .$$

Hence, for $s=1+it$, we have

$$-\sum_{i=r1}^{r2} \frac{\log p_i}{p_i^{s}\left(1-p_i^{-s}\right)} = \frac{e^{-(s-1)\log p_{r1}}}{(s-1)} - \frac{e^{-(s-1)\log p_{r2}}}{(s-1)} + O(1/\log p_{r1}) .$$

Appendix 5:

In Appendix 5, we will show that the sum $\sum_{\rho} E_1((s-\rho)\log p_r)$ is convergent if $\Re(s)>0.5$, $\Re(s-\rho)\geq 0$ and $s\neq\rho$ for every ρ. Also, if $|s-\rho| \geq \epsilon > 0$ for every ρ, then for sufficiently large p_r, we will show that

$$\sum_{\rho} E_1((s-\rho)\log p_r) = \frac{p_r^{-s}}{\log p_r} \sum_{\rho} \frac{p_r^{\rho}}{s-\rho} + O(1/(\log p_r)^2).$$

Furthermore, if *s* has a fixed value, then the sum $\sum_{\rho} E_1((s-\rho)\log p_r)$ is given by

$$\sum_{\rho} E_1((s-\rho)\log p_r) = -\frac{p_r^{-s}}{\log p_r} \sum_{\rho} \frac{p_r^{\rho}}{\rho} + O(1/\log p_r).$$

This task is achieved by noting that for sufficiently large $\log p_r$, $E_1((s-\rho)\log p_r)$ can be written as

(A5.1) $$E_1((s-\rho)\log p_r) = \frac{e^{-(s-\rho)\log p_r}}{(s-\rho)\log p_r} \left(1+O\left(\frac{1}{|s-\rho| \log p_r} \right) \right) .$$

The sum $\sum_{\rho} E_1((s-\rho)\log p_r)$ is then given by

(A5.2) $$\sum_{\rho} E_1((s-\rho)\log p_r) = \sum_{\rho} \frac{e^{-(s-\rho)\log p_r}}{(s-\rho)\log p_r} + \delta ,$$

where δ is contribution by the sum of the *O* terms in Equation (A5.1). It can be easily shown that if $|s-\rho| \geq \epsilon > 0$ for every ρ, then δ in Equation (A5.2) tends to zero as p_r approaches infinity. This result can be deduced by noting that $O(\delta)=(1/(\log p_r)^2)\sum_{\rho} 1/|s-\rho|^2$. Since the sum $\sum_{\rho} 1/|s-\rho|^2$ is bounded, therefore $O(\delta)=1/(\log p_r)^2$.

Equation (A5.2) can be then simplified to

$$\sum_{\rho} E_1((s-\rho)\log p_r) \;=\; \frac{p_r^{-s}}{\log p_r} \sum_{\rho} \frac{p_r^{\rho}}{s-\rho} \;+\; O(1/(\log p_r)^2)\,.$$

Let $s=\sigma+iT$ and $\rho_i=\beta_i+i\gamma_i$. We split ρ_i's into two groups. The first group comprises of the non-trivial zeros with γ_i's less than *mT*. The rest of the non-trivial zeros belong to the second group. Since the first group has a finite number of ρ_i's, thus the sum $\sum_{|\gamma_i|<mT} E_1((s-\rho)\log p_r)$ is bounded. Since $|p_r^{-s} p_r^{\rho_i}|<1$ for every ρ_i, therefore $\sum_{|\gamma_i|<mT} E_1((s-\rho)\log p_r)=O(1/\log p_r)$.

