Matrix Transformation and Transform the Generalized Wave Equation into the Maxwell Wave Equation and the Second Form of Wave Equation

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Abstract

For free electromagnetic field, there are two kinds of the wave equation, one is Maxwell wave equation, another is generalized wave equation. In the paper, according to the matrix transformation the author transform the general quadratic form into diagonal matrix. Then this can obtain both forms of wave equation. One is the Maxwell wave equation, another is the second form of the wave equation. In the half latter of the paper the author establish other two vibrator differential equations.

Key words: matrix transformation, wave equation.

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1 introduction

Starting from the generalized wave equation and the matrix transformation, when transform the generalized wave equation, the general quadratic matrix become a diagonal matrix. The author obtain both wave equations, One equation is the Maxwell wave equation, another is the second form of the wave equation. This can deem that the both wave equations are two kinds of the free electromagnetic field wave equations. Two wave equations are both standard quadratic forms. For the solution of the Maxwell wave equation, the wave amplitude $\mathbb{F}_0$ does not matter with the wave number k and wave frequency ω. In the wave amplitude of the generalized wave equation, being wave number k and wave frequency ω, it is a function of the k and the ω. Solve the second form of the wave equation, there are two solutions $\mathbb{F}_1'$ and $\mathbb{F}_1''$, when multiply $\mathbb{F}_1'$ by $\mathbb{F}_1''$ become a function $\mathbb{F}$, the $\mathbb{F}$ is just the solution of the generalized wave equation. However the generalized wave equation is distinct from the second form wave equation in free electromagnetic field. Last the author establish a pair of vibrator differential equations, their solutions multiplication is solution of the generalized wave equation too.

2 In free electromagnetic field, generalized wave equation

In free electromagnetic field, there are two kinds of wave equations, one is Maxwell wave equation, another is the generalized wave equation [1].

$$\nabla^2 \mathbb{F} - \frac{1}{c^2} \mathbb{C} \cdot \nabla \frac{\partial \mathbb{F}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbb{F}}{\partial t^2} = 0 \tag{1}$$

the $\mathbb{F}$ is electromagnetic field, the solution of equation (1) is

$$\mathbb{F} = \mathbb{F}_0 exp(\lambda kr - \lambda\omega t) exp(i\lambda kr - i\lambda\omega t) \tag{2}$$

In the wave amplitude $\mathbb{F}_0 exp(\lambda kr - \lambda\omega t)$, there are the wave number k and the wave frequency ω. When the k and the ω are variables, the wave amplitude also change follow them. Obviously it is a function of the k and the ω. By matrix transformation [2], two wave equations are deduced by the generalized wave equation (1). one of two wave equations is that well know the Maxwell wave equation [3], have

$$\nabla^2\mathbb{F}_1 - \frac{1}{c^2}\frac{\partial^2\mathbb{F}_1}{\partial t^2} = 0 \tag{3}$$

there is the solution

$$\mathbb{F}_1 = \mathbb{F}_{10} exp(ikr - i\omega t) \tag{4}$$

for the equation (4), the wave amplitude $\mathbb{F}_{10}$ is a constant, it does not matter with the wave number k and wave frequency ω.

Other form of two wave equations is second form of the wave equation, get

$$\nabla^2\mathbb{F}_1 + \frac{1}{c^2}\frac{\partial^2\mathbb{F}_1}{\partial t^2} = 0 \tag{5}$$

By separating of the spatial r and the temporal t of variants [4], for the electromagnetic field $\mathbb{F}_1$

$$\mathbb{F}_1(r,t) = \mathbb{R}(r)T(t) \tag{6}$$

in the equation (5), there is the first form

$$\frac{c^2}{\mathbb{R}(r)}\frac{\partial^2\mathbb{R}(r)}{\partial r^2} = -\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \lambda^2\omega^2 \tag{7}$$

here we will make the ω is a constant and the λ is a invariant. In the equation (7), there is a solution

$$\mathbb{F}_1' = \mathbb{R}_0 T_0 exp(\pm\lambda kr) exp(\pm i\lambda\omega t) \tag{8}$$

or rewrite, have

$$\mathbb{F}_1' = \mathbb{F}_0' exp(\pm\lambda kr) exp(\pm i\lambda\omega t) \tag{9}$$

For the equation (5), there is the second form

$$-\frac{c^2}{\mathbb{R}(r)}\frac{\partial^2\mathbb{R}(r)}{\partial r^2} = \frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \lambda^2\omega^2 \tag{10}$$

in (10), there is a solution

$$\mathbb{F}_1'' = \mathbb{R}_0 T_0 exp(\pm\lambda\omega t) exp(\pm i\lambda kr) \tag{11}$$

or rewrite (11), have

$$\mathbb{F}''_1 = \mathbb{F}_0'' exp(\pm\lambda\omega t) exp(\pm i\lambda kr) \tag{12}$$

Now the equation (9) $\mathbb{F}_1'$ is used to multiply the (12) $\mathbb{F}_1''$, get

$$\mathbb{F} = \mathbb{F}'_1\mathbb{F}''_1 = \mathbb{F}_0 exp(\lambda kr - \lambda\omega t) exp(i\lambda kr - i\lambda\omega t) \tag{13}$$

In the (13) the wave amplitude $\mathbb{F}_0 exp(\lambda kr - \lambda\omega t)$, there are wave number k and wave frequency ω. When the k and the ω are variable, the wave amplitude change follow them, it is a function of the k and the ω. For the wave amplitude (2) of the equation (1), the wave amplitude (13) compare with the equation (2), both the functions are same.

