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COLOR AND ISOSPIN WAVES FROM TETRAHEDRAL SHUBNIKOV GROUPS

Bodo Lampe

Abstract

This note supplements a recent article [1] in which it was pointed out that the observed spectrum of quarks and leptons can arise as quasi-particle excitations in a discrete internal space. The paper concentrated on internal vibrational modes and it was only noted in the end that internal spin waves ('mignons') might do the same job. Here it will be shown how the mignon-mechanism works in detail. In particular the Shubnikov group $A_4 + S(S_4 - A_4)$ will be used to describe the spectrum, and the mignetic ground state is explicitly given.

This is a supplement to section 7 of the article [1] where 8 internal spins were considered, whose collective excitations ('mignons' or 'i-spin waves') were used to construct quark and lepton states. The scenario to start with is a spinor model in a (6+2)-dimensional spacetime which by some unknown compactification process splits into a (3+1)-dimensional Minkowski space plus a (3+1)-dimensional internal space, in such a way that the internal space reappears as a finite and discrete internal 'crystal' on each point of the physical base space [1].

Generally, in $SO(d_1, d_2)$ the spinor dimensions viewed over complex space coincide with the case of the (d_1+d_2)-dimensional Euclidean space. Therefore spinors in 6+2 dimensions can be considered as SO(8) spinors by a suitable Wick rotation. Note that the Lie algebra of SO(8) is extremely symmetric - the key word here is triality [2] - and has an intimate connection to the nonassociative algebra of octonions [3, 4]. Furthermore, it is known that SO(8)-spinors can appear as 8-dimensional right-handed states $\mathbf{8_R}$ as well as 8-dimensional left-handed states $\mathbf{8_L}$ [15]. Both representations can be combined to a 16-dimensional Dirac spinor $\mathbf{8_L} + \mathbf{8_R}$ in a similar way as a Dirac spinor in 3+1 dimensions can be written as a sum of two Weyl spinors $\mathbf{2_L} + \mathbf{2_R}$.

When going to $SO^{ph}(3,1) \times SO^{in}(3,1)$ the SO(6,2) spinor will split into a product of a Dirac spinor in internal space and a Dirac spinor on Minkowski space according to [5]

$$\mathbf{8_L} + \mathbf{8_R} = (\mathbf{2_L^{ph}} + \mathbf{2_R^{ph}}, \mathbf{2_L^{in}} + \mathbf{2_R^{in}}) \tag{1}$$

The corresponding field has therefore 2 spinor indices a and i both running from 1 to 4 and corresponding to a particle with Dirac properties both in physical and in internal space.

The next step is to assume that after the compactification process a copy of the internal space is fixed to each point of physical space, so that internal Lorentz symmetry is broken and the induced i-spin structure can be analyzed as a nonrelativistic system of strongly correlated 3-dimensional i-spin vectors. This is not only supported by phenomenological observations (see below) but as a benefit the methods of solid state physics for the description of magnetic systems can be applied.

The internal system is described in second quantization language by creation and annihilation operators satisfying the canonical anti-commutation relations

$$[c_\alpha(m), c_\beta^\dagger(m')]_- = \delta_{\alpha\beta}\delta_{m,m'} \tag{2}$$

where $\alpha = \pm 1/2$ denotes the spin and $m, m' = 1, ..., 8$ denotes the sites of the internal crystal. The appropriate Hamiltonian for the free i-spin system is given by [9]

$$H_0 = -\sum_{m,m'} t(m, m')[c_\alpha^\dagger(m)c_\alpha(m') + h.c.] \tag{3}$$

where the sum is over all crystal sites $m \neq m'$ and t is the tunneling rate between the spins.

When it comes to interactions a suitable framework for discussion is given by Heisenberg spin models. These have been considered in statistical and solid state physics for a long time [6, 7, 8], and they have been used to describe magnetic phase transitions and excitations as well as many other phenomena. The basic variables are spin vectors $\mathbf{S}$ defined on each site of the internal crystal. They are related to the original creation and annihilation operators eq. (2) via

$$\mathbf{S}(m) = \frac{1}{2}c_\alpha^\dagger(m)\boldsymbol{\tau}_{\alpha\beta}c_\beta(m) \tag{4}$$

where $\boldsymbol{\tau}$ is the triplet of internal Pauli matrices.