The sum over the second group is then given by

$$\sum_{|\gamma_i|>mT} E_1((s-\rho)\log p_r) \;=\; -\frac{p_r^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i} + s\sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^2} + s^2\sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^3} + \ldots..\right).$$

It is well know that the first term $\sum_{|\gamma_i|>mT} p_r^{\rho}/\rho_i$ is convergent. The upper bound for the second term can be determined as follows

$$\frac{p_r^{-s}}{\log p_r}\, s \sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^2} \;\le\; \frac{|p_r^{-s}|}{\log p_r}\,|s| \sum_{|\gamma_i|>mT} \frac{|p_r^{\rho_i}|}{|\rho_i^2|}$$

Since for sufficiently large *T*, $|s|$ is given by *T* and the density of the non-trivial zeros is given by log *t*, thus

$$\frac{p_r^{-s}}{\log p_r}\, s \sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^2} \;\le\; \frac{|p_r^{-\min|\sigma-\beta_i|}|\,T}{\log p_r} \int_{mT}^{\infty} \frac{\log t}{t^2}\, dt\,,$$

or,

$$\frac{p_r^{-s}}{\log p_r}\, s \sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^2} \;\le\; \frac{|p_r^{-\min|\sigma-\beta_i|}|\,T}{\log p_r}\, \frac{O(\log T)}{m}$$

Similarly,

$$\frac{p_r^{-s}}{\log p_r}\, s^2 \sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^3} \;\le\; \frac{|p_r^{-\min|\sigma-\beta_i|}|\,T^2}{\log p_r}\, \frac{O(\log T)}{m^2}\,.$$

and so on. Consequently,

$$\left|\frac{p_r^{-s}}{\log p_r}\left(s\sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^2} + s^2\sum_{|\gamma_i|>mT} \frac{p_r^{\rho_i}}{\rho_i^3} + \ldots..\right)\right| \;\le\; \frac{|p_r^{-\min|\sigma-\beta_i|}|\,T^2}{\log p_r} \sum_{i=1}^{\infty} \frac{1}{m^i}\,,$$

or

$$\left|\frac{p_r^{-s}}{\log p_r}\left(s\sum_{|\gamma_i|>mT}\frac{p_r^{\rho_i}}{\rho_i^2}+s^2\sum_{|\gamma_i|>mT}\frac{p_r^{\rho_i}}{\rho_i^3}+.....\right)\right| = O(1/\log p_r).$$

Hence the sum $\sum_\rho E_1((s-\rho)\log p_r)$ is convergent and it is given by

$$\sum_\rho E_1((s-\rho)\log p_r) = \frac{p_r^{-s}}{\log p_r}\sum_\rho\frac{p_r^\rho}{s-\rho} + O(1/\log p_r)$$

Furthermore, if $\Im(s)\le T$ then

$$\sum_{|\gamma_i|>mT} E_1((s-\rho)\log p_r) = -\frac{p_r^{-s}}{\log p_r}\sum_{|\gamma_i|>mT}\frac{p_r^{\rho_i}}{\rho_i} + O(1/\log p_r)$$

Since the sum $(p_r^{-s}/\log p_r)\sum_{|\gamma_i|\le mT}(p_r^\rho/\rho_i)$ is given by $O(1/\log p_r)$ when $\Im(s)\le T$, therefore for s with $\Im(s)\le T$, we have

$$\sum_\rho E_1((s-\rho)\log p_r) = -\frac{p_r^{-s}}{\log p_r}\sum_\rho\frac{p_r^\rho}{\rho} + O(1/\log p_r)$$

It is well known that on RH, the density of the root on the critical line is given by $O(1/\log t)$. Consequently, if all the roots are located on the critical line then $\sum_\rho p_r^\rho/\rho = O(\sqrt{p_r}(\log p_r)^2)$ (refer to [1], section 5.5). The derivation of the estimate $O(\sqrt{p_r}(\log p_r)^2)$ is based on splitting the roots into two groups. The first group comprises of the roots with $\Im(\rho)\le p_r$. The contribution by this group to the estimate is governed by the sum $p_r^{1/2}\sum_{\Im\rho\le p_r} 1/\Im(\rho)$. The second group comprises of the roots with $\Im(\rho)>p_r$. The contribution by this group to the estimate is governed by the sum $p_r^{3/2}\sum_{\Im\rho>p_r} 1/(\Im(\rho))^2$.

