ON GENERAL FORMULAS FOR GENERATING SEQUENCES OF PYTHAGOREAN TRIPLES ORDERED BY $c - b$

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ABSTRACT. General formulas for generating sequences of Pythagorean triples ordered by $c - b$ are studied in this paper. As computational proof, tables were made with a C++ script showing Pythagorean triples ordered by $c - b$ and included as text files and screenshots. Furthermore, to enable readers to check and verify them, the C++ script which will interactively generate tables of Pythagorean triples from the computer console command line is attached. It can be run in Cling and ROOT CINT C/C++ interpreters or compiled.

1. INTRODUCTION

This is a study on general formulas that would generate sequences of Pythagorean triples. In particular, formulas [1] given by the ancient Greek mathematicians, Plato and Pythagoras, that enables us to generate sequences of Pythagorean triples ordered by $c-b = \{1, 2\}$. Another formula was given by Euclid. Although it generates all primitive Pythagorean triples, it produces unordered sequences.

These formulas imply that it is possible to order the sequences of all Pythagorean triples by $c - b > 2$. It was found that they are special cases of a much more general set of formulas.

2. PYTHAGOREAN TRIPLES

Definition. Let $a, b, c \in \mathbb{N}$ and $a < c,\ b < c$ then a Pythagorean triple is a triple of natural numbers such that $a^2 + b^2 = c^2$. It is primitive if it is pairwise relatively prime and the parities of a and b are always opposites while c is always odd.

Theorem 1. *If (a, b, c) is a Pythagorean triple then there exists $\alpha,\beta,\gamma \in \mathbb{Z}$ and $k \in \mathbb{N},\ k = 1, 2, 3, \ldots$ where $\alpha = b-a,\ \beta = c-a,\ \gamma = c-b,$ and $\beta = \alpha+\gamma,\ (a-\gamma)^2 = 2\gamma\beta,\ \gamma\beta = 2k^2$ such that $(a, b, c) = (\gamma+2k,\ \beta+2k,\ \gamma+\beta+2k)$.*

Proof. Substitute $b = a + \alpha$ and $c = a + \beta$ into $a^2 + b^2 = c^2$ then evaluate. We have

$$a^2 + (a + \alpha)^2 = (a + \beta)^2$$

$$a^2 + (a^2 + 2a\alpha + \alpha^2) = a^2 + 2a\beta + \beta^2$$

Rearrange terms and factorize

$$a^2 + 2a(\alpha - \beta) = \beta^2 - \alpha^2$$

$$a^2 - 2(\beta - \alpha)a = (\beta + \alpha)(\beta - \alpha)$$

But $\beta - \alpha = (c - a) - (b - a) = c - b$. This is γ thus $\alpha + \gamma = \beta$ and we have

$$a^2 - 2\gamma a = [(\alpha + \gamma) + \alpha]\gamma$$

$$a^2 - 2\gamma a = \gamma(2\alpha + \gamma)$$

Complete the square on the left hand side of the equation, arrange terms, and factorize

$$a^2 - 2\gamma a + (-\gamma)^2 - (-\gamma)^2 = \gamma(2\alpha + \gamma)$$

$$a^2 - 2\gamma a + (-\gamma)^2 = \gamma(2\alpha + \gamma) + (-\gamma)^2$$

$$(a - \gamma)^2 = 2\gamma\alpha + \gamma^2 + \gamma^2$$

$$(a - \gamma)^2 = 2\gamma(\alpha + \gamma)$$

$$\therefore (a - \gamma)^2 = 2\gamma\beta$$

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It is evident that $(a-\gamma)^2 = 2\gamma\beta$ has even parity and hence of the form $4k^2$. Thus $(a-\gamma)^2 = 4k^2$ and $\gamma\beta = 2k^2$. Take the square root of $(a-\gamma)^2 = 4k^2$ and we see that $a = \gamma \pm 2k$. We take $a = \gamma + 2k$ to avoid negative values. From $b - a = \alpha$ we get $b = a + \alpha = \beta + 2k$ and since $c - b = \gamma$ we also get $c = b + \gamma = \gamma + \beta + 2k$. □

