A new method to express sums of power of integers as a polynomial equation

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ABSTRACT

This is a new method to express The power sums of (n^m) as a (m+1)th-degree polynomial equation of n, there is some methods like "Faulhaber's formula" and others gives the solution of the power sums. nevertheless my method is doing the same but it does not depend on Bernoulli numbers or integrals as it is a simple algebraic way giving a polynomial equation and proved in a elementary algebraic logic.

INTRODUCTION

It is clear that summation of (n^m) is not simple to calculate for large number of n, so it is necessary to find a way to calculate the sum in a simple elementary formula.

The following summation

$$S_m(n) = \sum_{k=1}^{n} k^m \qquad m \in N \tag{1}$$

Is provided that m & n are always a positive natural number so that my formula can give a correct results.

THE FORMULA

The following equations are the concluded formulas from the proof , which is used together to carry out the elementary polynomial equation of n,

they are 3 formulas as below:

$$S_m(n) = \sum_{k=1}^{m+1} a_k n^k \qquad (the\ main\ polynimial\ equation) \tag{2}$$

$$a_{m+1} = \frac{1}{m+1} \qquad (\ n^{m+1}\ \ parameter) \tag{3}$$

$$a_k = \frac{1}{k!} * \sum_{j=1}^{m+1-k} (-1)^{j+1} \frac{a_{k+j}(k+j)!}{j+1!} \qquad (0 < k < m+1) \quad (all\ other\ parameters\ ) \tag{4}$$

Which can also expressed as one formula:

$$S_m(n) = \frac{n^{m+1}}{m+1} + \sum_{k=m}^{1} \left(\left(\frac{1}{k!} * \sum_{j=1}^{m+1-k} (-1)^{j+1} \frac{a_{k+j}(k+j)!}{j+1!} \right) n^k \right) \quad , \quad a_{m+1} = \frac{1}{m+1} \tag{5}$$

PROOF OF THE FORMULA

The sum of (n^m) and ($(n\text{-}1)^m$) are

$$S_m(n) = \sum_{k=1}^{n} k^m = k^1 + k^2 + k^3 + \cdots + k^{n-1} + k^n \tag{6}$$

$$S_m(n-1) = \sum_{k=1}^{n-1} k^m = k^1 + k^2 + k^3 + \cdots + k^{n-1} \tag{7}$$

So,

$$S_m(n) - S_m(n-1) = n^m \tag{8}$$

As a matter of fact, while $m \in N$ the obtained equation is always brought out as a polynomial equation of higher degree (m+1) as below:

$$S_m(n) = a_1 n + a_2 n^2 + a_3 n^3 + \cdots + a_m n^m + a_{m+1} n^{m+1} \tag{9}$$

Considering that $a_1, a_2, a_3, \dots, a_m, a_{m+1}$ are values.

So, the method is to find the values of equation No.(9)'s parameters $a_1, a_2, a_3, \dots, a_m, a_{m+1}$ in the form of $a_k = f_k(m) \qquad (k \in N \quad and \quad 0 < k \le m+1).$ (10)

That is what I will achieve by proof in the following steps :

$$S_m(n) = \sum_{k=1}^{m+1} a_k n^k \tag{11}$$

$$S_m(n-1) = \sum_{k=1}^{m+1} a_k (n-1)^k \tag{12}$$

$$S_m(n) - S_m(n-1) = \sum_{k=1}^{m+1} a_k n^k - \sum_{k=1}^{m+1} a_k (n-1)^k = n^m \tag{13}$$

after resolving $(n-1)^k$ by binomial theorem the result equation must be a polynomial equation as the following:

$$S_m(n) - S_m(n-1) = b_1 n + b_2 n^2 + b_3 n^3 + \cdots + b_m n^m + b_{m+1} n^{m+1} = n^m \tag{14}$$

$$b_k = f(m, a_1, a_2, a_3, \dots, a_m, a_{m+1}) \quad (0 < k \leq m+1) \tag{15}$$

all b_k values will equal to zero accept $b_m = 1$ to have the result equaling the other side of the equation n^m.because $a_1, a_2, a_3, \dots, a_m, a_{m+1}$ are not variables, they are in the form of $f(m)$ and "m" should be a known natural number.