We now consider the case where there are non-trivial zeros off the critical line. For this case, Bohr Landau theorem (refer to [1], section 9.6) states that for every ϵ there is a constant K such that the number of roots ρ's in the range $\{\Re(s)\ge 0.5+\epsilon, 0\le\Im(s)\le T\}$ is less than KT for all T. In other words, the density of the roots off the critical line is less than a constant K (or $O(1)$) compared with $\log(T)$ on the critical line). If we denote $N(\sigma, T)$ as the number of zeros $\beta+i\gamma$ such that $\beta>\sigma$ and $0<\gamma\le T$, then using Bohr-Landua theorem $N(\sigma,T)=O(T)$. Several theorems are presented in Titchmarch [7] that provide smaller bounds for $N(\sigma,T)$. Theorem 9.17 shows that $N(\sigma,T)=O(T^{4\sigma(1-\sigma)+\epsilon})$, where $0.5<\sigma<1$. Using this theorem, one may show that if $0.5<a<1$, $\Re(\rho)\le a$ for all 's ρ's and $\Re(\rho)=a$ for a finite or infinite number of ρ's, then if $\Re(s)=a$ and

$|s-\rho| \geq \epsilon > 0$ for every ρ, we have

$$\sum_{\rho} E_1\left((s-\rho)\log p_r\right) \;=\; O(1/p_r).$$

In other words; if $0.5<a<1$, $\Re(\rho)\leq a$ and $|s-\rho| \geq \epsilon > 0$ for every ρ, then the the sum $\sum_{\rho} E_1\left((s-\rho)\log p_r\right)$ approaches zero as p_r approaches infinity.

On the other hand, if $\Re(\rho)=0.5$ for all ρ's, then for $\Re(s)=0.5$ and $|s-\rho| \geq \epsilon > 0$, we have

$$\sum_{\rho} E_1\left((s-\rho)\log p_r\right) \;=\; O(\log p_r).$$

In other words; if all the non-trivial zeros are on the critical line and $\Re(s)=0.5$, then the $\sum_{\rho} E_1\left((s-\rho)\log p_r\right)$ diverges as p_r approaches infinity.

Appendix 6:

In Appendix 6, we will derive a formula, based on Theorem 3, for the growth of the difference between the actual prime numbers p_i and their ideal values of $\mathrm{Li}^{-1}(i)$. This will be achieved by first noting that for $s=1+it$, we have

$$\zeta(s) = \lim_{r\to\infty}\left\{\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)^{-1}\right\}.$$

Furthermore, Theorem 3 states that for $s=1+it$, we have

$$\log\,\zeta(s) \;=\; E_1((s-1)\log p_{a2}) - \sum_{i=1}^{a2}\log\left(1-\frac{1}{p_i^s}\right),$$

where the equality of both sides is attained as p_{a2} approaches infinity. For $a1<a2$, we have

(A6.1) $$\log\,\zeta(s) \;=-\sum_{i=1}^{a1-1}\log\left(1-\frac{1}{p_i^s}\right) - \sum_{i=a1}^{a2}\log\left(1-\frac{1}{p_i^s}\right) + E_1((s-1)\log p_{a2}).$$

Subtracting the term $E_1((s-1)\log p_{a1})$ from both sides of Equation (A6.1), we obtain

$$\log\,\zeta(s) - E_1((s-1)\log p_{a1}) = -\sum_{i=1}^{a1-1}\log\left(1-\frac{1}{p_i^s}\right) - \sum_{i=a1}^{a2}\log\left(1-\frac{1}{p_i^s}\right) - E_1((s-1)\log p_{a1}) + E_1((s-1)\log p_{a2})$$

