NP-Hardness of optimizing the sum of Rational Linear Functions over an Asymptotic-Linear-Program

Deepak Ponvel Chermakani

deepakc@pmail.ntu.edu.sg deepakc@e.ntu.edu.sg deepakc@ed-alumni.net deepakc@myfastmail.com deepakc@usa.com

Abstract: - **We convert, within polynomial-time and sequential processing, an NP-Complete Problem into a real-variable problem of minimizing a sum of Rational Linear Functions constrained by an Asymptotic-Linear-Program. The coefficients and constants in the real-variable problem are 0, 1, -1, K, or -K, where K is the time parameter that tends to positive infinity. The number of variables, constraints, and rational linear functions in the objective, of the real-variable problem is bounded by a polynomial function of the size of the NP-Complete Problem. The NP-Complete Problem has a feasible solution, if-and-only-if, the real-variable problem has a feasible optimal objective equal to zero. We thus show the strong NP-hardness of this real-variable optimization problem.**

1. Introduction

An Asymptotic-Linear-Program (ALP) is a linear program over real variables, whose coefficients and constants in the objective and constraints, are rational polynomial functions of K, the time parameter. It has been proved [1] that as K tends to positive infinity, the ALP demonstrates a steady-state behaviour in its feasibility (or infeasibility) and in its optimal basis of variables (if feasible).

It has been shown [2] that optimizing a single rational polynomial function of real variables, is NP-hard. It has also been shown [3] that optimizing a single rational linear function of binary variables, can be accomplished within polynomial-time.

Consider the problem of optimizing a sum of rational linear functions of real variables, over an ALP. We shall denote this problem as $O_{rational\_linear\_functions\_ALP}$. Denote $P_{rational\_linear\_functions\_ALP}$ as the problem of deciding whether or not the optimal objective value of $O_{rational\_linear\_functions\_ALP}$ is equal to a target integer.

In our paper [4], we showed the NP-Completeness of the problem $P_{linear\_eq\_binary\_1}$ of deciding the feasibility of a set of linear equations over binary variables, with coefficients and constants that are 0, 1, or -1. Consider an instance of problem $P_{linear\_eq\_binary\_1}$ having M linear equations, over a binary variable vector $< b_1, b_2, \ldots b_N >$, i.e. each variable b_i is allowed to be either 0 or 1, for all integers i in $[1,N]$:

$a_{1,1}\, b_1 + a_{1,2}\, b_2 + \ldots + a_{1,N}\, b_N = c_1$
$a_{2,1}\, b_1 + a_{2,2}\, b_2 + \ldots + a_{2,N}\, b_N = c_2$
…
$a_{M,1}\, b_1 + a_{M,2}\, b_2 + \ldots + a_{M,N}\, b_N = c_M$

where each of $a_{i,j}$ and c_i is given to be 0, 1, or -1, for all integers j in $[1,N]$, and all integers i in $[1,M]$.

In the subsequent sections of this paper, we will show how to convert an instance of $P_{linear\_eq\_binary\_1}$ into $P_{rational\_linear\_functions\_ALP}$.

2. Modelling Binary Variables using Rational Linear Equations over real variables

<u>Definitions</u>: Let x be a real variable such that $0 \leq x \leq 1$. Let $<x_1, x_2, \ldots x_N>$ be a vector of real variables, such that $0 \leq x_i \leq 1$ for all integers i in $[1,N]$. Let K tend to positive infinity.

<u>Theorem-1</u>: It is strongly NP-Hard to decide feasibility of Rational Linear Inequalities over real variables

<u>Proof</u>: Consider the Rational Equation $(x + ((1-x)/2)) = (1/(2-x))$, which when simplified yields $((x(1-x)) / (2(2-x))) = 0$. Because $0 \leq x \leq 1$, and because $x=2$ is the only point of discontinuity of the Rational Linear Function, we can use this to express a binary variable. So the following set of Rational Linear Inequalities has a real solution, if and only if, $P_{linear\_eq\_binary\_1}$ has a binary vector solution:

$a_{1,1}\, b_1 + a_{1,2}\, b_2 + \ldots + a_{1,N}\, b_N = c_1\,;$
$a_{2,1}\, x_1 + a_{2,2}\, x_2 + \ldots + a_{2,N}\, x_N = c_2\,;$
…
$a_{M,1}\, x_1 + a_{M,2}\, x_2 + \ldots + a_{M,N}\, x_N = c_M\,;$

$0 \leq x_1 \leq 1$; $(x_1 + ((1-x_1)/2)) = (1/(2-x_1))$;
$0 \leq x_2 \leq 1$; $(x_2 + ((1-x_2)/2)) = (1/(2-x_2))$;
... ...
$0 \leq x_N \leq 1$; $(x_N + ((1-x_N)/2)) = (1/(2-x_N))$;

Using the technique mentioned in paper [4], we can express (within polynomial-time) all these Rational Linear Inequalities with coefficients *0, 1,* or *-1*. Thus, it is strongly NP-hard to decide the feasibility of a set of Rational Linear Inequalities, over real variables.

