MATHEMATICAL THEORY OF MAGNETIC FIELD

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Abstract. The study of magnetic fields produced by steady currents is a full-valued physical theory which like any other physical theory employs a certain mathematics. This theory has two limiting cases in which source of the field is confined on a surface or a curve. It turns out that mathematical methods to be used in these cases are completely different and differ from from that of the main of the main part of this theory, so, magnetostatics actually consists of three distinct theories. In this work, these three theories are discussed with special attention to the case current carried by a curve. In this case the source serves as a model of thin wire carrying direct current, therefore this theory can be termed magnetostatics of thin wires. The only mathematical method used in this theory till now, is the method of Green's functions. Critical analysis of this method completed in this work, shows that application of this method to the equation for vector potential of a given current density has no foundation and application of this method yields erroneous results

1. Introduction

Magnetostatics is an integral part of classical electrodynamics which describes magnetic fields produced by steady current densities. Usually, magnetic field is represented by its strength $\vec{H}$ which obeys a shortened version of one of Maxwell equations

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} \tag{1.1}$$

where $\vec{J}$ stands for the current density. This vector is constrained with the electric charge conservation law, which in stationary case takes the form

$$\nabla \cdot \vec{J} = 0. \tag{1.2}$$

Besides, magnetic strength obeys another Maxwell equation

$$\nabla \cdot \vec{H} = 0$$

which signifies non-existence of magnetic monopoles. This equation signifies also that there exists a vector potential $\vec{A}$ defined by the equation

$$\vec{H} = \nabla \times \vec{A}. \tag{1.3}$$

Hereafter, magnetic field is presented by the vector field $\vec{A}$.

Substituting the equation (1.3)into the shortened Maxwell equation (1.2) turns it into the field equation for the vector potential

$$\nabla \times \nabla \times \vec{A} = \frac{4\pi}{c}\vec{J}. \tag{1.4}$$

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Since this moment on, one can consider magnetostatics to be a full-valued physical theory because it possesses a certain physical contents and a certain mathematical structure. The earlier is the study of stationary magnetic fields produced by steady sources and the latter is specified by the field equation and some linear functional space of its solutions.

2. The structure of magnetostatics

There exist three kinds of supports of current density which have dimensions ranging from 1 to 3 that actually divides magnetostatics into three distinct theories. Below we discuss all of them and describe their mathematical structures. Thereby, we show that two of them are built properly and then pass to the third one. It turns out that mathematical structure of the third one is based on the notion of Green's function, that gives one a reason to explore the problem of existence of Green's function for the field equation (1.4). Solution of this problem is presented in the end of this work.

In case of dimension 3 it is convenient to employ orthogonal sets of solutions of the vector wave equation

$$\nabla \times \nabla \times \vec{u} = \lambda \vec{u}. \tag{2.1}$$

It is evident that solutions of this equation are particular solutions of the field equation 1.4 with current density and vector potential of same form. These vector fields constitute an orthogonal set in a functional space of magnetic strengths and current densities. An arbitrary current density can be decomposed over this set that turns the field equation into an algebraic equation for the decomposition coefficients. This fact provides an approach to the field equation and specifies mathematical structure of the main part of magnetostatics. Besides, this theory possesses two limiting cases in which this mathematical structure does not work. They are discussed below.

Under some physical conditions electric currents flow only on a surface. In this case it can only be represented as surface current density that completely changes mathematical structure of this limiting case of the theory. Since a surface has measure zero, magnetic strength in question is curl-free almost everywhere, hence, there exists a scalar potential Φ for it so that $\vec{H} = \nabla\Phi$ almost everywhere. Evidently, this potential obeys Laplace equation which thus underlies magnetostatics of surface currents. Any simple surface either divides the space into two parts or has an edge. In any case, the most convenient approach is to obtain the field from solutions of the Laplace equation and match their tangential and normal derivatives on the surface due to the theory of double layer. Consequently, mathematical structure of magnetostatics of surface currents is given by Cauchy problem for the Laplace equation rather than a linear space of eigenfunctions of the double curl operator. This limiting case can be considered as an independent theory because it possesses its own mathematical structure not shared with the main theory.

There exists another limiting case in which support of current density is a curve. In this case magnetic strength also possesses a scalar potential, but solutions of the Laplace equation are useless due to lack of surface on which boundary conditions could be specified. The case of straight line as the support of the current density will be considered below, but it must be noted that magnetic field of a straight wire carrying direct current is known from the Ampere law itself. In any case, mathematical structure of magnetostatics of thin wires differs from those considered above. This mathematical structure will be discussed below, but before doing it, we need to clarify the problem definition.