What are the symmetries of this system? There are 2 continuous symmetries, associated with the conservation of internal charge and spin, respectively. Namely, the Hamiltonian H_0 is invariant under U(1) transformations $c_\alpha(m) \to \exp(i\theta_0)c_\alpha(m)$ for arbitrary (constant) angle θ_0 and under (constant) SU(2) transformations $c_\alpha(m) \to \exp(i\boldsymbol{\theta\tau})c_\alpha(m)$. The point is that without interaction the i-spins are fixed to the crystal sites but otherwise can rotate freely so that the system shows an overall internal SU(2) symmetry. When the interaction is switched on, the SU(2)-breaking mignetic ground state will be formed and the mignons will appear as spin vector fluctuations around the ground state.

There are also several discrete symmetries. First of all, there is the point group symmetry dictated by the discrete nature of the internal crystal. To be specific a tetrahedral arrangement of spins is chosen with point group symmetry S_4 and 8 spin vectors forming an internal 'mignetic molecule' [10, 11]. Counting the degrees of freedom one easily sees that there are 3×8=24 quasi-particle states to be expected, which can be ordered according to the symmetry of the (mignetic) ground state, cf. eq. (5) below. Note that S_4 is not a chiral symmetry because it contains improper rotations in the form of reflections of a plane. However, the spin vector is a pseudovector, and this implies that the combined point and mignetic transformations can form a chiral group - the Shubnikov group to be introduced later. Finally, the Hamiltonian is real ($H_0 = H_0^*$), a signature of time reversal invariance. Just as SU(2) and the reflection symmetries, the time reversal invariance will be broken by the mignetic ground state. At this point the existence of an internal time variable s which describes processes in the internal crystal and differs from physical time t, is mandatory, not only because it naturally leads to internal spin waves of antifermion spins which can be used to describe the quantum numbers of antiquarks and antileptons but also because the breaking of internal time reversal invariance will play an important role for the discussion of the ordered mignetic structures presented below. That is the reason why I started with SO(6,2) as the complete symmetry of the whole space before the compactification - instead of SO(6,1) as was done in ref. [1].

It is well known that one should use Shubnikov groups (which are sometimes called black-and-white groups) [12, 13] instead of ordinary point groups to classify the spectrum of spin wave excitations. More concretely, we shall use the Shubnikov group $A_4 + S(S_4 - A_4)$ instead of the pyritohedral group $A_4 \times Z_2$ considered in the article [1]. Here S_4 is the symmetry group of a regular tetrahedron, and A_4 its subgroup of proper rotations (i.e. without reflections).[1] S denotes the (internal) time inversion operation, which in an elegant way replaces the rather unnatural Z_2-factor in $A_4 \times Z_2$. Instead of eq. (4) of the paper, the 24 mignon states are then given by

$$\begin{aligned} &A(\nu_e) + A'(\nu_\mu) + A''(\nu_\tau) + T(d) + T(s) + T(b) + \\ &\quad A_s(e) + A_s'(\mu) + A_s''(\tau) + T_s(u) + T_s(c) + T_s(t) \end{aligned} \tag{5}$$

where A, A', A'' and T are singlet and triplet representations of A_4 and the index s denotes genuine representations of the Shubnikov group $A_4 + S(S_4 - A_4)$ [12, 14, 8].

[1] These groups have been discussed in connection with neutrino and family mixing by many authors [16].