Now the following equation will be resolved to get $a_1, a_2, a_3, \dots, a_m, a_{m+1}$ values.

$$\sum_{k=1}^{m+1} a_k n^k - \sum_{k=1}^{m+1} a_k (n-1)^k = n^m \tag{16}$$

$$\sum_{k=1}^{m+1} a_k n^k = a_1 n + a_2 n^2 + a_3 n^3 + \cdots + a_m n^m + a_{m+1} n^{m+1} \tag{17}$$

$$\sum_{k=1}^{m+1} a_k (n-1)^k = a_1(n-1) + a_2(n-1)^2 + a_3(n-1)^3 + \cdots + a_m(n-1)^m + a_{m+1}(n-1)^{m+1} \tag{18}$$

$$\begin{aligned}\sum_{k=1}^{m+1} a_k (n-1)^k = {} & a_1(n-1) + a_2(n^2 - 2n + 1) + a_3(n^3 - 3n^2 + 3n - 1) + \cdots \\ & + a_m\left(n^m - mn^{m-1} + \frac{m(m-1)n^{m-2}}{2} - \cdots\right) + a_{m+1}\left(n^{m+1} - (m+1)n^m \right.\\ & \left. + \frac{(m+1)mn^{m-1}}{2} - \cdots\right)\end{aligned} \tag{19}$$

$$\begin{aligned}\sum_{k=1}^{m+1} a_k n^k - \sum_{k=1}^{m+1} a_k (n-1)^k \\ = {} & (a_{m+1} - a_{m+1})n^{m+1} + (a_m - (a_m - a_{m+1}(m+1)))n^m \\ & + \left(a_{m-1} - \left(a_{m-1} - a_m m + \frac{a_{m+1}(m+1)m}{2}\right)\right)n^{m-1} + \cdots\end{aligned} \tag{20}$$

b_k will be the summation of all parameters of the same power "k" of "n" $\quad (0 < k \leq m+1)$

Also $b_k = 0\; when\;\; k \neq m$ and $b_k = 1\; when\;\; k = m$

See equation No. (14)

So, $b_{m+1} = a_{m+1} - a_{m+1} = 0$ (21)

$$b_m = a_m - (a_m - a_{m+1}(m+1)) = 1 \tag{22}$$

$$a_{m+1}(m+1) = 1 \;\; \rightarrow \;\; a_{m+1} = \frac{1}{m+1} \quad (m \neq -1) \tag{23}$$

$$b_{m-1} = a_{m-1} - \left(a_{m-1} - a_m m + \frac{a_{m+1}(m+1)m}{2}\right) = 0 \tag{24}$$

$$a_m m - \frac{a_{m+1}(m+1)m}{2} = 0 \tag{25}$$

$$a_m = \frac{a_{m+1}(m+1)m}{2m} = \frac{\frac{1}{m+1}(m+1)}{2} = \frac{1}{2} \quad (m \neq 0, m \neq -1) \tag{26}$$

$$b_{m-2} = a_{m-2} - \left(a_{m-2} - a_{m-1}(m-1) + \frac{a_m m(m-1)}{2} - \frac{a_{m+1}(m+1)m(m-1)}{6}\right) = 0 \tag{27}$$

$$a_{m-1}(m-1) - \frac{a_m m(m-1)}{2} + \frac{a_{m+1}(m+1)m(m-1)}{6} = 0 \tag{28}$$

$$a_{m-1} - \frac{a_m m}{2} + \frac{a_{m+1}(m+1)m}{6} = 0 \quad (m \neq 1) \tag{29}$$

$$a_{m-1} - \frac{\frac{1}{2}m}{2} + \frac{\frac{1}{m+1}(m+1)m}{6} = 0 \qquad (m \neq -1,0,1) \tag{30}$$

$$a_{m-1} - \frac{m}{4} + \frac{m}{6} = 0 \;\rightarrow a_{m-1} = \frac{m}{4} - \frac{m}{6} = \frac{m}{12} \quad (m \neq -1,0,1) \tag{31}$$

$$\begin{aligned} b_{m-3} = a_{m-3} - \Bigg(a_{m-3} - a_{m-2}(m-2) + \frac{a_{m-1}(m-1)(m-2)}{2} - \frac{a_m m(m-1)(m-2)}{6} \\ + \frac{a_{m+1}(m+1)m(m-1)(m-2)}{24}\Bigg) = 0 \end{aligned} \tag{32}$$

$$\begin{aligned} b_{m-3} = a_{m-2}(m-2) - \frac{a_{m-1}(m-1)(m-2)}{2} + \frac{a_m m(m-1)(m-2)}{6} \\ - \frac{a_{m+1}(m+1)m(m-1)(m-2)}{24} = 0 \end{aligned} \tag{33}$$