Theorem 2. *If (a,b,c) is a Pythagorean triple then there exists $\alpha,\beta,\gamma \in \mathbb{Z},\ k,n \in \mathbb{N},\ k = n,\ n = 1,2,3,\ldots$ where $\alpha = b-a,\ \beta = c-a,\ \gamma = c-b,$ and $\beta = \alpha+\gamma,\ (a-\gamma)^2 = 2\gamma\beta,\ \gamma\beta = 2k^2$ such that*

$$(a,b,c) = \begin{cases} (1+2k,\ 2k^2+2k,\ 1+2k^2+2k), & \text{if } \gamma = 1,\ \beta = 2k^2 \\ (2+2k,\ k^2+2k,\ 2+k^2+2k), & \text{if } \gamma = 2,\ \beta = k^2 \end{cases}$$

Proof. We have $\beta = \dfrac{2k^2}{\gamma}$ and so Pythagorean triples are generated by

$$(a,b,c) = \left(\gamma + 2k,\ \frac{2k^2}{\gamma} + 2k,\ \gamma + \frac{2k^2}{\gamma} + 2k\right)$$

Clearly, integral values can be obtained for $\gamma = \{1,2\}$. Let $\gamma = 1$ thus $\beta = 2k^2$ and $a = 1+2k,\ b = 2k^2+2k,\ c = 1+2k^2+2k$. Now let $\gamma = 2$ thus $\beta = k^2$ and $a = 2+2k,\ b = k^2+2k,\ c = 2+k^2+2k$. □

Corollary 2.1. *Primitive Pythagorean triples are generated by*

$$(a,b,c) = \begin{cases} (1+2n, 2n^2+2n, 1+2n^2+2n), & \text{if } \gamma = 1,\ \beta = 2n^2 \\ (4n, 4n^2-1, 4n^2+1), & \text{if } \gamma = 2,\ \beta = n^2 \end{cases}$$

Proof. From Theorem 2, let $\gamma = 1,\ \beta = 2k^2$ thus $a \nmid b,\ a \nmid c,\ b \nmid c\ \forall\ k$ and $\gcd(a,b,c) = 1$. Hence primitive Pythagorean triples are found for all $k = n$. For $\gamma = 2,\ \beta = k^2$ we consider when k is even and when it is odd.

If k is even let $k = 2n$ thus $a = 2(2n+1),\ b = 4n(n+1),\ c = 2(2n^2+2n+1)$ and $a \mid b,\ a \mid c,\ c \mid b, \gcd(a,b,c) = 2$. Hence non-primitive Pythagorean triples are found when k is even.

If k is odd let $k = 2n-1$ thus $a = 4n,\ b = 4n^2-1,\ c = 4n^2+1$ and $a \nmid b,\ a \nmid c,\ b \nmid c, \gcd(a,b,c) = 1$. Hence primitive Pythagorean triples are found when k is odd. □

3. General formulas for Pythagorean Triples

Theorem 3. *If (a,b,c) is a Pythagorean triple then there exists $\alpha,\beta,\gamma \in \mathbb{Z},\ k,m,n \in \mathbb{N},\ k = mn,\ m = 1,2,3,\ldots,$ $n = 1,2,3,\ldots$ where $\alpha = b-a,\ \beta = c-a,\ \gamma = c-b$ and $\beta = \alpha+\gamma,\ (a-\gamma)^2 = 2\gamma\beta,\ \gamma\beta = 2k^2$ such that*

$$(a,b,c) = \begin{cases} (m^2+2mn, 2n^2+2mn, m^2+2n^2+2mn), & \text{if } \gamma = m^2,\ \beta = 2n^2 \\ (2m^2+2mn, n^2+2mn, 2m^2+n^2+2mn), & \text{if } \gamma = 2m^2,\ \beta = n^2 \end{cases}$$

Proof. Let $k = mn$ then since $\gamma\beta = 2k^2$ we have $\gamma\beta = 2m^2n^2$. Clearly, the set of permutation for $\gamma\beta$ is $\{(m^2)(2n^2), (2m^2)(n^2)\}$. Thus by Theorem 1 we get the general formulas.

