Linear Programming using NNLS

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ABSTRACT

We discuss a new simple method to solve linear programming (LP) problems, based on the so called duality theory and nonnegative least squares method. The success for this method as far efficiency is concerned depends upon the success one may achieve by further research in finding efficient method to obtain nonnegative solution for a system of linear equations. Thus, the suggested method points the need to devise better methods, if possible, of finding nonnegative solution for a system of linear equations. Because, it is shown here that the problem of linear programming reduces to finding **nonnegative** solution of certain system of linear equations, if and when it exists, and this system of equations consists of 1) the equation representing duality condition; 2) the equations representing the constraints imposed by the given primal problem; and 3) the equations representing the constraints imposed by its corresponding dual problem. In this paper we have made use of well known method of nonnegative least squares (NNLS), [1], as a primary start for finding nonnegative solution for a system of linear equations. Two simple MATLAB codes for testing method by implementing it to solving some simple problems are provided at the end.

1. Introduction: The linear programming problem consists of objective function $c^T x$, which is to be maximized or minimized subject to plurality of inequality or equality constraints imposed by the problem.

A typical symmetric linear programming problem is expressed as follows:

Maximize: $c^T x$

Subject to: $Ax \leq b$

$x \geq 0$

Or

Minimize: $c^T x$

Subject to: $Ax \geq b$

$x \geq 0$

Where x is a column vector of size $n \times 1$ of unknowns, c is a column vector of size $n \times 1$ of profit/cost coefficients, and c^T is a row vector of size $1 \times n$ obtained by matrix transposition of c, A is a matrix of constraints coefficients of size $m \times n$, b is a column vector of constants of size $m \times 1$ representing the boundaries of constraints.

By introducing the appropriate slack/surplus variables, the above mentioned typical linear programming problem gets converted into canonical form as:

Maximize: $c^T x$

Subject to: $Ax + s = b$

$x \geq 0, \; s \geq 0$

where s is slack variable vector of size $m \times 1$

Or

Minimize: $c^T x$

Subject to: $Ax - t = b$

$x \geq 0, \; t \geq 0$

where t is surplus variable vector of size $m \times 1$.

2. Duality Theory: The basic idea behind the duality theory is that every linear programming problem has an associated linear program called its dual such that a solution to the original linear program also gives a solution to its dual. When the given original (generally called as a primal) problem is a maximization (minimization) problem, then its corresponding dual problem will be minimization (maximization) respectively.

The duality theory states that if an optimal solution exists to either the primal or the corresponding dual program, then the other problem also has an optimal solution and **the two objective functions have the same optimal value.**

The duality theorem implies that, in order to find the maximum value of the objective function of the primal, we could equally well find the minimum value of the objective function of the dual, which is easier in some circumstances. It is also a well-known fact that the dual of the dual is the primal itself.

Few important facts are in order:

for a given maximization primal and its corresponding minimization dual linear programming problem:

(i) If any one of the problems have optimal solution then maximum value of the objective function of the primal is equal to the minimum value of the objective function of the dual, that is

$$c^T x = b^T w.$$

(ii) If the maximization problem is unbounded then the minimization problem is infeasible.

(iii) If the minimization problem is unbounded then the maximization problem is infeasible.

3. The Method: A general representation of the primal and its corresponding dual problem in canonical form can be given as:

Maximize : $c^T x$

Subject to : $Ax+s=b$

$x \geq 0, \; s \geq 0$

Where s is slack variable vector of size $m \times 1$, and where $Ax+s=b$ represent together m equations obtained by adding m number of slack variables s in the m inequalities imposed by the primal problem.

Its corresponding dual is

Minimize : $b^T w$

Subject to : $A^T w - t = c$

$w \geq 0, \; t \geq 0$

where t is surplus variable vector of size $n \times 1$.

By duality theorem when x is optimal solution of primal and w is optimal solution of dual then : $c^T x = b^T w$.

The method essentially consists of simultaneously solving the following system of equations and obtaining non-negative solution which will be a directly obtained optimal solution for both the primal as well as its corresponding dual problem.

$$c^T x - b^T w = 0$$

$$Ax + s = b$$

$$A^T w - t = c$$

Subject to non-negativity constraints inequalities:

$$x \geq 0, \; s \geq 0, \; w \geq 0, \; t \geq 0.$$

The non-negative feasible solution of the above mentioned system of equations fulfills the following criteria:

(1) $c^T x - b^T w = 0$ observes the known criterion of duality theorem about the maximal value of objective function ($c^T x$) of primal and minimal value of the objective function ($b^T w$) of the corresponding dual problem which are equal only at the optimality.

(2) $Ax + s = b$ observes the inequality constraints (≤) imposed in the primal problem.

(3) $A^T w - t = c$ observes the inequality constraints (≥) imposed in the dual of the corresponding primal.