If we denote C as

(A6.2) $$C \;=\; -\sum_{i=1}^{a1}\log\left(1-\frac{1}{p_i^s}\right) + E_1((s-1)\log p_{a1}),$$

then we have the following equation

$$-\Re\left(\log\left(\zeta(s)\right)\right) = -\sum_{i=al}^{a2}\sum_{n=1}^{\infty}\frac{\cos\left(nt\log\left(p_i\right)\right)}{n\,p_i^{n\sigma}} - \Re\left(E_1\left(\left(s-1\right)\log p_{a2}\right)\right) + \Re\left(E_1\left(\left(s-1\right)\log p_{al}\right)\right) + \Re\left(C\right)$$

Consider, instead of the set of prime numbers p_1 , p_2 , .., p_n (where $n=\pi(p_n)$), the set of numbers m_1 , m_2 , .. m_n where $n=[\mathrm{Li}(m_n)]$ (where $[x]$ is the integer value of *x)*. Let k_i be the difference between the prime number p_i and its ideal value m_i (i.e. $p_i=m_i+k_i$). Thus, by the virtue of the Prime Number Theorem k_i/m_i and k_i/p_i approach zero as *i* approaches infinity.

Thus, referring to Appendix 7 and noting that $\log p_i=\log m_i(1+k_i/m_i)=\log m_i+\log(1+k_i/m_i)$ (where $\log(1+k_i/m_i)= k_i/m_i+O(k_i^2/m_i^2)$ approaches zero as *i* approaches infinity), one may then write

$$-\Re\left(E_1\left(\left(s-1\right)\log p_{a2}\right)\right) + \Re\left(E_1\left(\left(s-1\right)\log p_{al}\right)\right) = \sum_{i=al}^{a2}\Re\left(m_i^{-s}\right) + \varepsilon_1 = \sum_{i=al}^{a2}\frac{\cos\left(t\log m_i\right)}{m_i^{\sigma}} + \varepsilon_1 ,$$

where, $\varepsilon_1=O(p_{al}^{-\sigma})$. Thus

$$-\Re\left(\log\left(\zeta(s)\right)\right) = -\sum_{i=al}^{a2}\sum_{n=1}^{\infty}\frac{\cos\left(nt\log\left(p_i\right)\right)}{n\,p_i^{n\sigma}} + \sum_{i=al}^{a2}\frac{\cos\left(t\log m_i\right)}{m_i^{\sigma}} + C_1 ,$$

where $C_1 = \Re(C)+\varepsilon_1$ is bounded.

Let C_2 be defines as

$$C_2 = -\sum_{i=al}^{a2}\sum_{n=2}^{\infty}\frac{\cos\left(nt\log\left(p_i\right)\right)}{n\,p_i^{n\sigma}} + C_1 .$$

Thus, C_2 is also bounded for $\sigma>0.5$. Hence, we have the following equation,

$$\text{(A6.3)}\qquad -\Re\left(\log\left(\zeta(s)\right)\right) = -\sum_{i=al}^{a2}\frac{\cos\left(t\log p_i\right)}{p_i^{\sigma}} + \sum_{i=al}^{a2}\frac{\cos\left(t\log m_i\right)}{m_i^{\sigma}} + C_2$$

Substituting m_i+k_i for p_i , one may then obtain,

$$-\Re\left(\log\left(\zeta(s)\right)\right)=-\sum_{i=al}^{a2}\exp\left(-\sigma\log\left(m_i+k_i\right)\right)\cos\left(t\log\left(m_i+k_i\right)\right)+\sum_{i=al}^{a2}\exp\left(-\sigma\log m_i\right)\cos\left(t\log m_i\right)+C_2$$

Let $\delta_i=k_i/m_i$, where δ_i approaches 0 as *i* approaches infinity, then one may write,

$$\log\left(m_i+k_i\right) = \log m_i\left(1+\delta_i\right) = \log m_i + \log\left(1+\delta_i\right)$$