This is the general form of Pythagoras' and Plato's formulas. □

Corollary 3.1. *If $\gamma = m^2$ and $\beta = 2n^2$ then primitive Pythagorean triples are generated if m is odd and* $\gcd(m,n) = 1$.

Proof. Let $q,t \in \mathbb{N},\ q = 1,2,3,\ldots,\ t = 1,2,3,\ldots$. When m is odd then $m = 2t-1$ and $a = (2t-1)[(2t-1)+2n],\ b = 2n[n+(2t-1)],\ c = (2t-1)^2 + 2n[n+(2t-1)],\ \gcd(a,b,c) = 1$. When m is even then $m = 2t$ and $a = 4(t)(t+n),\ b = 2(n)(2t+n),\ c = 2(2t^2+2tn+n^2),\ \gcd(a,b,c) = 2$.

When $n = qm$, then $a = m^2(1+2q),\ b = 2qm^2(1+q),\ c = m^2(1+2q+2q^2)$ and $gcd(a,b,c) = m^2$ thus primitive Pythagorean triples are generated if m is odd and $\gcd(m,n) = 1$. □

Corollary 3.2. *If $\gamma = 2m^2$ and $\beta = n^2$ then primitive Pythagorean triples are generated if n is odd and* $\gcd(m,n) = 1$.

Proof. Let $q,t \in \mathbb{N},\ q = 1,2,3,\ldots,\ t = 1,2,3,\ldots$. When m is odd then $m = 2t-1$ and $a = 2(2t-1)[(2t-1)+n],\ b = n[n+2(2t-1)],\ c = 2(2t-1)[(2t-1)+n]+n^2, \gcd(a,b,c) = 1$. When m is even then $m = 2t$ and $a = 2(2t)(2t+n),\ b = n[n+2(2t)],\ c = 2(2t)(2t+n)+n^2, \gcd(a,b,c) = 1$. Since both have $\gcd(a,b,c) = 1$ we consider n. If n is even, let $n = 2t$ thus $a = 2m(m+2t),\ b = 4t(t+m),\ c = 2(m^2+2t^2+2mt)$ and $\gcd(a,b,c) = 2$. If n is odd let $n = 2t-1$ thus $a = 2m[m+(2t-1)],\ b = (2t-1)[(2t-1)+2m],\ c = 2m^2+(2t-1)^2+2m(2t-1)$ and $\gcd(a,b,c) = 1$.

When $n = qm$ then $a = 2m^2(1+q), b = m^2q(2+q), c = 2m^2(2+2q+q^2)$. We see that $gcd(a,b,c) = m^2$ thus primitive Pythagorean triples are generated if n is odd and $\gcd(m,n) = 1$. □

Corollary 3.3. *If $a < b$ then $n > \left\lfloor \frac{m}{\sqrt{2}} \right\rfloor$ for $\gamma = m^2,\ \beta = 2n^2$ and $n > \left\lfloor m\sqrt{2} \right\rfloor$ for $\gamma = 2m^2,\ \beta = n^2$.*

Proof. If $a < b$ then $\alpha > 0$ and so $\beta > \gamma$. Thus for $\gamma = m^2,\ \beta = 2n^2$ we have $2n^2 > m^2$ which is $n > \left\lfloor \frac{m}{\sqrt{2}} \right\rfloor$. And for $\gamma = 2m^2,\ \beta = n^2$ we have $n^2 > 2m^2$ which is $n > \left\lfloor m\sqrt{2} \right\rfloor$. □

