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A Note on Fractional Electrodynamics

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Abstract

We investigate the time evolution of the fractional electromagnetic waves by using the time fractional Maxwell's equations. We show that electromagnetic plane wave has amplitude which exhibits an algebraic decay, at asymptotically long times.

Keywords: Fractional Calculus; Fractional Electrodynamics; Fractional Maxwell's Equations

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1. Introduction

Fractional calculus is a very useful tool in describing the evolution of systems with memory, which typically are dissipative and to complex systems. In recent decades fractional calculus and in particular fractional differential equations have attracted interest of researches in several areas including mathematics, physics, chemistry, biology, engineering and economics. Applications of fractional calculus in the field of physics and astrophysics have gained considerable popularity and many important results were obtained during the last years [1-15]. In classical mechanics, as we can see in Ref. [7-9] the fractional formalism leads to relaxation and oscillation processes that exhibit memory and delay. This fractional nonlocal formalism is also applicable on materials and media that have electromagnetic memory properties. So the generalized fractional Maxwell's equations can give us new models that can be used in these complex systems. The aim of this work is to investigate the time evolution of the fractional electromagnetic waves by using the time fractional Maxwell's equations. In particular, we show that electromagnetic plane wave has an amplitude which exhibits an algebraic decay , at asymptotically large times. For this purpose in the following section we briefly review fractional electrodynamics theory [7, 19].

2. Fractional electrodynamics

In classical electrodynamics, behavior of electric fields ($\vec{E}$), magnetic fields ($\vec{B}$) and their relations to their sources, charge density ($\rho(\vec{r},t)$) and current density ($\vec{j}(\vec{r},t)$), are described by the following Maxwell's equations:

$$\vec{\nabla}.\vec{E} = \frac{4\pi}{\varepsilon}\rho(\vec{r},t) \tag{1}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{2}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{3}$$

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$$\vec{\nabla}\times\vec{B}=\frac{4\pi\mu}{c}\vec{j}(\vec{r},t)+\frac{\varepsilon\mu}{c}\frac{\partial\vec{E}}{\partial t} \tag{4}$$

Where ε and μ are electric permittivity and magnetic permeability, respectively. Now, introducing the potentials, vector $\vec{A}(x_i,t)$ and scalar $\varphi(x_i,t)$

$$\vec{B}=\vec{\nabla}\times\vec{A} \tag{5}$$

$$\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi \tag{6}$$

and using the Lorenz gauge condition we obtain the following decoupled differential equations for the potentials:

$$\Delta\vec{A}(\vec{r},t)-\frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}=-\frac{4\pi}{c}\vec{j}(\vec{r},t) \tag{7}$$

$$\Delta\varphi(\vec{r},t)-\frac{\varepsilon\mu}{c^2}\frac{\partial^2\varphi(\vec{r},t)}{\partial t^2}=-\frac{4\pi}{\varepsilon}\rho(\vec{r},t) \tag{8}$$

where $\frac{\varepsilon\mu}{c^2}=\frac{1}{v^2}$ and v is the velocity of the wave. Furthermore, for a particle motion with charge q in the presence of electric and magnetic field we can write the Lorentz force as

$$\vec{F}_L=q(\vec{E}+\vec{v}\times\vec{B}) \tag{9}$$

here, v is the particle's velocity. In terms of scalar and vector potentials, Eq. (5, 6) we may write the Lorentz force as

$$\vec{F}_L=q(-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi+\vec{v}\times(\vec{\nabla}\times\vec{A})) \tag{10}$$

As we can see in Ref. [7-9], in classical mechanics, the fractional formalism leads to relaxation and oscillation processes that exhibit memory and delay. This fractional nonlocal formalism is also applicable on materials and media that have electromagnetic memory properties. So the generalized fractional Maxwell's equations can give us new models that can be used in these complex systems. Up to now, several different kinds of fractional electrodynamics based on the different approaches to fractional vector calculus have been investigated [20-25]. For instance a fractional-dimensional space approach to the electrodynamics is presented [20] using the fractional Laplacian operator Δ_D [17, 18]:

$$\Delta_D=\frac{\partial^2}{\partial x^2}+\frac{\alpha_1-1}{x}\frac{\partial}{\partial x}+\frac{\partial^2}{\partial y^2}+\frac{\alpha_2-1}{y}\frac{\partial}{\partial y}+\frac{\partial^2}{\partial z^2}+\frac{\alpha_3-1}{z}\frac{\partial}{\partial z} \tag{11}$$

where, three parameters $(0<\alpha_1\le 1,\ \ 0<\alpha_2\le 1\ \ and\ \ 0<\alpha_3\le 1)$ are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $D=\alpha_1+\alpha_2+\alpha_3$.