Due to the charge conservation law, the value of current at each point of the curve is same, so, this value can be put equal to unity. Therefore, the only specification of the source of the field is the shape of the curve. Hence, in fact, the task is to find magnetic field of a given curve. The only condition the curve must meet, is that it has no endpoints. All we know about the field is that its strength $\vec{H}$ is continuous everywhere but the curve, and its contour integrals over all simple circuits

$$\oint \vec{H}\,\mathrm{d}\vec{l} \tag{2.2}$$

are equal either to 2π if the circuit surrounds the curve, or to zero otherwise. The only approach to the problem used till now is based on the method of Green's functions.

3. Sufficient condition of existence of Green's function for the shortened Maxwell equation

The method of Green's functions is believed to be applicable to any linear non-uniform equation and was successfully applied to numerous problems of this kind. This method is demonstrated the best way in electrostatics by applying it to the Poisson equation

$$\triangle\Phi = 4\pi\rho.$$

Let P and P' be two arbitrary points in the space and $\Psi(P, P')$ be potential produced by a unit point-like charge placed in the point P', and measured in the point P. Then the entire potential produced by a charge density ρ and measured in the point P is

$$\Phi(P) = \int \mathrm{d}P'\rho(P')\Psi(P, P'), \tag{3.1}$$

where $\mathrm{d}P'$ stands for the measure of volume integration over the space. The kernel of this integral transformation $\Psi(P, P')$ is termed the Green's function for the Poisson equation. Note that existence of this integral transformation is deduced from the assumption that each point of the source of the field contributes independently, consequently, the source can be divided into arbitrarily small parts so that one can pass to the limit in which the sum of contributions turns into an integral. In other words, existence of independent contributions of parts of the entire source constitutes a sufficient condition of existence of the integral transformation (3.1) and, thereby, of the Green's function itself. In this section we explore existence of Green's function for the field equation (1.4).

The reason to explore existence of Green's function in this case comes from the additional condition which the source must meet, namely, the charge conservation law presented in the form of the equation (1.2). In other words, the question to be answered is, whether the charge conservation law is compatible with the assumption that the entire current density can be divided into parts so that each of them contributes the entire vector potential independently. Therefore, non-existence of Green's function would signify that in general the method gives wrong results. This may well happen because of the following. It is evident enough that if the current density $\vec{J}$ does not satisfy the equations (1.2), the equation (1.4) with this current density as the right-hand side has no solutions. This fact reveals if one takes divergence of both sides and obtains a nonsense equation of the form $0 \neq 0$ that signifies non-existence of vector potential to be produced by such a source. At the same time, an integral transformation with any two-point Green's function always exists. The vector potential of a given current density $\vec{J}(\vec{r})$ obtained by the method of Green function is presented in the book [1] and has the form

$$\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\, \mathrm{d}\vec{r}'. \tag{3.2}$$

In case of circular current loop, the method yields the following expression for the vector potential in spherical coordinates:

$$A_\varphi = \frac{I}{ca}\int\limits_0^{2\pi} \frac{\cos\phi'\,\mathrm{d}\phi'}{(a^2+r^2-2ar\sin\theta\cos\phi')^{1/2}}. \tag{3.3}$$

If in this particular case the method gives a correct result it must be confirmed by substituting the vector potential obtained into the field equation (1.4). So should be done in case of an arbitrary curve, but the integral is too complicated, so, in general the method cannot be verified this way. However, since this integral exists also for an arc of the loop which has endpoints and hence, under no circumstances produces any vector potential, we have to test foundations of the method.

If division of the current density into parts by dividing the space into pieces breaks the charge conservation law, the integral transformation of the current density into the vector potential produced by it, is not justified a priori. Now we show that indeed, such a division of the source breaks the charge conservation law. For this end we introduce the following notations. Let B be a piece of the space and ∂B be its boundary which is a simple (sphere-like) surface in form. We introduce a step function θ as follows:

$$\theta(P) = \begin{cases} 1, & P \in B \\ 0, & P \notin B \end{cases} \tag{3.4}$$

and its gradient $\vec{\delta} = \nabla\theta$. Evidently, the vector field $\vec{\delta}$ is singular. It has the δ-function singularity on the surface ∂B and points inwards B being strictly orthogonal to the surface.