Theorem 4. *If (a, b, c) is a Pythagorean triple then there exists $\alpha, \beta, \gamma \in \mathbb{Z},\ k, m, n \in \mathbb{N},\ k = mn,\ m = 1, 2, 3, \dots ,$ $n = 1, 2, 3, \dots$ where $\alpha = b - a,\ \beta = c - a,\ \gamma = c - b$ and $\beta = \alpha + \gamma,\ (a - \gamma)^2 = 2\gamma\beta,\ \gamma\beta = 2k^2$ such that*

$$(a, b, c) = \begin{cases} ((m+n)^2 - n^2, 2(m+n)n, (m+n)^2 + n^2), & \text{if } \gamma = m^2,\ \beta = 2n^2 \\ (2(m+n)m, (m+n)^2 - m^2, (m+n)^2 + m^2), & \text{if } \gamma = 2m^2,\ \beta = n^2 \end{cases}$$

Proof. The general formulas from Theorem 3 can be rearranged to

$\gamma = m^2,\ \beta = 2n^2$:

$$a = m^2 + 2mn + (n^2 - n^2) = (m+n)^2 - n^2$$
$$b = 2n^2 + 2mn = 2(m+n)n$$
$$c = m^2 + 2mn + (n^2 + n^2) = (m+n)^2 + n^2$$

$\gamma = 2m^2,\ \beta = n^2$:

$$a = 2m^2 + 2mn = 2(m+n)m$$
$$b = n^2 + 2mn + (m^2 - m^2) = (m+n)^2 - m^2$$
$$c = (m^2 + m^2) + 2mn + n^2 = (m+n)^2 + m^2$$

This is the general form of Euclid's formula. □

4. Addendum

4.1. Pythagorean Triples in the sum of two like powers where $t > 2$.

Remark. Let (a, b, c) be a Pythagorean triple and $t = 3$ then

$\gamma = m^2, \beta = 2n^2$:

$$a^3 = m^6 + 6m^5n + 12m^4n^2 + 8m^3n^3$$
$$b^3 = 8n^6 + 24n^5m + 24n^4m^2 + 8n^3m^3$$
$$c^3 = m^6 + 6m^5n + 18m^4n^2 + 32m^3n^3 + 36m^2n^4 + 24mn^5 + 8n^6$$
$$a^3 + b^3 = m^6 + 6m^5n + 12m^4n^2 + 16m^3n^3 + 24m^2n^4 + 24mn^5 + 8n^6$$

$\gamma = 2m^2, \beta = n^2$:

$$a^3 = 8m^6 + 24m^5n + 24m^4n^2 + 8m^3n^3$$
$$b^3 = n^6 + 6n^5m + 12n^4m^2 + 8n^3m^3$$
$$c^3 = 8m^6 + 24m^5n + 36m^4n^2 + 32m^3n^3 + 18m^2n^4 + 6mn^5 + n^6$$
$$a^3 + b^3 = 8m^6 + 24m^5n + 24m^4n^2 + 16m^3n^3 + 12m^2n^4 + 6mn^5 + n^6$$

Let $t = 4$ then

$\gamma = m^2, \beta = 2n^2$:

$$a^4 = m^8 + 8m^7n + 32m^6n^2 + 32m^5n^3 + m^4n^4$$
$$b^4 = 16n^8 + 64n^7m + 128n^6m^2 + 64n^5m^3 + 16n^4m^4$$
$$c^4 = m^8 + 8m^7n + 32m^6n^2 + 80m^5n^3 + 136m^4n^4 + 160m^3n^5 + 128m^2n^6 + 64mn^7 + 16n^8$$
$$a^4 + b^4 = m^8 + 8m^7n + 32m^6n^2 + 32m^5n^3 + 17m^4n^4 + 64m^3n^5 + 128m^2n^6 + 64mn^7 + 16n^8$$

$\gamma = 2m^2, \beta = n^2$:

$$a^4 = 16m^8 + 64m^7n + 128m^6n^2 + 64m^5n^3 + 16m^4n^4$$
$$b^4 = n^8 + 8n^7m + 32n^6m^2 + 32n^5m^3 + 16n^4m^4$$
$$c^4 = 16m^8 + 64m^7n + 128m^6n^2 + 160m^5n^3 + 136m^4n^4 + 80m^3n^5 + 32m^2n^6 + 8mn^7 + n^8$$
$$a^4 + b^4 = 16m^8 + 64m^7n + 128m^6n^2 + 64m^5n^3 + 32m^4n^4 + 32m^3n^5 + 32m^2n^6 + 8mn^7 + n^8$$

We see that $a^3 + b^3 \neq c^3$ and $a^4 + b^4 \neq c^4$ in both sets. The pattern indicates that the terms around the middle of the expansions for $a^t + b^t$ will always differ with those of c^t for all $t > 2$.