Hence Proved

Theorem-2: For any positive integer i, $(((x/(K+2i-1)) + ((1-x)/(K+2i))) = 1/(K+2i-x)) \leftrightarrow$ (x is either *0* or *1*)

Proof: A Boolean statement P↔Q can be proved by showing Q→P and P→Q. For $x=0$, the value of the Left-Hand-Side (LHS) in the equation, is $1/(K+2i)$, which is equal to the value of the Right-Hand-Side (RHS). For $x=1$, the value of the LHS, is $1/(K+2i-1)$, which is equal to the value of the RHS. So Q→P. Next, $((x/(K+2i-1)) + ((1-x)/(K+2i))) = 1/(K+2i-x)$ implies that $(((x/(K+2i-1)) + ((1-x)/(K+2i))) - 1/(K+2i-x)) = 0$. Simplifying this expression yields $(x(1-x)) / ((K+2i-1)(K+2i)(K+2i-x)) = 0$. As K tends to positive infinity, the denominator of the LHS of this expression is always positive, so the only way for this equation to be satisfied is that $x(1-x) = 0$, which implies that x is either *0* or *1*. So P→Q.

Hence Proved

Theorem-3: Let < w_1 , w_2 ,… w_N > and < u_1 , u_2 ,… u_N > be two vectors of real numbers, such that $w_i \neq 0$ for all integers i in [*1,N*], and such that $w_i \neq w_j$,for all $i \neq j$. There exists a positive real γ that is a function of real numbers < w_1 , w_2 ,… w_N, > and real numbers < u_1 , u_2 ,… u_N >, such that for all $K > \gamma$, the following statement is true:
$((K((u_1/(K+w_1)) + (u_2/(K+w_2)) + ... + (u_N/(K+w_N))) = 0) \leftrightarrow (u_1 = u_2 = \ldots = u_N = 0))$

Proof: This is a generalization of Theorem-1 of the paper [5]. A Boolean statement P↔Q can be proved by showing Q→P and P→Q. As Q→P is obvious, we will focus on proving P→Q. Expressing $((u_1/(K+w_1)) + (u_2/(K+w_2)) + ... + (u_N/(K+w_N)))$ as a single rational expression, we obtain: $((u_1 A_1 + u_2 A_2 + ... + u_N A_N) / ((K+w_1)(K+w_2)... (K+w_N)))$, where, for all integers i in $[1,N]$, A_i = (product of $(K+w_j)$, over all integers j in $[1,N]$ and $j \neq i$). We can write the expression $(u_1 A_1 + u_2 A_2 + ... + u_N A_N)$ as ($K^{N-1} B_{N-1} + K^{N-2} B_{N-2} + ... + K^0 B_0$), where, for all integers i in $[0, N-1]$, B_i represents the coefficient of K^i in the expression $(u_1 A_1 + u_2 A_2 + ... + u_N A_N)$. We have:

$B_{N-1} = u_1 + u_2 + ... + u_N$
$B_{N-2} = (w_2+ w_3+...+ w_N)u_1 + (w_1+ w_3+ w_4+... w_N)u_2 + (w_1+ w_2+ w_4+ w_5+...+ w_N)x_3 + ... + (w_1+ w_2+ w_3+...+ w_{(N-1)})u_N$
$B_{N-3} = (w_2 * w_3+ w_2 * w_4+...+ w_2 * w_N + w_3 * w_4 + w_3 * w_5 + ... w_3 * w_N + ... w_{(N-1)} * w_N)u_1 +$
$(w_1 * w_3 + w_1 * w_4 ... w_1 * w_N + w_3 * w_4 + w_3 * w_5 + ... w_3 * w_N + ... w_{(N-1)} * w_N)u_2 +$
$...+$
$(w_1 * w_2+ w_1 * w_3+...+ w_1 * w_{(N-1)} + w_2 * w_3+ w_2 * w_4+...+ w_2 * w_{(N-1)} + ... + w_{(N-2)} * w_{(N-1)})u_N$
...
$B_0 = (w_2 * w_3 *...* w_N)u_1 + (w_1 * w_3 * w_4 *...* w_N)u_2 + (w_1 * w_2 * w_4 * w_5 *...* w_N)u_3 + ... + (w_1 * w_2 * w_3 *...* w_{(N-1)})u_N$

Generalizing the pattern in the above coefficients, $B_{N-1} = u_1 + u_2 + ... + u_N$, and, for all integers i in $[0,(N-2)]$, B_i = (Summation over all integers j in $[1,N]$, of $(u_j$*(summation of all combinations of product terms from Set of elements $\{\{ w_1 , w_2 , ... w_N\} - \{ w_j \}\}$, having $(N-i-1)$ elements in each product term))).