However, in this paper we study a new approach on this area [19]. The idea is in fact, to write the ordinary differential wave equations in the fractional form with respect to t, by replacing the time derivative with a fractional derivative of order α $(0<\alpha\le 1)$ namely:

$$\vec{\nabla}.\vec{E}=\frac{4\pi}{\varepsilon}\rho(\vec{r},t) \tag{12}$$

$$\vec{\nabla}.\vec{B}=0 \tag{13}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{1}{\eta^{1-\alpha}}\frac{\partial^{\alpha}\vec{B}}{\partial t^{\alpha}} \tag{14}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi\mu}{c}\vec{j}(\vec{r},t) + \frac{\varepsilon\mu}{c}\frac{1}{\eta^{1-\alpha}}\frac{\partial^{\alpha}\vec{E}}{\partial t^{\alpha}} \tag{15}$$

And the Eq. (5, 6) become

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{16}$$

$$\vec{E} = -\frac{1}{c\eta^{1-\alpha}}\frac{\partial^{\alpha}\vec{A}}{\partial^{\alpha}t} - \vec{\nabla}\varphi \tag{17}$$

And the Lorentz force Eq. (10) becomes

$$\vec{F}_L = q\left(-\frac{1}{c\eta^{1-\alpha}}\frac{\partial^{\alpha}\vec{A}}{\partial^{\alpha}t} - \vec{\nabla}\varphi + \vec{v}\times(\vec{\nabla}\times\vec{A})\right) \tag{18}$$

A simple example of application of the Eq. (18) is provided in Appendix. In above equations the fractional derivative of order α , $n-1<\alpha<n$, $n\in N$ is defined in the Caputo sense:

$$\frac{\partial^{\alpha}f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n f(\tau)}{\partial\tau^n}d\tau \tag{19}$$

Where Γ denotes the Gamma function. For $\alpha=n$, $n\in N$ the Caputo fractional derivative is defined as the standard derivative of order n . Also, note that we have introduced an arbitrary quantity η with dimension of [second] to ensure that all quantities have correct dimensions. As we can see from Eq. (19) Caputo derivative describes a memory effect by means of a convolution between the integer order derivative and a power of time that corresponds to intrinsic dissipation in the system. Now we can apply Lorentz gauge condition to obtain the corresponding time fractional wave equations for the potentials

$$\Delta\vec{A}(\vec{r},t) - \frac{\varepsilon\mu}{c^2}\frac{1}{\eta^{2(1-\alpha)}}\frac{\partial^{2\alpha}\vec{A}(\vec{r},t)}{\partial t^{2\alpha}} = -\frac{4\pi}{c}\vec{j}(\vec{r},t) \tag{20}$$

$$\Delta\varphi(\vec{r},t) - \frac{\varepsilon\mu}{c^2}\frac{1}{\eta^{2(1-\alpha)}}\frac{\partial^{2\alpha}\varphi(\vec{r},t)}{\partial t^{2\alpha}} = -\frac{4\pi}{\varepsilon}\rho(\vec{r},t) \tag{21}$$

If $\rho=0$ and $\vec{j}=0$, we have the homogeneous fractional differential equations

$$\Delta\vec{A}(\vec{r},t) - \frac{\varepsilon\mu}{c^2}\frac{1}{\eta^{2(1-\alpha)}}\frac{\partial^{2\alpha}\vec{A}(\vec{r},t)}{\partial t^{2\alpha}} = 0 \tag{22}$$

$$\Delta\varphi(\vec{r},t) - \frac{\varepsilon\mu}{c^2}\frac{1}{\eta^{2(1-\alpha)}}\frac{\partial^{2\alpha}\varphi(\vec{r},t)}{\partial t^{2\alpha}} = 0 \tag{23}$$

We are interested in the analysis of the electromagnetic fields starting from the equations. Now we can write the fractional equations in following compact form

$$\frac{\partial^2 Z(x,t)}{\partial x^2} - \frac{\varepsilon\mu}{c^2}\frac{1}{\eta^{2(1-\alpha)}}\frac{\partial^{2\alpha}Z(x,t)}{\partial t^{2\alpha}} = 0 \tag{24}$$

where $Z(x,t)$ represents both $\vec{A}(\vec{r},t)$ and $\varphi(\vec{r},t)$. We consider a polarized electromagnetic wave, then $A_x=0, A_y\neq 0, A_z\neq 0$. A particular solution of this equation may be found in the form

$$Z(x,t) = Z_0 e^{-ikx} u(t) \tag{25}$$

where k is the wave vector in the x direction and Z_0 is a constant. Substituting Eq. (25) into Eq. (24) we obtain

$$\frac{d^{2\alpha} u(t)}{dt^{2\alpha}} + {\Omega_f}^2 u(t) = 0 \tag{26}$$

where

$${\Omega_f}^2 = v^2 k^2 \eta^{2(1-\alpha)} = \Omega^2 \eta^{2(1-\alpha)} \tag{27}$$