Theorem 5. *If (a, b, c) is a Pythagorean triple and $t \in \mathbb{N},\ t = 1, 2, 3, \dots$ then $a^t + b^t \neq c^t\ \forall\ t > 2$.*

Proof. By Theorem 1 and the Binomial Theorem we have

$$a^t + b^t = (\gamma + 2k)^t + (\beta + 2k)^t \text{ and } c^t = [(\gamma + \beta) + 2k]^t$$
$$\sum_{i=0}^{t} \binom{t}{i} (\gamma)^{(t-i)} (2k)^i + \sum_{i=0}^{t} \binom{t}{i} (\beta)^{(t-i)} (2k)^i \neq \sum_{i=0}^{t} \binom{t}{i} (\gamma + \beta)^{(t-i)} (2k)^i$$

Thus by Theorem 3

$\gamma = m^2,\ \beta = 2n^2;$

$$\sum_{i=0}^{t} \binom{t}{i} (m^2)^{(t-i)} (2mn)^i + \sum_{i=0}^{t} \binom{t}{i} (2n^2)^{(t-i)} (2mn)^i \neq \sum_{i=0}^{t} \binom{t}{i} (m^2 + 2n^2)^{(t-i)} (2mn)^i$$

$\gamma = 2m^2,\ \beta = n^2;$

$$\sum_{i=0}^{t} \binom{t}{i} (2m^2)^{(t-i)} (2mn)^i + \sum_{i=0}^{t} \binom{t}{i} (n^2)^{(t-i)} (2mn)^i \neq \sum_{i=0}^{t} \binom{t}{i} (2m^2 + n^2)^{(t-i)} (2mn)^i$$

where $i \in \mathbb{Z}$ and $k, m, n \in \mathbb{N},\ k = mn,\ m = 1, 2, 3, \ldots\ ,\ n = 1, 2, 3, \ldots$ □

Corollary 5.1. *If (a, b, c) is a Pythagorean triple and $p, q, x, y, z \in \mathbb{N}$ where $p > 2$ is a prime number and $q = 1, 2, 3, \ldots$ then $x^p + y^p \neq z^p$ for $x = a^q,\ y = b^q,\ z = c^q$.*

Proof. If (a, b, c) is a Pythagorean triple and $t \in \mathbb{N}$ where $t = pq,\ p > 2$ is a prime and $q = 1, 2, 3, \ldots$ then by Theorem 5 we have $a^{pq} + b^{pq} \neq c^{pq}$ which is also $(a^q)^p + (b^q)^p \neq (c^q)^p$. Therefore $x^p + y^p \neq z^p$ for $x = a^q,\ y = b^q,\ z = c^q$. □

4.2. **Sum of two like powers where $t > 2$.**

Theorem 6. *If $x, y, z \in \mathbb{R},\ t \in \mathbb{N}$ and $x < y < z,\ t = 1, 2, 3, \ldots$ then there are no solutions to $x^t + y^t = z^t,\ t > 2$ where x, y, z are positive integers.*

Proof. Assume that there are integers x, y, z as solutions. Let $\alpha, \beta, \gamma \in \mathbb{Z}$ and $\alpha = y - x,\ \beta = z - x,\ \gamma = z - y$ then we get $\beta = \alpha + \gamma$ and $(x, y, z) = (x, x + \alpha, x + \beta)$.

