A Note on the Wave-Particle Duality

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The wave-particle duality is related with the decreasing of the wavelength. It explains why an atomic electron does not radiate (Bohr atom). But there are discrepancies with the Pauli exclusion principle and the Fermi-Dirac statistics. The spin-orbit coupling resolves these discrepancies producing the fine structure splitting of the atom energy levels.

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The light is an electromagnetic wave, hence the phenomena of interference and diffraction; but it also behaves as a particle called the photon, for example, on the photoelectric effect. De Broglie generalized this wave-particle (or wave-corpuscle) duality of the light to the matter, hence the figures of interference and diffraction of the electrons, typical of the waves.

We know that the light behaves more like a particle than as a wave the greater is its frequency f and its energy, $E = hf$, where h is the Planck constant, and the lesser is its wavelength λ, since the speed of the light in the vacuum is $c = \lambda f$. The so-called effective mass of the photon would be $m = E/c^2 = hf/c^2 = h/c\lambda$. And to lesser wavelength, more particle appearance.

For the matter would be: $E = mc^2$, as particle, where $m = \gamma m_0$ is the moving mass, $\gamma = (1 - v^2/c^2)^{-1/2}$ and v the velocity, and m_0 the rest mass; but, $E = hf$, as wave, and $v_f = \lambda f$ would be the velocity of the wave or phase velocity and λ the wavelength and f the frequency. Therefore, $v_f = \lambda f = (h/mv)(mc^2/h) = c^2/v$, where we have applied the de Broglie postulate: $mv = h/\lambda$. As $v < c$ (special relativity), $v_f > c$. And the lesser is v the greater is v_f. Now, $v_f = \lambda f = 2\pi f/(2\pi/\lambda) = \omega/k$, where $\omega = 2\pi f$ is the angular frequency and $k = 2\pi/\lambda$ the wave number. The argument of the wave packet or wave group is: $\varphi = \omega t - kr + \theta$, where t is the time, r the distance and θ the phase angle; and for the center of the packet, it is $d\varphi/dk = 0$, then, $(d\omega/dk)t - r + d\theta/dk = 0$, equation that corresponds to the movement of the center of the packet with a group velocity $v_g = d\omega/dk$. From $E = mc^2$, $p = mv$, where p is the momentum, and $m = \gamma m_0$, it is obtained that $E^2 = m_0^2c^4 + p^2c^2$, then $2EdE = 0 + 2pc^2dp$ and $dE/dp = pc^2/E = mvc^2/mc^2 = v$; but also, $E = hf$ and $p = h/\lambda$ (since $p = mv$), then, $v = dE/dp = hdf/hd(1/\lambda) = df/d(1/\lambda)$. But, $v_g = d\omega/dk = 2\pi df/2\pi d(1/\lambda) = df/d(1/\lambda)$; therefore, $v = v_g$. Consequently, we can assume that the matter behaves more like a wave than as a particle the greater is its velocity v and its energy, $E = \gamma m_0c^2$. The phase velocity v_f would decrease, since $v_f = c^2/v$, the frequency f would increase, since also is $E = hf$, and the wavelength λ would decrease, since $v_f = \lambda f$, approaching to the particle dimensions, to its diameter. Thus for the electron, for example, it would be a wave, of very small wavelength, that may give rise to interference and diffraction patterns when interacts with objects with dimensions of the same order of magnitude.

A very interesting example is its application to the Bohr atom. Due to the wave-particle duality, an electron can exhibit both wave and particle properties, but only one of both simultaneously due to the complementary postulate of Bohr. If the atomic electron would exhibit particle properties, this one would radiate electromagnetic energy, because of its acceleration, falling to the atomic nucleus. But, it is known that a closed string can be deformed for obtaining a stationary wave. As a stationary wave continues indefinitely, then there is not electromagnetic radiation. Therefore, the atomic electron does not radiate electromagnetic energy because exhibits stationary wave properties, with the condition $2\pi r = n\lambda$, where r is the radius of the orbit, $n = 1, 2, 3, ...$ (it is the first quantum number, called the principal quantum number) and λ the wavelength of the stationary wave. This one was solved by Bohr with his postulate[1]: $mvr = nh/2\pi$. If we apply the de Broglie postulate: $mv = h/\lambda$, we obtain the previous relation: $2\pi r = n\lambda$.