Now consider the expression ($K^{N-1} B_{N-1} + K^{N-2} B_{N-2} + ... + K^0 B_0$) as a univariate Polynomial in K. For a given set of scalars $\{u_1, u_2, ... u_N\}$, it is obvious that there exists an upper bound γ on the real root of this Polynomial, given by Lagrange's Theorem [1]. Hence for all $K > \gamma$, the only possibility for (($K^{N-1} B_{N-1} + K^{N-2} B_{N-2} + ... + K^0 B_0$) = 0) to be true, is ($B_i = 0$, for all integers i in $[0,N-1]$). This gives us a set of N linear equations in $\{u_1, u_2, ... u_N\}$, mentioned in Lemma-1:

Lemma-1: We aim to prove that the following N linear equations in $\{u_1, u_2, ... u_N\}$ are unique:
$B_{N-1} = u_1 + u_2 + ... + u_N = 0$, and,
for all integers i in $[0,(N-2)]$, B_i = (Summation over all integers j in $[1,N]$, of $(u_j * Comb_{(N-i-1)}(\{j\}))) = 0$.

Here ($Comb_{(N-i-1)}(\{j\})$) denotes summation of all combinations of product terms from Set of elements $\{\{ w_1 , w_2 , ... w_N\} - \{ w_j \}\}$, having $(N-i-1)$ elements in each product term. We denote: Set$\{a,b,c,d\}$ – Set$\{b,d\}$ = Set $\{a,c\}$.

Proof: (These N linear equations are unique) ↔ (determinant of matrix Ω_1, formed from coefficients of the linear equations, is non-zero). Ω_1 is shown in the Figure 1. We know that (determinant of a matrix is *0*) ↔ (determinant of its transpose is *0*). We also know that multiplying a row or column by a real number (equivalent to multiplying the determinant by that same real number), and, adding two rows or columns together, do not change the result of its determinant being zero or non-zero.

1	1	...	...	1	1
$Comb_{(1)}(\{w_1\})$	$Comb_{(1)}(\{w_2\})$	...	...	$Comb_{(1)}(\{w_{N-1}\})$	$Comb_{(1)}(\{w_N\})$
$Comb_{(2)}(\{w_1\})$	$Comb_{(2)}(\{w_2\})$	...	...	$Comb_{(2)}(\{w_{N-1}\})$	$Comb_{(2)}(\{w_N\})$
...	...	...	...	...	...
...	...	...	...	...	...
$Comb_{(N-2)}(\{w_1\})$	$Comb_{(N-2)}(\{w_2\})$	...	...	$Comb_{(N-2)}(\{w_{N-1}\})$	$Comb_{(N-2)}(\{w_N\})$
$Comb_{(N-1)}(\{w_1\})$	$Comb_{(N-1)}(\{w_2\})$	...	...	$Comb_{(N-1)}(\{w_{N-1}\})$	$Comb_{(N-1)}(\{w_N\})$

Figure 1: The square matrix Ω_1 of dimension N

Denoting Column i in the matrix as C_i, we apply column operations $C_{i\_next} = C_i - C_{i+1}$ on Ω_1, for all integers i in $[1,N-1]$. This eliminates one dimension, and we get the next square matrix Ω_2 of dimension $N-1$, shown in Figure 2.

(w_2-w_1)	(w_3-w_2)	...	...	$(w_{N-1}-w_{N-2})$	(w_N-w_{N-1})
$(w_2-w_1)*Comb_{(1)}(\{w_1,w_2\})$	$(w_3-w_2)*Comb_{(1)}(\{w_2,w_3\})$	...	...	$(w_{N-1}-w_{N-2})*Comb_{(1)}(\{w_{N-2},w_{N-1}\})$	$(w_N-w_{N-1})*Comb_{(1)}(\{w_{N-1},w_N\})$
$(w_2-w_1)*Comb_{(2)}(\{w_1,w_2\})$	$(w_3-w_2)*Comb_{(2)}(\{w_2,w_3\})$	...	...	$(w_{N-1}-w_{N-2})*Comb_{(2)}(\{w_{N-2},w_{N-1}\})$	$(w_N-w_{N-1})*Comb_{(2)}(\{w_{N-1},w_N\})$
...	...	...	...	...	...
...	...	...	...	...	...
$(w_2-w_1)*Comb_{(N-3)}(\{w_1,w_2\})$	$(w_3-w_2)*Comb_{(N-3)}(\{w_2,w_3\})$	...	...	$(w_{N-1}-w_{N-2})*Comb_{(N-3)}(\{w_{N-2},w_{N-1}\})$	$(w_N-w_{N-1})*Comb_{(N-3)}(\{w_{N-1},w_N\})$
$(w_2-w_1)*Comb_{(N-2)}(\{w_1,w_2\})$	$(w_3-w_2)*Comb_{(N-2)}(\{w_2,w_3\})$	...	...	$(w_{N-1}-w_{N-2})*Comb_{(N-2)}(\{w_{N-2},w_{N-1}\})$	$(w_N-w_{N-1})*Comb_{(N-2)}(\{w_{N-1},w_N\})$