(4) $x \geq 0, \; s \geq 0, \; w \geq 0, \; t \geq 0$ observe the non-negativity constraints imposed in the primal as well as in the corresponding dual and thus maintain the feasibility of the solution for primal as well as its dual problem.

When the non-negative solution exists, the part of this solution vector x, s represent the optimal solution for the primal problem, while the other part w, t represent the optimal solution for the corresponding dual problem. When such a non-negative solution does not exist, then the problem at hand is either unbounded or infeasible.

Theorem 3.1: (x, s) and (w, t) are optimal solutions of the primal and the corresponding dual problem respectively if and only if they satisfy

$$c^T x - b^T w = 0$$

$$Ax + s = b$$

$$A^T w - t = c$$

Such that $x \geq 0$, $s \geq 0$, $w \geq 0$, $t \geq 0$.

Proof: Straightforwardly follows from the above discussion.

□

For the discussion of NNLS algorithm one should see [1]. The solution using the above method for small sized problems (up to 100 variables) can be easily verified using the simple Matlab programs given below.

We now provide two simple MATLAB PROGRAMS, first for maximization problem and second for minimization respectively, for reader's checking:

(i) Maximization LP Problem:

```
%M-file for Symmetric Maximization Problem
%Requirement: All inequalities must be of Type  <=  i.e. Ax<=b
% The program solves Linear Programming Maximization Problem
%The program uses "LSQNONNEG" algorithm
%Important Note: If the Value of Norm is LARGE then the problem
%at hand is either Unbounded or Infeasible
%The Proper value 0f norm is usually very small like: 1e-014
A=input('Enter Matrix of Constraints Coefficients >')
cT=input('Enter Row-vector of Profit Coefficients >')
b=input('Enter Column-vector of Limits of Constraints >')
[m,n]=size(A);

D1=[cT zeros(1,m) -b' zeros(1,n)];
D2=[A eye(m) zeros(m,m+n)];
D3=[zeros(n,m+n) A' -eye(n)];
E=[D1;D2;D3];
f=[0;b;cT'];
z=lsqnonneg(E,f)
% For the student version of MATLAB replace "lsqnonneg" by "nnls"
disp('THE MAX VALUE OF OBJECTIVE FUNCTION IS')
d=cT*z(1:n);
```

```
disp(d)
disp('THE VALUE OF NORM IS')
disp(norm(f-E*z))
```

(ii) Minimization LP Problem:

```
%M-file for Symmetric Minimization Problem
%Requirement : All inequalities must be of Type  >=  i.e. Ax>=b
% The program solves Linear Programming Minimization Problem
%The program uses "LSQNONNEG" algorithm
%Important Note : If the Value of Norm is LARGE then the problem
%at hand is either Unbounded or Infeasible
%The Proper value of norm is usually very small like: 1e-014
A=input('Enter Matrix of Constraints Coefficients >')
cT=input('Enter Row-vector of Cost Coefficients >')
b=input('Enter Column-vector of Limits of Constraints >')
[m,n]=size(A);

D1=[cT zeros(1,m) -b' zeros(1,n)];
D2=[A -eye(m) zeros(m,m+n)];
D3=[zeros(n,m+n) A' eye(n)];
E=[D1;D2;D3];
f=[0;b;cT'];
z=lsqnonneg(E,f)
% Users of the student version should replace “lsqnonneg” by “nnls”

disp('THE MIN VALUE OF OBJECTIVE FUNCTION IS')
d=cT*z(1:n);
disp(d)
disp('THE VALUE OF NORM IS')
disp(norm(f-E*z))
```

4. Examples: We give an example solved using the above simple programs.

(1) Maximize: $2x_1 + 10x_2 + x_3$

Subject to: $5x_1 + 2x_2 + x_3 \le 7$

$$2x_1 + x_2 + 7x_3 \le 9$$

$$x_1 + 3x_2 + 2x_3 \le 5$$

$$x_1 \le 1$$

$$x_2 \le 1$$

$$x_3 \le 1$$

$$x_1, x_2, x_3 \ge 0$$

Solution:

PRIMAL SOLUTION:

```
    0.8889
    1.0000
    0.5556
         0
    2.3333
         0
    0.1111
         0
    0.4444
```

DUAL SOLUTION:

```
    0.3333
         0
    0.3333
         0
    8.3333
         0
         0
         0
         0
```

THE MAX VALUE OF OBJECTIVE FUNCTION IS
12.3333

THE VALUE OF NORM IS
1.8829e-014

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5. Conclusion: We suggest here to make direct use of the equation implied by the duality theorem. We have seen here that the efficiency of solving the Linear Programming problem by the suggested method can be improved by devising efficient methods for finding nonnegative solution, if and when exists, for a system of linear equations.

References

1. Charles L. Lawson and Richard J. Hanson, 'Solving Least Squares Problems, Prentice-Hall, Inc., Engle-Wood Cliffs, N.J., 1974.