We now have $x^t + (x + \alpha)^t = (x + \beta)^t$ which by the Binomial Theorem expands to

$$x^t + x^t + \binom{t}{1} \alpha x^{t-1} + \binom{t}{2} \alpha^2 x^{t-2} + \cdots + \binom{t}{t-1} \alpha^{t-1} x + \alpha^t = x^t + \binom{t}{1} \beta x^{t-1} + \binom{t}{2} \beta^2 x^{t-2} + \cdots + \binom{t}{t-1} \beta^{t-1} x + \beta^t$$

and can be rearranged to

$$x^t - \binom{t}{1} (\beta - \alpha) x^{t-1} - \binom{t}{2} \left(\beta^2 - \alpha^2\right) x^{t-2} - \cdots - \binom{t}{t-1} \left(\beta^{t-1} - \alpha^{t-1}\right) x - \left(\beta^t - \alpha^t\right) = 0 \qquad (1)$$

If it has at least one positive integral root then $x + \alpha$ and $x + \beta$ are also positive integers and we have a solution.

Observe that equation 1 is monic and all coefficient are integers hence all rational roots must be integral and a factor of $-(\beta^t - \alpha^t)$. We also see that there is only one change of signs which by Descartes' Rule of Signs [3][4] indicates one positive real root or none. Therefore we try to determine if this positive real root is an integer.

Now let $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ be a polynomial where all coefficients are integers and $a_n = 1$ and let d be an integral root then $d^n + a_{n-1} d^{n-1} + \ldots + a_0$. Since d divides all terms preceding the last term then it must also divide a_0. See [3]. This can be used to check whether a monic polynomial with integral coefficients have an integral root.

Looking at equation 1, we see that the only possible roots that divides all the coefficients after the first term and preceding the last term are ± 1 and $\pm(\beta - \alpha)$. But when substituted, the left hand side does not evaluate to zero. This shows that there are no positive or negative integral roots and implies that if there are real roots and if it is neither integral nor rational then it must be irrational.

By assuming there are integers x, y, z as solutions, we are led to equation 1 which was shown to have no integral roots. We have a contradiction thus the assumption is false and the theorem is true. □

Corollary 6.1. *If $x^t + y^t = z^t,\ t > 2$ and $x < y < z$ such that x, y, z^t are positive integers then z is always irrational.*

Proof. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and x, y be positive integers then α is also a positive integer. Theorem 5 implies that at least one of (x, y, z) is irrational and that would be z in this case. Hence $\beta = z - x$ is irrational. Therefore z will always be irrational because $(x, y, z) = (x, x + \alpha, x + \beta)$. □

5. Screenshots of C++ script table output

```
File  Edit  View  Terminal  Go  Help
********************************************************************************
                 A C++ script to generate Pythagorean triples
      Eduardo Calooy Roque / eddieboyroque@gmail.com / November 17, 2012
********************************************************************************
Set 1: m = 77, n > 54, Gamma = 5929
--------------------------------------------------------------------------------
   n |      a ,      b ,      c |      Remarks
--------------------------------------------------------------------------------
7000 |
7001 | 1084083 , 99106156 , 99112085 | Pythagorean , Primitive
7002 | 1084237 , 99134316 , 99140245 | Pythagorean , Primitive
7003 | 1084391 , 99162480 , 99168409 | Pythagorean , Primitive
7004 | 1084545 , 99190648 , 99196577 | Pythagorean , Primitive
7005 | 1084699 , 99218820 , 99224749 | Pythagorean , Primitive
7006 | 1084853 , 99246996 , 99252925 | Pythagorean , Primitive
7007 |
7008 | 1085161 , 99303360 , 99309289 | Pythagorean , Primitive
7009 | 1085315 , 99331548 , 99337477 | Pythagorean , Primitive
7010 | 1085469 , 99359740 , 99365669 | Pythagorean , Primitive
7011 | 1085623 , 99387936 , 99393865 | Pythagorean , Primitive
7012 | 1085777 , 99416136 , 99422065 | Pythagorean , Primitive
7013 | 1085931 , 99444340 , 99450269 | Pythagorean , Primitive
7014 |
7015 | 1086239 , 99500760 , 99506689 | Pythagorean , Primitive
7016 | 1086393 , 99528976 , 99534905 | Pythagorean , Primitive
7017 | 1086547 , 99557196 , 99563125 | Pythagorean , Primitive
7018 |
7019 | 1086855 , 99613648 , 99619577 | Pythagorean , Primitive
7020 | 1087009 , 99641880 , 99647809 | Pythagorean , Primitive
7021 |
7022 | 1087317 , 99698356 , 99704285 | Pythagorean , Primitive
7023 | 1087471 , 99726600 , 99732529 | Pythagorean , Primitive
7024 | 1087625 , 99754848 , 99760777 | Pythagorean , Primitive
7025 | 1087779 , 99783100 , 99789029 | Pythagorean , Primitive
7026 | 1087933 , 99811356 , 99817285 | Pythagorean , Primitive
7027 | 1088087 , 99839616 , 99845545 | Pythagorean , Primitive
7028 |
7029 |
Generate another table?(y/n):
```