Figure 2: The square matrix Ω_2 of dimension *(N-1)*

In Ω_2, divide Column C_i by $(w_{i+1} - w_i)$ for all integers i in $[1,N-2]$, then again apply $C_{i\_next} = C_i - C_{i+1}$ for all integers i in $[1,N-2]$, to eliminate another dimension to get square matrix Ω_3 in Figure 3.

(w_3-w_1)	(w_4-w_2)	...	...	$(w_{N-1}-w_{N-3})$	(w_N-w_{N-2})
$(w_3-w_1)*Comb_{(1)}(\{w_1,w_2,w_3\})$	$(w_4-w_2)*Comb_{(1)}(\{w_2,w_3,w_4\})$	...	...	$(w_{N-1}-w_{N-3})*Comb_{(1)}(\{w_{N-3},w_{N-2},w_{N-1}\})$	$(w_N-w_{N-2})*Comb_{(1)}(\{w_{N-2},w_{N-1},w_N\})$
$(w_3-w_1)*Comb_{(2)}(\{w_1,w_2,w_3\})$	$(w_4-w_2)*Comb_{(2)}(\{w_2,w_3,w_4\})$	...	...	$(w_{N-1}-w_{N-3})*Comb_{(2)}(\{w_{N-3},w_{N-2},w_{N-1}\})$	$(w_N-w_{N-2})*Comb_{(2)}(\{w_{N-2},w_{N-1},w_N\})$
...	...	...	...	...	...
...	...	...	...	...	...
$(w_3-w_1)*Comb_{(N-4)}(\{w_1,w_2,w_3\})$	$(w_4-w_2)*Comb_{(N-4)}(\{w_2,w_3,w_4\})$	...	...	$(w_{N-1}-w_{N-3})*Comb_{(N-4)}(\{w_{N-3},w_{N-2},w_{N-1}\})$	$(w_N-w_{N-2})*Comb_{(N-4)}(\{w_{N-2},w_{N-1},w_N\})$
$(w_3-w_1)*Comb_{(N-3)}(\{w_1,w_2,w_3\})$	$(w_4-w_2)*Comb_{(N-3)}(\{w_2,w_3,w_4\})$	...	...	$(w_{N-1}-w_{N-3})*Comb_{(N-3)}(\{w_{N-3},w_{N-2},w_{N-1}\})$	$(w_N-w_{N-2})*Comb_{(N-3)}(\{w_{N-2},w_{N-1},w_N\})$

Figure 3: The square matrix Ω_3 of dimension *(N-2)*

For Ω_j where $2 \le j \le (N-1)$, we now attempt to complete the Induction proof that dividing Column C_i by $(w_{i+j-1} - w_i)$ for all integers i in $[1,N-j]$, and subsequently applying operations $C_{i\_next} = C_i - C_{i+1}$ for all integers i in $[1,N-j]$, gives square matrix Ω_{j+1} of dimension $(N-j)$ where the k^{th} element of Column C_i , is equal to $((w_{i+j} - w_i)* Comb_{(k-2)}(\{w_i, w_{i+1},...,w_{i+j}\}))$.

Consider any column vector C_i in Ω_j where ($1 \le i \le (N-j)$). Assume that the first element in C_i is $(w_{i+j-1} - w_i)$, and the k^{th} element where ($2 \le k \le (N-j+1)$) in C_i is $((w_{i+j-1} - w_i)*Comb_{(k-1)}(\{w_i, w_{i+1},...,w_{i+j-1}\}))$. Further assume that the first element in C_{i+1} is $(w_{i+j} - w_{i+1})$, and the k^{th} element ($2 \le k \le (N-j+1)$) in C_{i+1} is $((w_{i+j} - w_{i+1})*Comb_{(k-1)}(\{w_{i+1}, w_{i+2},...,w_{i+j}\}))$. In Ω_j , after dividing C_i by $(w_{i+j-1} - w_i)$ for all integers i in $[1,(N-j+1)]$, the value of the k_{th} element in ($C_i - C_{i+1}$) becomes:

$$= ((Comb_{(k-1)}(\{w_i, w_{i+1},...,w_{i+j-1}\})) - (Comb_{(k-1)}(\{w_{i+1}, w_{i+2},...,w_{i+j}\})))$$
$$= (w_{i+j} *Comb_{(k-2)}(\{w_i, w_{i+1},...,w_{i+j}\}) - w_i *Comb_{(k-2)}(\{w_i, w_{i+1},...,w_{i+j}\}))$$
$$= ((w_{i+j} - w_i)* Comb_{(k-2)}(\{w_i, w_{i+1},...,w_{i+j}\})),$$ which is equal to the k^{th} element of Column C_i in Ω_{j+1}

