Exponential-geometric mean

Farrukh Ataev Rakhimjanovich
November 2012

Abstract: A new concept of exponential-geometric mean is introduced and its properties are analyzed.

The concepts and properties of means of a set of numbers are well studied in calculus. In [1], a mean μ of a set of numbers $x_i,\ i=1,2,...,n$ is defined as the value that satisfies the condition:

$$\min(x_1,x_2,...,x_n) \le \mu \le \max(x_1,x_2,...,x_n).$$

In this short note I introduce a new so-called exponential-geometric mean and give some of its properties.

Definition: A lower (upper) exponential-geometric mean μ of two positive numbers a and b is:

$$\mu_0 = \sqrt[a+b]{a^b b^a} \quad \left(\mu_1 = \sqrt[a+b]{a^a b^b}\right).$$

We can see that the exponential-geometric mean μ of any two positive numbers a and b conforms to the general definition of the mean. Indeed, without loss of generality, let's assume $a \le b$, then:

$$a \le \sqrt[a+b]{a^b b^a} \le b \Rightarrow a^{a+b} \le a^b b^a \le b^{a+b} \Rightarrow \begin{cases} a^a \le b^a \\ a^b \le b^b \end{cases}$$

$$\left(a \le \sqrt[a+b]{a^a b^b} \le b \Rightarrow a^{a+b} \le a^a b^b \le b^{a+b} \Rightarrow \begin{cases} a^a \le b^b \\ a^a \le b^a \end{cases} \right).$$

Note that the other two combinations $\sqrt[a+b]{a^a b^a}$ and $\sqrt[a+b]{a^b b^b}$ may not suit the general definition of the mean.

Examples: The lower and upper exponential-geometric means of numbers

1) 2 and 3 are $\mu = \sqrt[2+3]{2^3 \cdot 3^2} \approx 2.35$ and $\mu = \sqrt[2+3]{2^2 \cdot 3^3} \approx 2.55$.
2) 2 and 2.5 are $\mu = \sqrt[2+2.5]{2^{2.5} \cdot 2.5^2} \approx 2.21$ and $\mu = \sqrt[2+2.5]{2^2 \cdot 2.5^{2.5}} \approx 2.26$.
3) 0.5 and 0.7 are $\mu = \sqrt[0.5+0.7]{0.5^{0.7} \cdot 0.7^{0.5}} \approx 0.58$ and $\mu = \sqrt[0.5+0.7]{0.5^{0.5} \cdot 0.7^{0.7}} \approx 0.61$.

It is well known the following relationships between harmonic, geometric and arithmetic means:

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}.$$

Lemma 1: For the lower and upper exponential-geometric means of two positive numbers a and b the following holds true:

$$\sqrt[a+b]{a^b b^a} \le \frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt[a+b]{a^a b^b}.$$

Proof: It is sufficient to prove the leftmost inequality, since it is equivalent with the rightmost shown below.

$$\sqrt[a+b]{a^b b^a} \le \frac{2ab}{a+b} \Rightarrow a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} \le \frac{2ab}{a+b} \Rightarrow \frac{a+b}{2} \le a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}.$$

Now, without loss of generality, let us denote $b = ac,\ c \ge 1$. Then,

$$\frac{a+ac}{2} \le a^{\frac{a}{a+ac}} (ac)^{\frac{ac}{a+ac}} \Rightarrow \frac{a(1+c)}{2} \le ac^{\frac{c}{1+c}} \Rightarrow 1 \le \frac{2c^{\frac{c}{1+c}}}{1+c}.$$

For the function $y(x)=\frac{2x^{\frac{x}{1+x}}}{1+x}$, $y(1)=1$ and it is monotonously increasing at $x \geq 1$. Its limit for $x \to +\infty$ is $\lim_{x\to+\infty} \frac{2x^{\frac{x}{1+x}}}{1+x} = 2$. □

Lemma 2: For the lower exponential-geometric mean (LEGM), harmonic mean (HM), geometric mean (GM), arithmetic mean (AM) and upper exponential-geometric mean (UEGM) the following relations hold true:

1) $LEGM \cdot UEGM = AM \cdot HM = GM^2$;
2) $LEGM \cdot AM \leq HM \cdot UEGM$;
3) $UEGM - AM \geq HM - LEGM$;
4) $UEGM - LEGM \geq AM - HM$;
5) $LEGM + UEGM \geq GM$.

Proof: The 1st is rather straightforward:

$$LEGM \cdot UEGM = \sqrt[a+b]{a^b b^a} \cdot \sqrt[a+b]{a^a b^b} = ab = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = AM \cdot HM = \left(\sqrt{ab}\right)^2 = GM^2.$$

The 2nd is obvious since $LEGM \leq HM$ and $AM \leq UEGM$.
The 3rd and 4th can be proved by the method used in Lemma 1.
The 5th is also obvious. If we take into account the 1st relation, then

$$LEGM + UEGM \geq \sqrt{LEGM \cdot UEGM}. \ \square$$

The exponential-geometric means and their properties allow to estimate cumbersome and inconvenient expressions (and their limits). For example:

1) $\left(\sin^2\alpha\right)^{\sin^2\alpha} \cdot \left(\cos^2\alpha\right)^{\cos^2\alpha} \geq \frac{1}{2}$ *for* $\alpha \neq \frac{\pi k}{2}, k \in Z$.

2) $\sqrt[\tan x+\cot x]{(\tan x)^{\cot x} \cdot (\cot x)^{\tan x}} \leq 1$ *for* $x \in \left(\pi k,\ k \in Z; \frac{\pi}{2} + \pi k,\ k \in Z\right) \Rightarrow$

$$(\tan x)^{\cot x} (\cot x)^{\tan x} \leq 1 \ \textit{for} \ x \in \left(\pi k,\ k \in Z; \frac{\pi}{2} + \pi k,\ k \in Z\right).$$

3) $\sqrt[x+1]{x^x \cdot 1^1} \geq \frac{x+1}{2}$ *for* $x > 0 \Rightarrow x^{\frac{x}{x+1}} \geq \frac{x+1}{2}$ *for* $x > 0$.

4) $\sqrt[e^x+1]{\left(e^x\right)^1 \cdot 1^{e^x}} \leq \frac{2e^x}{e^x+1}$ *for* $x > 0 \Rightarrow \exp\left(\frac{x}{e^x+1}\right) \leq \frac{2e^x}{e^x+1}$ *for* $x > 0$.

5) $\sqrt[x^a+1]{\left(x^a\right)^1 \cdot 1^{x^a}} \leq \frac{x^a+1}{2}$ *for* $x, a > 0 \Rightarrow x \wedge \left(\frac{a}{x^a+1}\right) \leq \frac{x^a+1}{2}$ *for* $x, a > 0$.

Reference

[1] Eric Weisstein, "CRC Concise Encyclopedia of Mathematics", 2-edition, CRC Press LLC

About author: Farrukh Ataev R., Lecturer at Westminster International University in Tashkent, Uzbekistan, Tashkent, Istikbol Street 12.
Email: farruhota@gmail.com.