Since $\log(1+\delta_i) = \delta_i+O(\delta_i^2)$, therefore,

$$\log(m_i+k_i) = \log m_i + \delta_i+O(\delta_i^2)$$

Hence,

$$\exp(-\sigma\log(m_i+k_i)) = (1-\sigma\delta_i+O(\delta_i^2))\exp(-\sigma\log m_i)$$

Similarly,

$$\cos(t\log(m_i+k_i)) = \cos(t\log m_i)\cos(t\delta_i+tO(\delta_i^2)) - \sin(t\log m_i)\sin(t\delta_i+tO(\delta_i^2))$$

if we choose $a1$ so that $t\delta_i \ll 1$, then,

$$\cos(t\log(m_i+k_i)) = (1+O(\delta_i^2))\cos(t\log m_i) - t\delta_i(1+O(\delta_i^2))\sin(t\log m_i)$$

where by the virtue of the Prime Number Theorem, the term $(1+O(\delta_i^2))$ approaches 1 as *i* approaches infinity. Hence, Equation (A6.3) can be written as:

$$-\Re\left(\log(\zeta(s))\right) = \sum_{i=a1}^{a2}\frac{\delta_i}{m_i^{\sigma}}\left(t\sin(t\log m_i)+\sigma\cos(t\log m_i)\right)+C_3$$

where C_3 is equal to C_2 plus the sum of the terms that contain $O(\delta_i^2)$ (Since $\delta_i=k_i/m_i$, the sum of these terms is comparable with the integral $\int 1/x^{1+\sigma}dx$). For each *t*, the sum of these terms is bounded and can be chosen to be less than 1 for sufficiently large p_{a1} . Thus,

$$\text{(A6.4)} \qquad -\Re\left(\log(\zeta(s))\right) = \sum_{i=a1}^{a2}\frac{k_i}{m_i^{1+\sigma}}\left(t\sin(t\log m_i)+\sigma\cos(t\log m_i)\right) + C_3$$

where $m_i=\mathrm{Li}^{-1}(i)$ represents the ideal value for the prime number $p_i=\pi^{-1}(i)$ and k_i is the deference between p_i and m_i . Thus, Equation (A6.4) establishes a relationship between the $\zeta(s)$ and k_i . If all k_i 's are zero, then the value of $\log|\zeta(s)|$ will be given by the C_3 which bounded for each *t*. It should be pointed out that Equation (A6.4) is valid only in the right section of the the critical strip that is void of non-trival zeros. Therefore, Equation (A6.4) establishes the relationship between k_i 's and the non-trival zeros of $\zeta(\sigma+it)$ with the highest value of σ . Therefore, one can then use this equation to determine the growth of k_i by first writing Equation (A6.4) as the following integral

$$-\Re\left(\log(\zeta(s))\right) = \int_{x=m_{a1}}^{m_{a2}}\frac{f(x)}{x^{1+\sigma}}\left(t\sin(t\log x)+\sigma\cos(t\log x)\right)d\,\mathrm{Li}(x) + C_3\,,$$

where $f(x)=k_{\mathrm{Li}(x)}$ .Thus,

$$-\Re\left(\log(\zeta(s))\right) = \int_{x=m_{a1}}^{m_{a2}}\frac{f(x)}{x^{1+\sigma}\log x}\left(t\sin(t\log x)+\sigma\cos(t\log x)\right)dx + C_3\,.$$

Using the substitutions $y = \log x$ and $dx = e^y\, dy$, one may write the above integral as

$$\text{(A6.5)} \quad -\Re\left(\log\left(\zeta(s)\right)\right) \;=\; \int_{y=\log m_{a1}}^{\log m_{a2}} \frac{h(y)}{y}\, e^{-\sigma y}\left(t \sin(t\,y) + \sigma \cos(t\,y)\right) dy \;+\; C_3\,,$$

where, $f(x) = f(e^y) = h(y)$. Equation (A5.5) should be compared with product formula [1]

$$\text{(A6.6)} \qquad \log\zeta(s) \;=\; \log\zeta(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) - \log\prod\left(\frac{s}{2}\right) + \frac{s}{2}\log\pi - \log(s-1)\,,$$

where the above sum is performed over the zero's (ρ) of $\zeta(s)$ in the critical strip.