This completes the Induction Proof. The loss of dimension (between Ω_j and Ω_{j+1}) is obvious after applying $C_{i\_next} = (C_i - C_{i+1})$, for all integers i in $[1,N-j]$, since the first row of Ω_j always has 1, after the division of Column C_i of Ω_j by $(w_{i+j-1} - w_i)$.

We proceed to iteratively obtain square matrices of smaller dimensions, until Ω_{N-1} of dimension 2 in Figure 4.

$(w_{N-1}-w_1)$	(w_N-w_2)
$(w_{N-1}-w_1)*w_N$	$(w_N-w_2)*w_1$

Figure 4: The square matrix Ω_{N-1} of dimension *2*

The final operation of dividing Column C_i by $(w_{N+i-2} - w_i)$ for all integers i in $[1,2]$ and applying the column operation $C_{1\_next} = C_1 - C_2$, yields the single element $(w_N - w_1)$. From all the divisions of the columns of the matrices performed so far, the value of the determinant Ω_1 is non-zero, if and only if, the following product of $(N(N-1)/2)$ terms is non-zero:

$$(w_2 - w_1)(w_3 - w_1) \dots (w_N - w_1)\ (w_3 - w_2)(w_4 - w_2) \dots (w_N - w_2)\ (w_4 - w_3)(w_5 - w_3) \dots (w_N - w_3) \dots (w_N - w_{N-1})$$

That is possible, if and only if, $w_i \neq w_j$ for all $i \neq j$ which is given to be true. <u>Hence proved Lemma-1</u>.

Thus, the only solution that satisfies the set of homogenous linear equations in Lemma-1, is $u_i = 0$ for all integers i in $[1,N]$.

<u>Hence Proved</u>

<u>Theorem-4</u>: Let $< w_1 , w_2 ,\dots w_N >$ and $< u_1 , u_2 ,\dots u_N >$ be two vectors of real numbers, such that $w_i \neq 0$ for all integers i in $[1,N]$, and such that $w_i \neq w_j$,for all $i \neq j$. There is a one-to-one mapping between every $< u_1 , u_2 ,\dots u_N >$ and $((u_1 /(K+w_1)) + (u_2 /(K+w_2)) + \dots + (u_N /(K+w_N)))$

<u>Proof</u>: Assume that there exists a non-trivial real vector $<\Delta_1, \Delta_2, \dots \Delta_N>$ such that $((u_1 /(K+w_1)) + (u_2 /(K+w_2)) + \dots + (u_N /(K+w_N))) = (((u_1 + \Delta_1)/(K+w_1)) + ((u_2 + \Delta_2) /(K+w_2)) + \dots + ((u_N + \Delta_N) /(K+w_N)))$. This would imply that $((\Delta_1 /(K+w_1)) + (\Delta_2 /(K+w_2)) + \dots + (\Delta_N /(K+w_N))) = 0$, which would contradict Theorem-3. This implies that every real vector $<u_1, u_2, \dots u_N>$ corresponds uniquely to $((u_1 /(K+w_1)) + (u_2 /(K+w_2)) + \dots + (u_N /(K+w_N)))$ and vice-versa.

<u>Hence Proved</u>

<u>Theorem-5</u>: For each integer i in $[1,N]$, denote $y_i = ((x_i /(K+2i-1)) + ((1-x_i)/(K+2i)))$, and denote $z_i = 1/(K+2i-x_i)$. Then, $(y_1 + y_2 + \dots + y_N = z_1 + z_2 + \dots + z_N) \leftrightarrow (<x_1, x_2, \dots x_N>$ is a binary vector)