and Ω is the fundamental frequency of the electromagnetic wave. Using the Laplace integral transformations, one obtains the solutions:

$$u(t) = u(0) E_{2\alpha}(-{\Omega_f}^2 t^{2\alpha}) \tag{28}$$

for the case of $0 < \alpha < \frac{1}{2}$ and

$$u(t) = u(0) E_{2\alpha}(-{\Omega_f}^2 t^{2\alpha}) + t u'(0) E_{2\alpha,2}(-{\Omega_f}^2 t^{2\alpha}) \tag{29}$$

for the case of $\frac{1}{2} < \alpha < 1$, where $u'(0) = \frac{\partial u(t)}{\partial t}\Big|_{t=0}$. So with the boundary conditions

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \tag{30}$$

the general solution of the Eq. (26) may be

$$\begin{cases} a) & u(t) = u_0 E_{2\alpha}(-{\Omega_f}^2 t^{2\alpha}) \qquad for \quad 0 < \alpha < \frac{1}{2} \\ b) & u(t) = u_0 E_{2\alpha}(-{\Omega_f}^2 t^{2\alpha}) + t u_1 E_{2\alpha,2}(-{\Omega_f}^2 t^{2\alpha}) \quad for \quad \frac{1}{2} < \alpha < 1 \end{cases} \tag{31}$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)} \tag{32}$$

is one-parameter Mittag-Leffler function. Substituting the Eq. (31a) in Eq. (25) we have a particular solution of the equation as

$$Z(x,t) = (Z_0 u_0) e^{-ikx} E_{2\alpha}(-{\Omega_f}^2 t^{2\alpha}) \tag{33}$$

We can easily see that in the case $\alpha = 1$, the solution to the equation is

$$Z(x,t) = \mathrm{Re}(Z_0 u_0 e^{i(\Omega t - kx)}) \tag{34}$$

which defines a periodic, with fundamental period $T = 2\pi\Omega$, monochromatic wave in the, x, direction and in time, t. This result is very well known from the ordinary electromagnetic waves theory. However for the arbitrary case of α ($0 < \alpha < 1$) the solution is periodic only respect to x and it is not periodic with respect to t. The solution represents a plane wave with time decaying amplitude. For example for the case $\alpha = \frac{1}{2}$ we have

$$u(t) = E_1(-\eta\Omega^2 t) = e^{-\eta\Omega^2 t} \tag{35}$$

where for simplicity, we have used $u_0 = 1$ initial condition. Therefore the solution is

$$Z(x,t) = (Z_0 e^{-\eta\Omega^2 t}) e^{-ikx} \tag{36}$$

Also for the case of $\alpha = \frac{1}{4}$ we have

$$u(t) = E_{\frac{1}{2}}(-\eta^{\frac{3}{2}}\Omega^2\sqrt{t}) = e^{\eta^3\Omega^4 t}(1+erf\,(-\eta^{\frac{3}{2}}\Omega^2\sqrt{t})) = e^{\eta^3\Omega^4 t}erfc(\eta^{\frac{3}{2}}\Omega^2\sqrt{t}) \tag{37}$$

where $erfc$ denotes the complimentary error function and the error function is defined as

$$erf\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt, \quad erfc(z) = 1 - erf\,(z), \quad z \in C \tag{38}$$

For large values of z ,the complimentary error function can be approximated as

$$erfc(z) \simeq \frac{1}{\sqrt{\pi z}}\exp(-z^2) \tag{39}$$

Substituting Eq. (37) into Eq. (25) leads to the solution

$$Z(x,t) = (Z_0 e^{\eta^3\Omega^4 t} erfc(\eta^{\frac{3}{2}}\Omega^2\sqrt{t}))e^{-ikx} \tag{40}$$

At asymptotically large times, and using Eq. (39) we have

$$Z(x,t) \simeq (\frac{Z_0}{\sqrt{\pi\eta^{\frac{3}{2}}\Omega^2\sqrt{t}}})e^{-ikx} \tag{41}$$

Then for these cases, the solutions are periodic only respect to x and they are not periodic with respect to t . In fact solutions represent plane waves with time decaying amplitude.