$$a_{m-2} - \frac{a_{m-1}(m-1)}{2} + \frac{a_m m(m-1)}{6} - \frac{a_{m+1}(m+1)m(m-1)}{24} = 0 \quad (m \neq 2) \tag{34}$$

$$a_{m-2} - \frac{\frac{m}{12}(m-1)}{2} + \frac{\frac{1}{2}m(m-1)}{6} - \frac{\frac{1}{m+1}(m+1)m(m-1)}{24} = 0 \qquad (m \neq -1,0,1,2) \qquad (35)$$

$$a_{m-2} - \frac{m(m-1)}{24} + \frac{m(m-1)}{12} - \frac{m(m-1)}{24} = 0 \qquad (m \neq -1,0,1,2) \qquad (36)$$

$$a_{m-2} - 0 = 0 \quad \rightarrow a_{m-2} = 0 \qquad (m \neq -1,0,1,2) \qquad (37)$$

I have got the values of $a_{m+1}, a_m, a_{m-1}, a_{m-2}$, as below:

$$a_{m+1} = \frac{1}{m+1} \qquad (m \neq -1) \qquad (38)$$

$$a_m = \frac{1}{2} \qquad (m \neq -1,0) \qquad (39)$$

$$a_{m-1} = \frac{m}{12} \qquad (m \neq -1,0,1) \qquad (40)$$

$$a_{m-2} = 0 \qquad (m \neq -1,0,1,2) \qquad (41)$$

To obtain a general equation of all a_k $(0 < k < m+1)$

from equation No.(20) I get.

$$\begin{aligned} b_{m-z} = a_{m-z} - \Bigg(& a_{m-z} - a_{m-z+1}(m-z+1) + \frac{a_{m-z+2}(m-z+2)(m-z+1)}{2!} \\ & - \frac{a_{m-z+3}(m-z+3)(m-z+2)(m-z+1)}{3!} + \cdots \\ & + \frac{a_{m+1}(m+1)m(m-1)(m-2)\ldots(m-z+2)(m-z+1)}{(z+1)!}\Bigg) = 0 \ , \\ & (0 < z < m) \end{aligned} \qquad (42)$$

So,

$$\begin{aligned} b_{m-z} = -a_{m-z+1}(m-z+1) & + \frac{a_{m-z+2}(m-z+2)(m-z+1)}{2!} \\ & - \frac{a_{m-z+3}(m-z+3)(m-z+2)(m-z+1)}{3!} + \cdots \\ & + \frac{a_{m+1}(m+1)m(m-1)(m-2)\ldots(m-z+2)(m-z+1)}{(z+1)!} = 0 \ , \\ & (0 < z < m) \end{aligned} \qquad (43)$$

Then, I can say that:

$$b_{m-z} = \sum_{s=1}^{z+1} (-1)^s \frac{a_{m-z+s}(m-z+s)!}{s!\,(m-z)!} = 0 \qquad (0 < z < m) \qquad (44)$$

I do not need b_k values because they are already known but I search for a_k values.

So, let $k-1 = m-z$

To replace z by k and simplify the formula.

$$b_{k-1} = \sum_{s=1}^{m+2-k} (-1)^s \frac{a_{k-1+s}(k-1+s)!}{s!\,(k-1)!} = 0 \qquad (0 < k < m+1) \tag{45}$$

$$b_{k-1} = -\frac{a_k k!}{k-1!} + \sum_{s=2}^{m+2-k} (-1)^s \frac{a_{k-1+s}(k-1+s)!}{s!\,(k-1)!} = 0 \qquad (0 < k < m+1) \tag{46}$$

$$b_{k-1} = -a_k k + \sum_{s=2}^{m+2-k} (-1)^s \frac{a_{k-1+s}(k-1+s)!}{s!\,(k-1)!} = 0 \qquad (0 < k < m+1) \tag{47}$$

By canceling (b_{k-1}) , the equation will be.

$$a_k k = \sum_{s=2}^{m+2-k} (-1)^s \frac{a_{k-1+s}(k-1+s)!}{s!\,(k-1)!} \qquad (0 < k < m+1) \tag{48}$$

$$a_k = \frac{1}{k} * \sum_{s=2}^{m+2-k} (-1)^s \frac{a_{k-1+s}(k-1+s)!}{s!\,(k-1)!} \qquad (0 < k < m+1) \tag{49}$$

$$a_k = \frac{1}{k!} * \sum_{s=2}^{m+2-k} (-1)^s \frac{a_{k-1+s}(k-1+s)!}{s!} \qquad (0 < k < m+1) \tag{50}$$