<u>Proof</u>: A Boolean statement P↔Q can be proved by showing Q→P and P→Q. For $x_i = 0$, the value of $y_i = 1/(K+2i)$, which is equal to the value of z_i . For $x_i = 1$, the value of $y_i = 1/(K+2i-1)$, which is equal to the value of z_i . So for each element of $<x_1, x_2, \dots x_N>$ being either 0 or 1, $(y_1 + y_2 + \dots + y_N = z_1 + z_2 + \dots + z_N)$. So Q→P. Next, from Theorem-2 of this paper, for any integer i in $[1,N]$, $(y_i = z_i)\rightarrow(x_i$ is either 0 or $1)$. We now focus on proving that $(y_1 + y_2 + \dots + y_N = z_1 + z_2 + \dots + z_N) \rightarrow ((y_i = z_i)$, for all integers i in $[1,N])$. Note now from Theorem-3, that it is not possible for $(y_1 + y_2 + \dots + y_N - z_1 - z_2 - \dots - z_N = 0)$ unless x_i takes on a value, such that the denominator of one of the terms of y_i is equal to the denominator of z_i . It is not possible for the balancing of y_i to be done by any other z_j $(j \neq i)$ because $0 \leq x_i \leq 1$, for all integers i in $[1,N]$. That is either $(K+2i-1) = (K+2i-x_i)$ or $(K+2i) = (K+2i-x_i)$. That is either $x_i = 1$ or $x_i = 0$, and in both these cases, we have $(y_i = z_i)$ as seen in Theorem-2. So $(y_1 + y_2 + \dots + y_N = z_1 + z_2 + \dots + z_N) \rightarrow ((y_i = z_i)$, for all integers i in $[1,N]) \rightarrow (x_i$ is either 0 or 1, for all integers i in $[1,N])$. So P→Q.

<u>Hence Proved</u>

<u>Theorem-6</u>: The globally minimum value of $(y_1 + y_2 + \dots + y_N - z_1 - z_2 - \dots - z_N)$ is 0. Also this global minimum is reached when $<x_1, x_2, \dots x_N>$ is a binary vector

<u>Proof</u>: From Theorem-5, it is obvious that $(y_1 + y_2 + \dots + y_N - z_1 - z_2 - \dots - z_N = 0) \leftrightarrow (<x_1, x_2, \dots x_N>$ is a binary vector$)$. We now focus on proving that the minimum value of the expression $(y_1 + y_2 + \dots + y_N - z_1 - z_2 - \dots - z_N)$ is 0. For any integer i in $[1,N]$, we see that $(y_i - z_i) = ((x_i (1-x_i)) / ((K+2i-1)(K+2i)(K+2i-x_i)))$. As K tends to positive infinity, and as $0 \leq x_i \leq 1$, the denominator of this expression is always positive, and the numerator $(x_i (1-x_i))$ is always non-negative. So the minimum value of $(y_i - z_i)$ is zero, which happens when x_i is either 0 or 1. Thus, the global minimum of $(y_1 + y_2 + \dots + y_N - z_1 - z_2 - \dots - z_N)$ is 0, which happens only when $<x_1, x_2, \dots x_N>$ is a binary vector.

<u>Hence Proved</u>

<u>Start of example illustrating Theorem-3</u>

Consider an example with $N=4$, the expression: $((u_1 / (K+w_1)) + (u_2 / (K+w_2)) + (u_3 / (K+w_3)) + (u_4 / (K+w_4)))$

$= ((K+w_2)(K+w_3)(K+w_4)u_1 + (K+w_1)(K+w_3)(K+w_4)u_2 + (K+w_1)(K+w_2)(K+w_4)u_3 + (K+w_1)(K+w_2)(K+w_3)u_4) / (K+w_1)(K+w_2)(K+w_3)(K+w_4)$

$= (K^3(u_1 + u_2 + u_3 + u_4) +$
$K^2((w_2+w_3+w_4)u_1 + (w_1+w_3+w_4)u_2 + (w_1+w_2+w_4)u_3 + (w_1+w_2+w_3)u_4) +$
$K ((w_2*w_3+w_2*w_4+w_3*w_4)u_1 + (w_1*w_3+w_1*w_4+w_3*w_4)u_2 + (w_1*w_2+w_1*w_4+w_2*w_4)u_3 + (w_1*w_2+w_1*w_3+w_2*w_3)u_4) +$
$((w_2*w_3*w_4)u_1 + (w_1*w_3*w_4)u_2 + (w_1*w_2*w_4)u_3 + (w_1*w_2*w_3)u_4)) / ((K+w_1)(K+w_2)(K+w_3)(K+w_4))$

The matrix Ω_1 is shown in Figure 5.

1	1	1	1
$w_2+w_3+w_4$	$w_1+w_3+w_4$	$w_1+w_2+w_4$	$w_1+w_2+w_3$
$w_2*w_3+w_2*w_4+w_3*w_4$	$w_1*w_3+w_1*w_4+w_3*w_4$	$w_1*w_2+w_1*w_4+w_2*w_4$	$w_1*w_2+w_1*w_3+w_2*w_3$
$w_2*w_3*w_4$	$w_1*w_3*w_4$	$w_1*w_2*w_4$	$w_1*w_2*w_3$

Figure 5: The square matrix Ω_1 for our example

Apply $C_{1\_next} = C_1 - C_2$, $C_{2\_next} = C_2 - C_3$, to get rid of first row and last column, so the resulting matrix Ω_2 is in Figure 6.