Let now $\vec{J}$ be a current density which satisfies the equation (1.2) and hence, produces a certain magnetic field. The question is, whether so does the part of this source confined inside the domain B. It does so if the current density restricted this way satisfies the equation (1.2). That is what we are going to check out. As seen, this part of the source has current density equal to $\theta\vec{J}$, so, it remains to substitute this vector into the equation. Taking divergence of this vector gives a δ-function singularity on the surface ∂B:

$$\nabla\cdot(\theta\vec{J}) = \vec{J}\cdot\vec{\delta} \neq 0.$$

Hence, division of a current density by dividing the space in parts entails a fast breakdown of the charge conservation law. This fact signifies that division of the current density by dividing the space into pieces is incompatible with the charge conservation law, consequently, an assumption of independent contributions of "elements of current" to the entire vector potential, is erroneous. Hence, sufficient condition of applicability of the method of Green's functions to the field equation (1.4) is broken. In the next section we formulate the necessary condition of existence of Green's function for vector potential.

4. Necessary condition

Till now there was no question of applicability of the method of Green's functions to the field equation (1.4) because the sufficient condition considered above, was always believed to be completed. Now, it turns out that this condition is broken, therefore we have to return to the very beginning and either justify the method or show that it does not work at all. In fact, failure of one sufficient condition does not mean anything. However, since this moment on, everything changes. In particular, the equation (3.3) which was believed to be correct due to that condition, now needs to be tested another way that is very difficult.

The main reason why yet the method might be working consists in the following. Since the equation (1.4) is linear, there exists a linear map $\vec{J} \to \vec{A}$. Since such a map exists only for an entire source, it can only be an integral transformation. If the form of such an integral transformation is given once and forever, the Green's function exists and the method does not need any justification. The main reason why the method might be wrong, is that there is no guarantee that the source enters the integrand of this integral transformation [1].

The method might well work properly, if the following condition is completed: in all particular cases the integral (3.2) provides a correct result. Evidently, this is the only necessary condition which can be suggested in the situation encountered. Hence, if no general proof the the method exists, only particular cases make sense. It must be pointed out that verification of the method becomes an extremely difficult task, because now it is not enough to write down an integral like that presented in the equation (3.3), it must be taken and substituted into the field equation (1.4) and the equation must either turn into an identity if the result is correct, or not otherwise. Every particular case which confirms it, confirms only that the method gives a correct result in this particular case, but does not justify the method in general case. At the same time, a single example in which the result obtained is wrong, provides a proof that the method does not work at all. In other words, it suffices to present a single example in which the method gives a wrong result to prove that the vector potential cannot be represented in the form of integral transformation like (3.2) that signifies non-existence of Green's functions for the field equation. This will be done in the next section.

5. Examples

In this section, we consider few examples of current density for which the integral (3.2) can be taken analytically. We start with the simplest example of magnetic field of a thin wire which is a straight line in form. Magnetic field of this source is well-known and the integral (3.2) for it can be taken in analytical form. The result can be substituted into the field equation (1.4). To complete the calculation we introduce round cylinder coordinates $\{z, \rho, \varphi\}$ in which the line coincides with the axis $\rho = 0$. The task is to take the integral for the current density in question and check out if the vector potential obtained this way

[1] the author knows that it is not so

yields the correct expression for the magnetic strength. Note that the vector potential has only z-component and solution of the field equation is

$$A_z = \pm \ln \frac{\rho}{a}$$

where a is an arbitrary constant of dimension of length. Now, let us calculate the integral (3.2) for the value of the z-component for a point with coordinates $z = 0$, an arbitrary value of $\varphi = 0$ and some certain value of ρ which needs no special symbol. The integral has the form

$$\int \frac{\mathrm{d}z}{\sqrt{z^2 + \rho^2}} = \operatorname{arcsinh} \frac{z}{\rho}$$

and substituting the limits from $z = -\infty$ to $z = +\infty$ we find, first, that the integral diverges and second, that arcsinh turns into natural logarithm. The earlier does not mean that the result is physically erroneous, because the vector potential obtained yields the correct expression for the strength after the following renormalization procedure is made:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{z^2 + \rho^2}} = \lim_{b\to\infty} \int_{-b}^{b} \frac{\mathrm{d}z}{\sqrt{z^2 + \rho^2}} = 2 \lim_{b\to\infty} \operatorname{arcsinh} \frac{b}{\rho} =$$

$$= 2 \lim_{b\to\infty} \ln \frac{b}{\rho} = 2 \lim_{b\to\infty} (\ln b - \ln \rho) \to \ln \rho$$

where an infinite constant which does not contribute the strength, is subtracted. Besides, we assume that a unit of length is used so that both a and ρ are dimensionless. The final result gives the correct expression for A_z.