To simplify the equation also,

Let $s = j+1 \quad \rightarrow \quad j = s-1$

$$a_k = \frac{1}{k!} * \sum_{j=1}^{m+1-k} (-1)^{j+1} \frac{a_{k+j}(k+j)!}{j+1!} \qquad (0 < k < m+1) \tag{51}$$

RESULTS

finally here is the formula for all a_k as below.

$$a_k = \frac{1}{k!} * \sum_{j=1}^{m+1-k} (-1)^{j+1} \frac{a_{k+j}(k+j)!}{j+1!} \qquad (0 < k < m+1) \tag{52}$$

$$a_k = \frac{1}{m+1} \qquad (k = m+1) \tag{53}$$

$$S_m(n) = \sum_{k=1}^{m+1} a_k n^k \tag{54}$$

If we join the above formulas we obtain one formula as below:

$$S_m(n) = \frac{n^{m+1}}{m+1} + \sum_{k=m}^{1} \left(\left(\frac{1}{k!} * \sum_{j=1}^{m+1-k} (-1)^{j+1} \frac{a_{k+j}(k+j)!}{j+1!} \right) n^k \right) \quad , \quad a_{m+1} = \frac{1}{m+1} \tag{55}$$

CONCLUSION

The resulted equations are the concluded formulas which should express the power summation of (n^m) as a simple polynomial equation of (n^{m+1})th degree provided that *(m & n)* are always a natural numbers.

Here is an example to explain how the method works and how to get parameters values.

it also shows that these parameters give the values of each other one by one.

Example:

$$S_3(n) = \sum_{k=1}^{n} k^3$$

Solution,

$m = 3$

$$S_3(n) = \sum_{k=1}^{4} a_k n^k = a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4$$

$$a_4 = \frac{1}{3+1} = \frac{1}{4}$$

$$a_k = \frac{1}{k!} * \sum_{j=1}^{4-k} (-1)^{j+1} \frac{a_{k+j}(k+j)!}{j+1!} \qquad 0 < k < 4$$

$$a_3 = \frac{1}{3!} * \sum_{j=1}^{1} (-1)^{j+1} \frac{a_{3+j}(3+j)!}{j+1!} = \frac{1}{3!} * \frac{a_4 * 4!}{2!} = \frac{\frac{1}{4} * 4!}{3! * 2!} = \frac{3!}{3! * 2!} = \frac{1}{2}$$

$$a_2 = \frac{1}{2!} * \sum_{j=1}^{2} (-1)^{j+1} \frac{a_{2+j}(2+j)!}{j+1!} = \frac{1}{2!}\left(\frac{a_3(3)!}{2!} - \frac{a_4(4)!}{3!}\right) = \frac{1}{2!}(3a_3 - 4a_4)$$
$$= \frac{1}{2}\left(3 * \frac{1}{2} - 4 * \frac{1}{4}\right) = \frac{1}{4}$$

$$a_1 = \frac{1}{1!} * \sum_{j=1}^{3} (-1)^{j+1} \frac{a_{1+j}(1+j)!}{j+1!} = \sum_{j=1}^{3} (-1)^{j+1} a_{1+j} = a_2 - a_3 + a_4 = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

$$S_3(n) = 0 * n + \frac{1}{4} * n^2 + \frac{1}{2} * n^3 + \frac{1}{4} * n^4 = \frac{n^2 + 2n^3 + n^4}{4}$$

Another solution,

$m = 3$

$$a_{m+1} = \frac{1}{m+1} \qquad m \neq -1 \qquad \rightarrow \qquad a_4 = \frac{1}{3+1} = \frac{1}{4}$$

$$a_m = \frac{1}{2} \qquad m \neq -1,0 \qquad \rightarrow \qquad a_3 = \frac{1}{2}$$

$$a_{m-1} = \frac{m}{12} \qquad m \neq -1,0,1 \qquad \rightarrow \qquad a_2 = \frac{3}{12} = \frac{1}{4}$$

$$a_{m-2} = 0 \qquad m \neq -1,0,1,2 \qquad \rightarrow \qquad a_1 = 0$$

$$S_3(n) = a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4$$

$$S_3(n) = 0 * n + \frac{1}{4} * n^2 + \frac{1}{2} * n^3 + \frac{1}{4} * n^4 = \frac{n^2 + 2n^3 + n^4}{4}$$

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