By the matrix transformation, both wave equation (3) and (5) are derived from the generalized wave equation (1), thus general quadratic form become a standard quadratic form. We consider that the generalized wave equation is perfect conformability with the Maxwell wave and the second form of wave equation.

3 two vibrator equations and their solutions

Now let us establish a pair of vibrator differential equations in the electromagnetic field, the first vibration equation in the temporal variable t, have

$$\frac{d^2\mathbb{F}_2}{dt^2} + 2(\lambda\omega)\frac{d\mathbb{F}_2}{dt} + 2\lambda^2\omega^2\mathbb{F}_2 = 0 \tag{14}$$

In the spatial variance, the second wave equation is

$$\nabla^2\mathbb{F}_2 - 2\lambda k\nabla \cdot \mathbb{F}_2 + 2\lambda^2 k^2\mathbb{F}_2 = 0 \tag{15}$$

the solution of the equation (14), get

$$\mathbb{F}_2 = \mathbb{F}_{20}exp(-\lambda\omega t)exp(-i\lambda\omega t) \tag{16}$$

only in the variable temporal t, rewrite the equation (16) as the temporal t function, have

$$T(t) = \mathbb{F}_{20}exp(-\lambda\omega t)exp(-i\lambda\omega t) \tag{17}$$

In the spheroidal coordinates, the equation (15) of spherically symmetric, become

$$\frac{d^2\mathbb{F}_2}{dr^2} - 2(\lambda k)\frac{d\mathbb{F}_2}{dr} + 2\lambda^2 k^2\mathbb{F}_2 = 0 \tag{18}$$

only in the spatial r of variance, for (18) there is the solution

$$\mathbb{F}_2 = \mathbb{F}_{20}exp(\lambda kr)exp(i\lambda kr) \tag{19}$$

rewrite (19) as a function of the radius r, get

$$\mathbb{R}(r) = \mathbb{F}_{20}exp(\lambda kr)exp(i\lambda kr) \tag{20}$$

Evidently in the free electromagnetic field $\mathbb{F}_2$, both equations (14) and (15) are vibrator equations, the equation (14) is alone the temporal t of variant, the (18) is a equation only in the spatial r of variant. The equation (20) $\mathbb{R}(r)$ is used to multiply the (17) $T(t)$, the function $\mathbb{F}(r,t)$ simultaneously there are the variable spatial r and the temporal t by (6) $\mathbb{F} = \mathbb{R}(r)T(t)$, become

$$\mathbb{F} = \mathbb{R}(r)T(t) = \mathbb{F}_0 exp(\lambda kr - \lambda\omega t)exp(i\lambda kr - i\lambda\omega t) \tag{21}$$

we compare the equation (21) with the solution (2), two forms (2) and (21) are completely identical, the equation (14) and (15) are regard as relation with the generalized wave equation (1). we know that the wave number k and the wave frequency ω have relation to $k = \frac{\omega}{c}$. Subtract the equation (14) from the equation (15), get

$$c^2\nabla^2\mathbb{F}_2 - 2\lambda\omega c\nabla\cdot\mathbb{F}_2 + 2\lambda^2\omega^2\mathbb{F}_2 - (\frac{d^2\mathbb{F}_2}{dt^2} + 2\lambda\omega\frac{d\mathbb{F}_2}{dt} + 2\lambda^2\omega^2\mathbb{F}_2) = 0 \tag{22}$$

due to meanwhile there are the temporal t and the spatial ∇ of variants, the (22) is arranged again, become

$$c^2\nabla^2\mathbb{F}_2 - \frac{\partial^2\mathbb{F}_2}{\partial t^2} - 2\lambda\omega(c\nabla\cdot\mathbb{F}_2 + \frac{\partial\mathbb{F}_2}{\partial t}) + 2\lambda^2\omega^2(\mathbb{F}_2 - \mathbb{F}_2) = 0 \tag{23}$$

When passive, there is continuity equation $\nabla\cdot c\mathbb{F}_2 + \frac{\partial\mathbb{F}_2}{\partial t} = 0$, so the equation (23) become $\nabla^2\mathbb{F}_2 - \frac{1}{c^2}\frac{\partial^2\mathbb{F}_2}{\partial t^2} = 0$, evidently it is just Maxwell wave equation.