3. Asymptotic behavior of the solution

The algebraic decay of the solutions of the fractional equations is the most important effect of the fractional derivative in the typical fractional equations contrary to the exponential decay of the usual standard form of the equations. To describe this algebraic decay in our case, we consider the integral form for the Mittag-Leffler function. The asymptotic expansion of $E_\alpha(z)$ based on the integral representation of the Mittag-Leffler function in the form [16]

$$E_\alpha(z) = \frac{1}{2\pi i}\int_\Upsilon \frac{t^{\alpha-1}\exp(t)}{t^\alpha - z}dt \tag{42}$$

where $\Re(\alpha) > 0, (\alpha, z) \in C$ and the path of integration Υ is a loop starting and ending at $-\infty$ and encircling the circular disk $|t| \le |z|^{\frac{1}{\alpha}}$ in the positive sense $|\arg t| < \pi$ on Υ . The integrand has a branch point at $t = 0$ .The complex t -plane is cut along the negative real axis and in the cut plane the integrand is single-valued the principal branch of t^α is taken in the cut plane. Eq. (42) can be proved by expanding the integrand in powers of t and integrating term by term by making use of the well-known Hankel's integral for the reciprocal of the gamma function, namely

$$\frac{1}{\Gamma(\beta)} = \frac{1}{2\pi i}\int_{H_a} \frac{e^\zeta}{\zeta^\beta}d\zeta \tag{43}$$

The integral representation Eq. (42) can be used to obtain the asymptotic expansion of the Mittag-Leffler function at infinity. Accordingly, the following cases are obtained.

If $0 < \alpha < 2$ and μ is a real number such that $\frac{\pi\alpha}{2} < \mu < \min[\pi, \pi\alpha]$ then for $N^* \in N$, $N^* \neq 1$ there holds the following asymptotic expansion:

$$E_{\alpha}(z)=\frac{1}{\alpha}z^{\frac{(1-\beta)}{\alpha}}\exp(z^{\frac{1}{\alpha}})-\sum_{r=1}^{N^*}\frac{1}{\Gamma(1-\alpha r)}\frac{1}{z^r}+O[\frac{1}{z^{N^*+1}}] \tag{44}$$

as $|z|\to\infty$, $|\arg z|\le\mu$ and

$$E_{\alpha}(z)=-\sum_{r=1}^{N^*}\frac{1}{\Gamma(1-\alpha r)}\frac{1}{z^r}+O[\frac{1}{z^{N^*+1}}] \tag{45}$$

as $|z|\to\infty$, $\mu\le|\arg z|\le\pi$. In our case, $z=-\Omega_f{}^2t^{2\alpha}$ and

$$E_{2\alpha}(-\Omega_f{}^2t^{2\alpha})\simeq\frac{1}{\Gamma(1-2\alpha)}\frac{1}{\Omega_f{}^2t^{2\alpha}} \tag{46}$$

Then, substitution of Eq. (46) into Eq. (33) gives

$$Z(x,t)\simeq[(\frac{Z_0}{\Omega_f{}^2\Gamma(1-2\alpha)})t^{-2\alpha}]e^{-ikx} \tag{47}$$

As we can see in this result, we arrive to the asymptotic solution for the electromagnetic wave equation which represents a plane wave with algebraic time-decaying amplitude. This is a direct consequence of the fractional time derivative in the system. In the other word fractional differentiation with respect to time can be interpreted as an existence of memory effects which correspond to intrinsic dissipation in our system.

4. Conclusion

The asymptotic behavior of Mittag-Leffler functions [16] plays a very important role in the interpretation and understanding of the solutions of various problems of physics connected with fractional phenomena that occur in complex systems. In this article we have studied the time evolution of the fractional electromagnetic waves by using the time fractional Maxwell's equations. We showed that electromagnetic plane wave has amplitude which exhibits an algebraic decay for $t\to\infty$ in our case (Eq. (41, 47)).

Appendix: Fractional dynamics of charged particles

For the simplest case we can consider motion of charged particles in a uniform electric field $\vec{E}=E_z\hat{k}$. So using the fractional Newton's second law we have

$${}_0^cD_t^{\alpha}p_z(t)=\eta^{1-\alpha}qE_z \tag{48}$$

if $p_z(0)=0$, so we have

$$p_z(t)=\frac{qE_z\eta^{1-\alpha}}{\Gamma(\alpha+1)}t^{\alpha} \tag{49}$$

where p_z is the z-component of particle's momentum. Also we can easily calculate z-component of particle's position as a function of time, i.e. $z(t)$ from

$$p_z(t) = \frac{m}{\eta^{1-\alpha}} ({}_0^c D_t^\alpha z(t)) \tag{50}$$

Taking into account the initial condition $z(0) = 0$ and substituting Eq. (49) into Eq. (50) leads to the solution

$$z(t) = \frac{qE_z \eta^{2-2\alpha}}{m\Gamma(2\alpha+1)} t^{2\alpha} \tag{51}$$

For the case of $\alpha = 1$ we can easily show that

$$z_{\alpha=1}(t) = \frac{qE_z}{m\Gamma(3)} t^2 = \frac{qE_z}{2m} t^2 \tag{52}$$

as expected from the standard electrodynamics.

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