The second and the third examples are presented by impossible current densities which do not satisfy the equation (1.2) and which, nevertheless, play an important role in our proof of non-existence of Green's function for the equation (1.4). Consider a straight wire with one endpoint, carrying a direct current. Though the charge conservation law is broken in its endpoint and no vector potential exists for such a source, the integral (3.2) exists and can be taken analytically. The conductor is presented by the half-line $\rho = 0$, $z \le 0$ in a round cylinder coordinate system. Consider the integral for a point with coordinates z and ρ and arbitrary value of the coordinate φ. The integral can be represented as follows:

$$A_z = \int_{-\infty}^{0} \frac{\mathrm{d}z}{\sqrt{(z-a)^2 + \rho^2}}$$

and it is convenient to divide the domain of integration into the half-line $(-\infty, a,)$ and the segment $[a, 0]$. The earlier undergoes renormalization and turns into logarithm as in the previous case and the latter is found in tables of integrals:

$$A_z = \ln \rho - \operatorname{arcsinh} \frac{a}{\rho}. \tag{5.1}$$

The vector $\vec{H} = \nabla \times \vec{A}$ has no meaning of magnetic strength, but we introduce it as an auxiliary object. This vector has only one non-zero component H_φ equal to

$$H_\varphi = \rho \frac{\partial A_z}{\partial \rho} = 1 - \frac{z}{\sqrt{z^2 + \rho^2}}. \tag{5.2}$$

In the third case the current with magnitude one half flows out from the point $z = 0$, $\rho = 0$ in two opposite directions, thus, towards $z = \pm\infty$ that is impossible because this point must be a source of electric charge. Nevertheless, we calculate the integral in question and find that it is equal to the difference of two integrals of this form taken over the half-lines $(-\infty, 0)$ and $(0, \infty)$ that gives

$$A_z = 2 \arcsin \frac{a}{\rho}. \tag{5.3}$$

Below we need representation of this vector in other coordinate systems. For example, if in Cartesian coordinates the conductor coincides with the y-axis, one needs to make the following substitutions:

$$z \to y, \quad \rho \to \sqrt{z^2 + x^2}$$

that gives

$$A_y = \operatorname{arcsinh} \frac{y}{\sqrt{z^2 + x^2}}.$$

Another coordinate system in which this result will be used is that of round cylinder coordinates $\{z, \rho, \varphi\}$ in which the line lies in the plane $z = 0$. In this system we have

$$y = \rho \sin\varphi, \quad \mathrm{d}y = \sin\varphi \, \mathrm{d}\rho + \rho\cos\varphi \, \mathrm{d}\varphi, \quad A_y \, \mathrm{d}y = A_\rho \, \mathrm{d}\rho + A_\varphi \, \mathrm{d}\varphi,$$

hence, we obtain from it that

$$A_\rho = \operatorname{arcsinh} \frac{\rho \sin\varphi}{\sqrt{z^2 + \rho^2 \cos^2\varphi}}, \quad A_\varphi = \operatorname{arcsinh} \frac{\rho^2 \sin\varphi \cos\varphi}{\sqrt{z^2 + \rho^2 \cos^2\varphi}}$$

Though A_φ has no meaning of vector potential, we denote its curl $\vec{H}$ and evaluate the only its component H_φ:

$$H_\varphi = \rho \frac{\partial A_\rho}{\partial z} = -\frac{2z\rho^2 \sin^2\varphi}{\sqrt{z^2 + \rho^2}(z^2 + \rho^2 \cos^2\varphi)}. \tag{5.4}$$

Now we can return to realistic current densities and find out the vector potential produced by a T-wire. A T-wire consists of one straight line and one half-line orthogonal to it, connected by its endpoint. Though this combination consists of impossible current densities, the resulting current density considered below, is realistic. In this construction, a current flows along the half-line up to the junction point and in two opposite directions on the straight line outwards from the junction, so that magnitude of current in each branch is equal to one half. Note that since the half-line and the straight line are orthogonal to each other, they produce orthogonal components of the vector potential.