From Equation (A6.5), one may be able to compute the derivative of $\log\zeta(s)$ using Cauchy-Riemann equations. Using Cauchy-Riemann equations, the derivative of a complex function $f(z)$ (where $z = x + iy$) that is analytic over the region Ω is given by

$$f'(z) = \frac{\partial\Re(f(z))}{\partial x} - i\frac{\partial\Re(f(z))}{\partial y}\,.$$

Thus,

$$\text{(A5.7)} \qquad \frac{d}{ds}\log\zeta(s) \;=\; \frac{\zeta'(s)}{\zeta(s)} \;=\; \frac{\partial\Re(\log\zeta(s))}{\partial\sigma} - i\frac{\partial\Re(\log\zeta(s))}{\partial t}\,.$$

Hence,

$$\text{(A6.8)} \quad \Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) \;=\; \int_{y=\log m_{a1}}^{\log m_{a2}} h(y)\, e^{-\sigma y}\left(t \sin(t\,y) + \sigma\cos(t\,y)\right) dy \;-\; \int_{y=\log m_{a1}}^{\log m_{a2}} \frac{h(y)}{y}\, e^{-\sigma y}\cos(t\,y)\, dy + C_4$$

$$\text{(A6.9)} \quad \Im\left(\frac{\zeta'(s)}{\zeta(s)}\right) \;=\; \int_{y=\log m_{a1}}^{\log m_{a2}} h(y)\, e^{-\sigma y}\left(t \cos(t\,y) - \sigma\sin(t\,y)\right) dy \;+\; \int_{y=\log m_{a1}}^{\log m_{a2}} \frac{h(y)}{y}\, e^{-\sigma y}\sin(t\,y)\, dy + C_5$$

where C_4 and C_5 are both bounded for values of s that are not in the vicinity of s=1. Equations (A6.8) and (A6.9) should be compared with the formula for $\zeta'(s)/\zeta(s)$ given by [1]

$$\text{(A6.10)} \qquad -\frac{\zeta'(s)}{\zeta(s)} \;=\; \frac{s}{s-1} - \sum_{\rho} \frac{s}{\rho(s-\rho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}\,.$$

To make this comparison, we first combine Equations (A6.8) and (A6.9) to obtain;

$$\frac{\zeta'(s)}{\zeta(s)} \;=\; \int_{y=\log m_{a1}}^{\log m_{a2}} \left(h(y)(\sigma + it) + \frac{h(y)}{y}\right) e^{-(\sigma+it)y}\, dy \;+\; C_6$$

or,

$$\frac{\zeta'(s)}{\zeta(s)} = \int_{y=\log m_{a1}}^{\log m_{a2}} \left(s\ h(y)+\frac{h(y)}{y}\right) e^{-sy}\, dy + C_6$$

Thus, by letting m_2 approaches infinity, one may then obtain,

$$\frac{\zeta'(s)}{\zeta(s)} = \int_{y=0}^{\infty} \left(s\ h(y)+\frac{h(y)}{y}\right) e^{-sy}\, dy + C_7$$

where C_7 is a bounded and analytic for every *s* that is to the right of the non trivial zero(s) with the maximum value for σ and is given by,

$$C_7 = \int_{y=0}^{\log m_1} \left(s\ h(y)+\frac{h(y)}{y}\right) e^{-sy}\, dy + C_6$$