However, there are discrepancies with the Pauli exclusion principle[2] and the Fermi-Dirac statistics. The Pauli exclusion principle can be summarized saying, for example, that two electrons in an atom cannot share the same orbit with the same four quantum numbers: n (principal), l (orbital), m_l (magnetic) and m_s (magnetic of spin). If the three first quantum numbers are the same for the two electrons, then, an electron has $m_s = +1/2$ and the other one $m_s = -1/2$. However, an electron would have a total angular momentum projection $|L_z| + |S_z|$ and the other one $|L_z| - |S_z|$ (spin-orbit coupling), where $L_z = m_l h/2\pi$ and $S_z = m_s h/2\pi$ are the angular momentum projections, orbital and intrinsic (or of spin), respectively. But, the electron with $|L_z| + |S_z|$ would have a little more energy than the one with $|L_z| - |S_z|$; therefore, both electrons would be in different near orbits (fine structure splitting of the atom energy levels).

Then, why are not there two electrons with the same four quantum numbers in the same orbit?. Because the electrons are fermions and they, according to the Fermi-Dirac statistics, have antisymmetric states, that is: $|\Psi\Phi> = -|\Phi\Psi>$, where Ψ and Φ are wave functions and the Dirac kets $|\Psi>$, $|\Phi>$, $|\Psi\Phi>$ and $|\Phi\Psi>$ single and two electron states, respectively. If $\Phi = \Psi$, then $|\Psi\Psi> = -|\Psi\Psi>$, $|\Psi\Psi> + |\Psi\Psi> = 2\,|\Psi\Psi> = 0$, and $|\Psi\Psi> = 0$, therefore this state cannot exist, and two electrons with the same four quantum numbers cannot share the same orbit. In quantum mechanics (QM), the wave function is complex, not real, and $|\Psi(x,y,z,t)|^2 = \Psi\Psi^*$, where Ψ^* is the complex conjugate of Ψ, gives the probability density of finding the particle at the point (x,y,z) at the instant t (Born postulate). In the QM, due to the Heisenberg uncertainty principle, there are no trajectories or orbits.

However, if we apply the wave-particle duality, we have that two electrons might stay, *a priori*, in the same orbit with the same four quantum numbers because: the two electrons would have the stationary waves $A \cos a$ and $B \cos b$, respectively, where A and B are the amplitudes and $a = 2\pi ft - kr + \theta$ and $b = 2\pi ft - kr + \varphi$ the arguments, and $r = n\lambda/2\pi$ is the radius of the orbit, n being the principal quantum number. We suppose single waves, although it implies that $v = v_f = c$. As the two electrons are in the same orbit, their energies, that is, the root mean square (rms) values of their waves, $A/2^{1/2}$ and $B/2^{1/2}$, are the same, then $B = A$. And the superposition of both waves would be: $A \cos a + B \cos b = A (\cos a + \cos b) = 2A \cos((a + b)/2) \cos((a - b)/2) = 2A\cos((\theta - \varphi)/2)$

$cos((a + b)/2)$. Due to the energy conservation, it would be: $(2A/2^{1/2})cos((\theta - \varphi)/2) = A/2^{1/2} + B/2^{1/2} = A/2^{1/2} + A/2^{1/2} = 2A/2^{1/2}$; then, $cos((\theta - \varphi)/2) = 1$ and $\theta - \varphi = j2\pi$ $(j = 0, 1, 2, ...)$, and the two waves would be in phase forming a two-electron stationary state. However, this state has not been observed because the two electrons are not interchangeable, since $|\Psi\Phi\rangle = -|\Phi\Psi\rangle$, then, due to the Pauli exclusion principle, an electron would have $m_s = +1/2$ and the other one $m_s = -1/2$ and they would have a very little difference between their energies (spin-orbit coupling), occupying two very close, but different, orbits (fine structure splitting). Note that we have used cosine functions because: $sin\ \alpha = cos(\pi/2 - \alpha) = cos\ a$ with $\pi/2 - \alpha = a$ and $sin\ \beta = cos(\pi/2 - \beta) = cos\ b$ with $\pi/2 - \beta = b$.

In summary, the wave-particle duality is related with the decreasing of the wavelength. It explains why an atomic electron does not radiate (Bohr atom). But there are discrepancies with the Pauli exclusion principle and the Fermi-Dirac statistics. The spin-orbit coupling resolves these discrepancies producing the fine structure splitting of the atom energy levels.

[1] The Bohr postulate was proposed and published before by Nicholson.

[2] The Pauli exclusion principle is based on the sequence of atomic electrons: *2, 8, 18, ...*; discovered by Stoner. From this sequence Pauli deduced the expression: $2n^2$, where n is the principal quantum number, that gives: $2(1^2), 2(2^2), 2(3^2), ...$; and devised a fourth quantum number with only two possible values. Kronig proposed to Pauli (and also to others) that this fourth quantum number would be the electron spin, but his idea was rejected by Pauli, and Kronig does not published it. Months later, and independently, Uhlenbeck and Gousmith discover the electron spin and published it.

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