Figure 1. Set 1 : $\gamma = m^2, \ \beta = 2n^2$

```
File  Edit  View  Terminal  Go  Help
******************************************************************************
                  A C++ script to generate Pythagorean triples
      Eduardo Calooy Roque / eddieboyroque@gmail.com / November 17, 2012
******************************************************************************
Set 2: m = 77, n > 108, Gamma = 11858
------------------------------------------------------------------------------
   n |      a ,      b ,       c |       Remarks
------------------------------------------------------------------------------
7000 |
7001 | 1090012 , 50092155 , 50104013 | Pythagorean , Primitive
7002 |
7003 | 1090320 , 50120471 , 50132329 | Pythagorean , Primitive
7004 |
7005 | 1090628 , 50148795 , 50160653 | Pythagorean , Primitive
7006 |
7007 |
7008 |
7009 | 1091244 , 50205467 , 50217325 | Pythagorean , Primitive
7010 |
7011 | 1091552 , 50233815 , 50245673 | Pythagorean , Primitive
7012 |
7013 | 1091860 , 50262171 , 50274029 | Pythagorean , Primitive
7014 |
7015 | 1092168 , 50290535 , 50302393 | Pythagorean , Primitive
7016 |
7017 | 1092476 , 50318907 , 50330765 | Pythagorean , Primitive
7018 |
7019 | 1092784 , 50347287 , 50359145 | Pythagorean , Primitive
7020 |
7021 |
7022 |
7023 | 1093400 , 50404071 , 50415929 | Pythagorean , Primitive
7024 |
7025 | 1093708 , 50432475 , 50444333 | Pythagorean , Primitive
7026 |
7027 | 1094016 , 50460887 , 50472745 | Pythagorean , Primitive
7028 |
7029 |
Generate another table?(y/n):
```