Now, if we consider that *h(y)* is the sum of two components $h_1(y)$ and $h_2(y)$, i.e.,

$$h(y) = h_1(y)+h_2(y),$$

where $h_1(y)$ is the component that is generating the term of the sum over ρ in Equation (A6.10) then,

$$s\, L(h_1(y)) + L\left(\frac{h_1(y)}{y}\right) = \sum_{\rho} \frac{s}{\rho(s-\rho)},$$

and

$$s\, L(h_2(y)) + L\left(\frac{h_2(y)}{y}\right) = -C_7 - \frac{s}{s-1},$$

where *L (f(y))* denotes the Laplace Transform of the function *f(y)*. It should be pointed out, that one of the components of C_7 is generated by the term $E_1((s-1)\log p_{a1})$ (Equation A6.3) that has a singularity at *s*=1. This singularity eliminates the singularity of $\zeta(s)$ at *s*=1 and consequently eliminates the pole at *s*=1. Therefore, the term $C_7+s/(s-1)$ has no singularities for $\sigma \geq 0.5$ and consequently, $h_2(y)$ grows no faster than $e^{(0.5y)}$.

Hence, we have

$$h_1(y) = \sum_{\rho} \frac{e^{y\rho}}{\rho} \left(1-\frac{1}{y\rho}\right)^{-1},$$

and for large values of y, we have

$$h(y) = \sum_{\rho} \frac{e^{y\rho}}{\rho} + \text{ lesser terms },$$

Therefore,

$$f(x) = \sum_{\rho} \frac{x^{\rho}}{\rho} \;+\; \text{lesser terms} ,$$

where $f(x)=k_{\mathrm{Li}(x)}$ and k_i is the deference between p_i and m_i ($m_i=\mathrm{Li}^{-1}(i)$ represents the ideal value for the prime number $p_i=\pi^{-1}(i)$).

Appendix 7:

For $\sigma>0.5$, we will show that

$$\sum_{i=a1}^{a2} \frac{1}{m_i^{\sigma+it}} \;=\; -E_1((s-1)\log p_{a2}) + E_1((s-1)\log p_{a1}) + O\left(p_{a1}^{-\sigma}\right) .$$

First, we write

$$\sum_{i=a1}^{a2} \frac{1}{m_i^{\sigma+it}} \;=\; \sum_{i=a1}^{a2} \frac{\cos(t\log m_i)}{m_i^{\sigma}} + i\sum_{i=a1}^{a2} \frac{\sin(t\log m_i)}{m_i^{\sigma}} .$$

The two sums on the right side can be written as integrals as we did in Appendix 3. However, instead of Equation (4.4), we use the following equation to represent $\pi_m(x)$ (which is the number of m_i's that are less than *x)*

$$\pi_m(x) = \mathrm{Li}(x)+\varepsilon ,$$

where ε can take any value between 0 and 1.

Thus, one may obtain

$$\sum_{i=a1}^{a2} \frac{\cos(t\log m_i)}{m_i^{\sigma}} \;=\; \int_{p_{a1}}^{p_{a2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx + O\left(p_{a1}^{-\sigma}\right) ,$$

and

$$\sum_{i=a1}^{a2} \frac{\sin(t\log m_i)}{m_i^{\sigma}} \;=\; \int_{p_{a1}}^{p_{a2}} \frac{e^{-\sigma\log x}\sin(t\log x)}{\log x}dx + O\left(p_{a1}^{-\sigma}\right) .$$

The above two integrals were computed in Appendix 3. Therefore,

$$\sum_{i=a1}^{a2} \frac{\cos(t\log m_i)}{m_i^{\sigma}} \;=\; -\Re\left[E_1((s-1)\log p_{a2})\right] + \Re\left[E_1((s-1)\log p_{a1})\right] + O\left(p_{a1}^{-\sigma}\right) ,$$

and

$$\sum_{i=a1}^{a2} \frac{\sin(t \log m_i)}{m_i^{\sigma}} = -\Im\left[E_1((s-1)\log p_{a2})\right] + \Im\left[E_1((s-1)\log p_{a1})\right] + O\left(p_{a1}{}^{-\sigma}\right).$$

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