6. Non-existence of Green's function for magnetic field

If the method of Green's functions works properly, then the sum of expressions (5.2) and (5.4) is equal to the φ-component of magnetic strength produced by the T-current considered in the previous section. This allows one to take contour integrals (2.2) over circles of constant ρ lying in planes $z = \text{const}$. Under $z < 0$ such a circuit surrounds a non-zero current and if the expressions obtained are correct, it must be equal exactly to 2π and under $z > 0$ it must be exactly zero. However, none of these results comes out. For example, the integral which should be equal to zero is equal to

$$\int \vec{H}\,\mathrm{d}\vec{l} = 2\pi\left(1 - \frac{z}{\sqrt{z^2+\rho^2}}\right) - \int\limits_0^{2\pi} \frac{2z\rho^2\sin^2\varphi\,\mathrm{d}\varphi}{\sqrt{z^2+\rho^2}(z^2+\rho^2\cos^2\varphi)}.$$

The second integral tends to zero under $0 < z \to 0$ whereas the first one does to 2π. In general, this expression cannot be equal to zero because, on one hand, none of two terms is zero and, on the other hand, one of them contains only even powers whereas another does only odd powers of z. Thus, we have found an example of current for which the integral (3.2) can be taken analytically and the curl of the vector field obtained has non-zero values of the integral (2.2) over circles which do not surround any current. This example leaves no hopes that the method of Green's functions can give correct solutions of the equation (1.4). Before, analysis of foundations of the method of Green's functions showed that it suffices to find out a single example in which it gives an erroneous result to draw a conclusion that this method is not applicable. Non-applicability of this method signifies only that Green's function for the field equation (1.4) does not exist. As this verification demonstrated an apparent breakdown of the equation, an example is found and non-existence of Green's function for the equation is proved.

7. Conclusions

Physically, electric current is always carried by a physical body which occupies a part of the space. However, some conductors have specific shapes which allow one to neglect their thickness so that the conductor can be considered to be a surface or a curve in form. These idealizations create the corresponding limiting cases of the study of magnetic fields of steady currents and it turns out that mathematical methods to be used in the original magnetostatics and in these limiting cases are completely different. As a result, the study divides into three distinct theories in which dimensions of the field source ranges from 1 to 3, which share common physical foundations, but have different mathematical structures.

Magnetostatics of surface currents has mathematical structure which consists of the theories of potential double layer and of the Cauchy problem for the Laplace equation. This part of magnetostatics is well-developed because its main equation is scalar. The point is that standard texts on mathematical physics teach only how to solve scalar equations, one of which the Laplace equation is. In other words, magnetic field of surface currents can always be obtained by using mathematical methods created long ago. Two other parts of magnetostatics have no well-known mathematical background, therefore, they are not developed so well.

Magnetostatics in its original form has mathematical structure given by a set of particular solutions of the vector wave equation (2.1), which constitute a basis in a linear functional space of magnetic fields and current densities. Though this equation is not scalar and standard texts on mathematical physics tell nothing about it, it can be solved in the most general form in various coordinate systems. Hence, the main part of magnetostatics can be developed properly as soon as all results on the theory of this equation are collected and presented. This work would be completed long ago, but till now no good exposition of theory of the equation (2.1) is presented in the literature.

Unlike these two theories, magnetostatics of thin wires has no mathematical structure. For long time it was believed that magnetic field of a thin wire can be obtained by the method of Green's function for the field equation (1.4), but, as was shown in this work, application of this method has no foundation. Till now, there was no question of applicability of the method to the equation because the method was believed to be applicable to any linear non-uniform equation. As was shown above, in case of the vector wave equation it is not so due to the charge conservation law. This fact, by itself does not mean that the method cannot be applied and the crucial argument would be demonstration of an apparently erroneous result which this method gives. Such a results was obtained in case of a T-shaped wire carrying a bifurcating current. As such a demonstration is presented, no question remain of inapplicability of the method. The result presented in this work, signifies that vector potential cannot be expressed in the form (3.2), or, in other words, the equation (2.1) has no Green's function. Since no other approaches to the problem of magnetic field of this source is known, failure of the only known approach deprives this theory of its mathematical contents. Thus, only one of three theories which constitute the whole body of magnetostatics, is developed properly whereas two others remain to be developed in the future.

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