FIGURE 2. Set 2 : $\gamma = 2m^2,\ \beta = n^2$

Implementation with GMP library:

```
File  Edit  View  Terminal  Go  Help
--------------------------------------------------------------------------------
                    A C++ script to generate Pythagorean triples
       Eduardo Calooy Roque / eddieboyroque@gmail.com / November 17, 2012
--------------------------------------------------------------------------------
               set 1: gamma = c - b = 982007569, m = 31337
--------------------------------------------------------------------------------
  n     |                (a,b,c)                      |       Remarks
--------------------------------------------------------------------------------
1000000 | 63656007569 , 2062674000000 , 2063656007569 | Pythagorean , Primitive
1000001 | 63656070243 , 2062678062676 , 2063660070245 | Pythagorean , Primitive
1000002 | 63656132917 , 2062682125356 , 2063664132925 | Pythagorean , Primitive
1000003 | 63656195591 , 2062686188040 , 2063668195609 | Pythagorean , Primitive
1000004 | 63656258265 , 2062690250728 , 2063672258297 | Pythagorean , Primitive
1000005 | 63656320939 , 2062694313420 , 2063676320989 | Pythagorean , Primitive
1000006 | 63656383613 , 2062698376116 , 2063680383685 | Pythagorean , Primitive
1000007 | 63656446287 , 2062702438816 , 2063684446385 | Pythagorean , Primitive
1000008 | 63656508961 , 2062706501520 , 2063688509089 | Pythagorean , Primitive
1000009 | 63656571635 , 2062710564228 , 2063692571797 | Pythagorean , Primitive
1000010 | 63656634309 , 2062714626940 , 2063696634509 | Pythagorean , Primitive
1000011 | 63656696983 , 2062718689656 , 2063700697225 | Pythagorean , Primitive
1000012 | 63656759657 , 2062722752376 , 2063704759945 | Pythagorean , Primitive
1000013 | 63656822331 , 2062726815100 , 2063708822669 | Pythagorean , Primitive
1000014 | 63656885005 , 2062730877828 , 2063712885397 | Pythagorean , Primitive
1000015 | 63656947679 , 2062734940560 , 2063716948129 | Pythagorean , Primitive
1000016 | 63657010353 , 2062739003296 , 2063721010865 | Pythagorean , Primitive
1000017 | 63657073027 , 2062743066036 , 2063725073605 | Pythagorean , Primitive
1000018 | 63657135701 , 2062747128780 , 2063729136349 | Pythagorean , Primitive
1000019 | 63657198375 , 2062751191528 , 2063733199097 | Pythagorean , Primitive
1000020 | 63657261049 , 2062755254280 , 2063737261849 | Pythagorean , Primitive
1000021 | 63657323723 , 2062759317036 , 2063741324605 | Pythagorean , Primitive
1000022 | 63657386397 , 2062763379796 , 2063745387365 | Pythagorean , Primitive
1000023 | 63657449071 , 2062767442560 , 2063749450129 | Pythagorean , Primitive
1000024 | 63657511745 , 2062771505328 , 2063753512897 | Pythagorean , Primitive
1000025 | 63657574419 , 2062775568100 , 2063757575669 | Pythagorean , Primitive
1000026 | 63657637093 , 2062779630876 , 2063761638445 | Pythagorean , Primitive
1000027 | 63657699767 , 2062783693656 , 2063765701225 | Pythagorean , Primitive
1000028 | 63657762441 , 2062787756440 , 2063769764009 | Pythagorean , Primitive
1000029 | 63657825115 , 2062791819228 , 2063773826797 | Pythagorean , Primitive
Generate another table?(y/n):
```

FIGURE 3. Set 1 : $\gamma = m^2,\ \beta = 2n^2$

6. Conclusion

It is evident from the general formulas that Pythagorean triples are infinite and can be grouped into two infinite sets of infinite sets ordered by $c - b$.

As a supplement, it was shown that they could be used to prove that $a^t + b^t \neq c^t$ for all $t \geqslant 3$ if (a, b, c) is a Pythagorean triple. Proofs for cases $t = \{3, 4\}$ were shown indicating that higher values for t is also valid. The reason is that the terms around the middle of the binomial expansions will always differ for $t \geqslant 3$.

A C++ script that can be run in Cling and ROOT CINT C/C++ interpreters is attached here instead of typesetting it verbatim. It is just a simple interactive command line interface program. If desired, it can be compiled by executing "g++ ptgko.cpp -o ptgko -DSTANDALONE" at the terminal console of any Linux distribution.

If big numbers are desired, the GMP - GNU Multi Precision Library can be utilized to also find large Pythagorean triples. Here is the script: . It can be compiled by executing "g++ ptgkogmp.cpp -o ptgkogmp -lgmpxx -lgmp -DSTANDALONE".

I also attached tables for primitive Pythagorean triples ordered by $c - b = \{1, 2, 2(2)^2, 3^2, 77^2, 2(77^2), 31337^2\}$ with n ranging from 1 to 100 thousand here:

I hope this humble work will in some way benefit fellowmen.

$- - - \sim\sim \Theta \sim\sim - - -$

Proverbs 3:13

Happy is the man that finds wisdom, and the man that gets understanding.

Proverbs 9:10

The fear of the Lord is the beginning of knowledge: and the knowledge of the Holy One is understanding.

$- - - \sim\sim \Theta \sim\sim - - -$

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