w_2-w_1	w_3-w_2	w_4-w_3
$(w_2-w_1)*(w_3+w_4)$	$(w_3-w_2)*(w_1+w_4)$	$(w_4-w_3)*(w_1+w_2)$
$(w_2-w_1)*w_3*w_4$	$(w_3-w_2)*w_1*w_4$	$(w_4-w_3)*w_1*w_2$

Figure 6: The square matrix Ω_2 for our example

In Ω_2, divide Column C_i by $(w_{i+1} - w_i)$ for all integers i in $[1,2]$, then apply $C_{i\_next} = C_i - C_{i+1}$ for all integers i in $[1,2]$, to eliminate another dimension to get square matrix Ω_3 in Figure 7.

w_3-w_1	w_4-w_2
$(w_3-w_1)*w_4$	$(w_4-w_2)*w_1$

Figure 7: The square matrix Ω_3 for our example

In Ω_3, divide Column C_i by $(w_{i+2} - w_i)$ for all integers i in $[1,2]$, then apply $C_{i\_next} = C_i - C_{i+1}$ for all integers i in $[1,2]$, to eliminate another dimension to get a single element whose value is $(w_4 - w_1)$. Thus, taking into account all the divisions performed so far, we have the following 4 true statements:

(Determinant of Ω_1 is non-zero) $\leftrightarrow$

$((w_2 - w_1)(w_3 - w_1)(w_4 - w_1)(w_3 - w_2)(w_4 - w_2)(w_4 - w_3) \neq 0) \leftrightarrow$

($w_i \neq w_j$ for all $i \neq j$) $\leftrightarrow$

$((K((u_1 /(K+ w_1)) + (u_2 /(K+ w_2)) + (u_3 /(K+ w_3)) + (u_4 /(K+ w_4))) = 0) \leftrightarrow (u_1 = u_2 = u_3 = u_4 = 0))$

End of Example illustrating Theorem-3

3. Expressing $P_{linear\_eq\_binary\_1}$ as $P_{rational\_linear\_functions\_ALP}$

3.1 Obtaining purely Linear constraints, and a sum of Rational Linear Functions for the Objective Function

We use Theorem-5 and Theorem-6 to express $P_{linear\_eq\_binary\_1}$ as $P_{rational\_linear\_functions\_ALP}$. We aim to minimize $(y_1 + y_2 + ... + y_N - z_1 - z_2 - ... - z_N)$, referred to as the objective function, over the constraints of $P_{linear\_eq\_binary\_1}$, replacing its binary variable vector $< b_1, b_2, ... b_N >$ with the real variable vector $< x_1, x_2, ... x_N >$. (The objective is minimized to zero) $\leftrightarrow$ (One of the 2^N possible binary vector solutions is allowed for $< x_1, x_2, ... x_N >$). Our intention is to allow the objective of $P_{rational\_linear\_functions\_ALP}$ to have a sum of rational linear functions. We also intend to allow the constraints of $P_{rational\_linear\_functions\_ALP}$ to have purely linear functions (and not rational linear functions). So we make appropriate substitutions for this, and add more linear constraints in the process. For each integer i in $[1,N]$, make the substitution $y_i = (x_i / p_i) + ((1-x_i) / q_i)$, and the substitution $z_i = 1/ r_i$, where:

$p_i = (K+2i-1)$; $q_i = (K+2i)$; $r_i = (K+2i-x_i)$

where each of p_i, q_i, and r_i is a real variable.

Note that the objective is a summation (over all integers i in $[1,N]$) of the term $((x_i (1-x_i)) / ((K+2i-1)(K+2i)(K+2i-x_i)))$. So we introduce a multiplicative term K^3 on the objective. Note that if this multiplicative term is not introduced, any tools that attempt to evaluate the value of the objective will always obtain a value of 0, since the value of $\text{Limit}_{K\rightarrow(\text{positive infinity})}(1/K)$

is considered to be 0. Also note that the value of $(K^3(y_1 + y_2 + ... + y_N - z_1 - z_2 - ... - z_N))$, as K tends to positive infinity, can either be equal to 0, or be equal to a non-zero positive real with a lower bound equal to some function of the coefficients and constants in the linear equations of $P_{linear\_eq\_binary\_1}$ (i.e. it cannot tend to 0 and remain positive).