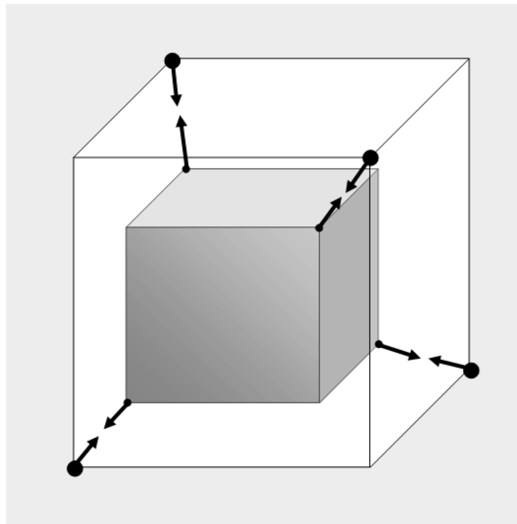

Figure 1: Mignetic ground state with 8 spin vectors arranged as follows: the corner points of the outer tetrahedron (big black dots) are given by the coordinate vectors $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$ and the spin vectors are chosen as to point to its centre $(0,0,0)$, i.e. the spin vectors sitting on the 4 sites are the negative of the coordinate vectors. The remaining 4 points lie on the inner tetrahedron (small black dots) which is obtained from the outer by multiplying the above coordinates by a common shrinking factor < 1, and the spinvectors of the inner tetrahedron are oriented opposite to those of the first one. The tetrahedra themselves have the tetrahedral group S_4 as point group symmetry. From the pseudovector property of the spin vectors it can be shown that the spin system has $A_4 + S(S_4 - A_4)$ symmetry.

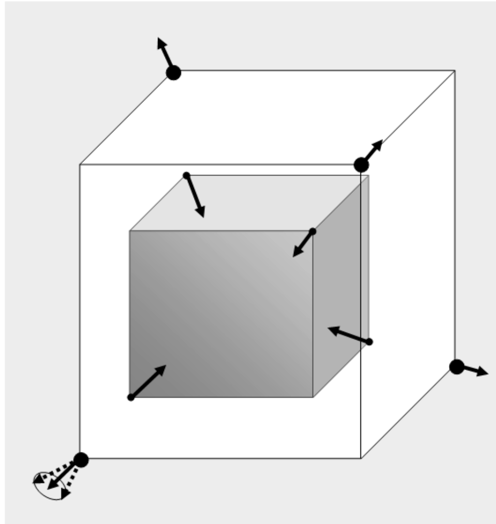

Figure 2: A ground state similar to fig. 1, however with opposite chirality. As compared to fig. 1 all spins point in opposite directions. Also shown is the behavior of one of the spin vectors in an excited state. Such an excitation is obviously not identical to its chiral counterpart, i.e. to the states derived from fig. 1

The 24 d.o.f. in eq. (5) correspond in fact to 8 spins each with 3 possible directions of the spin vector, as predicted above.

Note that the multiplet structure eq. (5) is quite unique. None of the other possible Shubnikov groups [12] show a pattern of the type eq. (5). Most of them do not even have triplets, and if so, one usually finds doublets as well.

To understand the physics it is useful trying to construct a ground state, for which the 24 states eq. (5) represent (internal spinwave) excitations. In other words, one is looking for a static system with 8 internal spins and symmetry group $A_4 + S(S_4 - A_4)$. The simplest 'mignetic molecule' [10, 11] of this kind consists of two regular tetrahedra with spin vectors arranged as in figure 1. Note that this ground state shows a rather strong type of antiferromagnetic order. Firstly, the spins in each tetrahedron add up to zero. Furthermore the spins appear in pairs with partners coming from both tetrahedra and which are oriented oppositely.

As a consequence, the (internal) Heisenberg spin SU(2) symmetry of the mignetic system is broken as well as the internal point symmetry S_4:

$$SU(2) \times S \times S_4 \to A_4 + S(S_4 - A_4) \tag{6}$$

The remaining Shubnikov symmetry $A_4+S(S_4-A_4)$ does not contain any reflections, because improper rotations S_4-A_4 appear only in combination with the time reversal operation S. Therefore it is a chiral group (just as $A_4 \times Z_2$), a property which was essential in the paper [1] to derive the parity violation of the weak interactions.

Note further that internal time reversal S is itself broken, as can be seen easily, because it is *not* an element of the Shubnikov group $A_4 + S(S_4 - A_4)$. Only combinations of the form SR, where $R \in S_4 - A_4$ is an improper rotation, are symmetries of the system. The point is that applying S (or R) to the ground state fig. 1 one will obtain a different state (fig. 2) with higher energy and opposite chirality, whose excitations have nothing to do with the excitations of fig. 1.

All states in eq. (5) are therefore chiral states, the difference between A and A_s, A' and A'_s etc excitations being mainly odd and even behavior under transformations SR. Since they are different multiplets, they will in general have different masses. On the level of quarks and leptons this gives different masses to weak isospin partners. How this is linked to the breaking of the Standard Model gauge group $\mathrm{SU}(2)_L$ will be discussed in a future publication.

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