4 Transform the generalized wave equation by the matrix

The equation (1) is a general quadratic form, now we transform equation (1) into diagonal matrix. For advantageous to the operate, we make $\mathbb{C}\cdot\nabla = p$ and $\frac{\partial}{\partial t} = \varepsilon$. Thus the equation (1) is rewritten

$$(p^2 - p\varepsilon - \varepsilon^2)\mathbb{F} = 0 \tag{24}$$

if alone take the bracket of the equation (24), have

$$p^2 - p\varepsilon - \varepsilon^2 = 0 \tag{25}$$

Now rewrite on the equation (25), become

$$p^2 - \frac{1}{2}p\varepsilon - \frac{1}{2}p\varepsilon - \varepsilon^2 = 0 \tag{26}$$

starting from the matrix transformation, we transform equation (26) into diagonal matrix

$$\begin{pmatrix} p & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} p \\ \varepsilon \end{pmatrix} = 0 \tag{27}$$

we take only the matrix as

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \tag{28}$$

and introduce into the characteristic polynomial of the equation (28), may be changed into

$$det(\lambda E - A) = 0 \tag{29}$$

in the equation (29), there are both characteristic values, one is $\lambda_1 = \frac{\sqrt{5}}{2}$ and another is $\lambda_2 = -\frac{\sqrt{5}}{2}$. For the equation (29), we get a equation of the matrix

$$(\lambda E - A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{30}$$

When spread out equation (30), we obtain an equation set

$$\begin{cases} (\sqrt{5} - 2)x_1 + x_2 = 0 \\ x_1 + (\sqrt{5} + 2)x_2 = 0 \end{cases} \tag{31}$$

Now substitute the first solution of the equation (29) $\lambda_1 = \dfrac{\sqrt{5}}{2}$ into the equation (31), meanwhile make the (31) solution are $x_1 = -1$ and $x_2 = \sqrt{5} - 2$, thus have a base solution

$$\alpha_1 = (-1 \quad \sqrt{5} - 2)^T \tag{32}$$

When we substitute the $\lambda_2 = -\dfrac{\sqrt{5}}{2}$ into the equation (31), that make the (31) solution $x'_1 = 1$ and $x'_2 = \sqrt{5}+2$, for another the solution of the equation (29) thus get another base solution

$$\alpha_2 = (1 \quad \sqrt{5} + 2)^T \tag{33}$$

By the base solutions (32) and (33), we take α_1 and α_2 to constitute a matrixes , have

$$Y = \begin{pmatrix} -1 & 1 \\ \sqrt{5} - 2 & \sqrt{5} + 2 \end{pmatrix} \tag{34}$$

and another matrix, get

$$Y^T = \begin{pmatrix} -1 & \sqrt{5} - 2 \\ 1 & \sqrt{5} + 2 \end{pmatrix} \tag{35}$$

by these matrixes (28), (34) and (35), the general quadratic matrix (28) become a diagonal matrix

$$Y^T A Y = \begin{pmatrix} -10 + 5\sqrt{5} & 0 \\ 0 & -10 - 5\sqrt{5} \end{pmatrix} = diag(\Lambda_1 \quad \Lambda_2) \tag{36}$$

Now we can transform the general matrix of the equation (27) into the diagonal matrix equation

$$\begin{pmatrix} p & \varepsilon \end{pmatrix} \begin{pmatrix} -10 + 5\sqrt{5} & 0 \\ 0 & -10 - 5\sqrt{5} \end{pmatrix} \begin{pmatrix} p \\ \varepsilon \end{pmatrix} \mathbb{F}_1 = 0 \tag{37}$$

After transformation, the electromagnetic field has changed from being $\mathbb{F}$ to the field $\mathbb{F}_1$. The equation (37) is spread out

$$5\sqrt{5}(p^2 - \varepsilon^2)\mathbb{F}_1 - 10(p^2 + \varepsilon^2)\mathbb{F}_1 = 0 \tag{38}$$

We had made $\mathbb{C} \cdot \nabla = p$ and $\dfrac{\partial}{\partial t} = \varepsilon$ front of the equation (3). only take the first bracket among the (38), have

$$\nabla^2 \mathbb{F}_1 - \frac{1}{c^2}\frac{\partial^2 \mathbb{F}_1}{\partial t^2} = 0 \tag{39}$$

The equation (39) become well known the Maxwell wave equation. Take the second bracket of (38), get

$$(p^2 + \varepsilon^2)\mathbb{F}_1 = 0 \tag{40}$$

for the equation (40) can write another formulation

$$\nabla^2 \mathbb{F}_1 + \frac{1}{c^2}\frac{\partial^2 \mathbb{F}_1}{\partial t^2} = 0 \tag{41}$$

namely there is the equation (5) before the (41) the second form of the wave equation.

References

[1] X.Z. Zhong. *Electromagnetic Field Equation and Field Wave Equation.* vixra. org: [84] 1101.0057., 2011.

[2] GeneH.Golub, CharlesF.Ven loan. *MATRIX COMPUTATIONS.* ISBN 0-8018-5414-8. 3rd ed © 1996. Hopkins University press.

[3] J.D. Jackson. *CLASSICAL ELECTRODYNAMICS*, 3rd ed. John-Wiley & Sons, Inc., 1999.

[4] M.D. Weir, R.L. Finney and F.R. Giordano. *THOMAS' CALCULUS*. 10th ed. Publishing as Pearson Education. Inc., 2004.