3.2 $O_{rational\_linear\_functions\_ALP}$ and $P_{rational\_linear\_functions\_ALP}$

We write out $O_{rational\_linear\_functions\_ALP}$ with the following Objective and Constraints:

Minimize the Objective:

$$K^3 ((x_1 / p_1) + (x_2 / p_2) + ... + (x_N / p_N) + ((1-x_1) / q_1) + ((1-x_2) / q_2) + ... + ((1-x_N) / q_N) - (1/r_1) - (1/r_2) - ... - (1/r_N))$$

Subject to Constraints:

$$a_{1,1} x_1 + a_{1,2} x_2 + ... + a_{1,N} x_N = c_1 ;$$
$$a_{2,1} x_1 + a_{2,2} x_2 + ... + a_{2,N} x_N = c_2 ;$$
$$\dots$$
$$a_{M,1} x_1 + a_{M,2} x_2 + ... + a_{M,N} x_N = c_M ;$$
$$0 \le x_1 \le 1; \quad p_1 = (K+1); \quad q_1 = (K+2); \quad r_1 = (K+2 - x_1);$$
$$0 \le x_2 \le 1; \quad p_2 = (K+3); \quad q_2 = (K+4); \quad r_2 = (K+4 - x_2);$$
$$... \quad ... \quad ... \quad ...$$
$$0 \le x_N \le 1; \quad p_N = (K+2N-1); \quad q_N = (K+2N); \quad r_N = (K+2N - x_N);$$

$O_{rational\_linear\_functions\_ALP}$ has $3N$ rational linear functions in its objective, $(M+4N)$ linear constraints, and $4N$ real variables. We state $P_{rational\_linear\_functions\_ALP}$ as: (($O_{rational\_linear\_functions\_ALP}$ is feasible) AND (Zero is the minimum objective of $O_{rational\_linear\_functions\_ALP}$)). Finally, ($P_{rational\_linear\_functions\_ALP}$ is TRUE) $\leftrightarrow$ (A feasible binary solution exists to $P_{linear\_eq\_binary\_1}$).

3.3 Strong NP-hardness of $P_{rational\_linear\_functions\_ALP}$

An ALP whose coefficients and constants are rational functions of K, can be expressed with coefficients and constants that are linear functions of K. Example, the constraint $(3K^2 + 2K + 5) x < (7/K)$, can be replaced with simultaneous constraints $(y_0 K < 7 ; y_0 = y_1 + y_2 + y_3 ; y_1 = 3K y_{11} ; y_{11} = Kx ; y_2 = 2Kx ; y_3 = 5x)$. Also these constraints may be further expressed with coefficients and constants that are $0, 1, -1, K,$ or $-K$. Example, replace $(y_2 = 2Kx)$ with $(y_2 = Kz_1 + Kz_2 ; x = z_1 ; x = z_2)$.

As the maximum magnitude of coefficients in $O_{rational\_linear\_functions\_ALP}$ is $2N$, it can be rewritten (within polynomial time) to have coefficients and constants are $0, 1, -1, K,$ or $-K$. This shows the strong NP-hardness of $P_{rational\_linear\_functions\_ALP}$.

4. Conclusion

In this paper, we converted an NP-Complete problem (over binary variables), within polynomial-time, into a decision problem (over real variables) of whether or not the minimum value of a sum of Rational Linear Functions, is zero, constrained by an Asymptotic-Linear-Program. The size (i.e. number of constraints and variables, and rational linear functions in the objective) in the obtained real-variable-problem is bounded by a polynomial function of the size of the given NP-Complete Problem. The real-variable problem can also be efficiently expressed (within polynomial-time) with coefficients and constants that are $0, 1, -1, K,$ or $-K$. We thus, showed that it is strongly NP-hard to optimize the sum of rational linear functions of real variables, constrained by an Asymptotic-Linear-Program.

References

[1] R.G. Jeroslow, *Asymptotic Linear Programming*, Operations Research, Volume 21, No: 5, pages 1128-1141, 1973.
[2] D. Jibetean, E. de Klerk, *Global optimization of rational functions: a semidefinite programming approach*, Mathematical Programming, Volume 106, No: 1, pages 93-109, 2006.
[3] N. Megiddo, *Combinatorial Optimization with Rational Objective Functions*, Maths of Operations Research, Vol 4, pp 414-424, 1979.
[4] Deepak Ponvel Chermakani, *NP-Completeness of deciding the feasibility of Linear Equations over binary variables with coefficients and constants that are 0, 1, or -1*, arXiv:1210.4120v2 [cs.CC].
[5] Deepak Ponvel Chermakani, *A Non-Triviality Certificate for Scalars and its application to Linear Systems*, arXiv:1204.1764v1 [cs.CC].

About the Author

I, Deepak Ponvel Chermakani, wrote this paper out of my own interest, during my spare time. In Sep-2010, I completed a fulltime one-year Master Degree in *Operations Research with Computational Optimization*, from University of Edinburgh UK (*www.ed.ac.uk*). In Jul-2003, I completed a fulltime four-year Bachelor Degree in *Electrical and Electronic Engineering*, from Nanyang Technological University Singapore (*www.ntu.edu.sg*). In Jul-1999, I completed fulltime schooling from National Public School in Bangalore in India.