PHYS-Nov-2012-95 v1

Underlying symmetry among the quark and lepton mixing angles (Five year update)

J. S. Markovitch
P.O. Box 2411
*West Brattleboro, VT 05303**
(Dated: November 9, 2012)

In 2007 a mathematical model encompassing both quark and lepton mixing was introduced. As five years have elapsed since its introduction it is timely to assess the model's accuracy. Despite large conflicts with experiment at the time of its introduction, five of six predicted angles now fit experiment fairly closely. The one angle incorrectly forecast necessitates a small change to the model's original framework (essentially, a sign is toggled). This change retains most of the model's original economy, while being interesting in its own right. The model's predicted mixing angles in degrees are 45, 33.210 911, and 8.034 394 (new) for leptons; and 12.920 966, 2.367 442, and 0.190 986 for quarks.

PACS numbers: 12.15.Ff, 14.60.Pq

*Electronic address: jsmarkovitch@gmail.com

Contents

I. INTRODUCTION

In 2007 a mathematical model encompassing both quark and lepton mixing was introduced [1] (most recently updated in 2011 [2]). As five years have elapsed since its introduction it is timely to issue an update:

- to present a clearer and simpler version of the model.
- to review the status of the model's predictions.
- to accommodate recent neutrino mixing results (requiring changing the sign of a key term's exponent).
- to note new results showing that a large portion of the mixing model arises independently in pure mathematics.

II. THE FINE STRUCTURE CONSTANT INVERSE DERIVED FROM $g_{12} = 1/10$ AND $g_{13} = 1/30\,000$

Experiment reveals that the quark and lepton mixing angles occupy a wide range [3] [4]

$$\sim\!45^\circ > \sim\!34^\circ > \sim\!13^\circ > \sim\!9^\circ > \sim\!2^\circ > \sim\!0.2^\circ \quad .$$

In order to produce model angles fitting such angles we begin by defining

$$g_{12} = \frac{1}{10}$$
$$g_{13} = \frac{1}{30\,000} \quad .$$

Importantly, the above definitions facilitate compactly reproducing the fine structure constant (FSC) inverse α^{-1} (to within about seven parts per billion) as follows

$$\left[\frac{1}{3g_{12}} - \frac{g_{13}}{3}\right]^3 + \left[\frac{1}{g_{12}} - g_{13}\right]^2 = \tag{1}$$
$$\left[\frac{10}{3} - \frac{1}{3\times 30\,000}\right]^3 + \left[10 - \frac{1}{30\,000}\right]^2 = \alpha^{-1} = 137.036\,000\,0023\ldots \quad ,$$

where the 2010 CODATA value for α^{-1} equals 137.035 999 074 [5]. Hence, g_{12} and g_{13} were *not* chosen exclusively to fit the mixing data. Moreover, new results from the author establish that the value 137.036, in the form of an equation very similar to the above equation, arises independently in pure mathematics [6][7].

TABLE I: The angles below are constrained by the requirement that: (a) the values of the first two rows sum to equal the values of the third row; (b) the values of each row fulfill Eqs. (2a)–(2d); and (c) the values of rows one, two, and three produce the identities of Eqs. (3g), (4g), and (5g), respectively. With the exception of θ_{13}^{L} the six angles in the table are the predicted quark and lepton mixing angles. As discussed in the text, the angle $\theta_{13\ \mathrm{large}}^{L}$ replaces θ_{13}^{L} as a prediction, where they help compactly reproduce the fine structure constant inverse, as in Eqs. (7a) and (7b), respectively.

g_{12}	g_{13}	θ_{23}^{L}	θ_{13}^{L} [a]	θ_{12}^{L}	θ_{23}^{Q}	θ_{13}^{Q}	θ_{12}^{Q}
1/10 [b]	0	$45° + 90°$	$0°$	$33.210\,911°$	$+ 90°$	$0°$	$12.920\,966°$
0 [c]	1/30 000	$- 90°$	$0.013\,665°$	$0°$	$2.367\,442°$	$0.190\,986°$	$0°$
1/10 [d]	1/30 000	$45°$	$0.013\,665°$	$33.210\,911°$	$2.367\,442° + 90°$	$0.190\,986°$	$12.920\,966°$

[a] But it is $\theta_{13\ \mathrm{large}}^{L}$, equaling about $8.034\,394°$, which is expected to match experiment. See Secs. VI and VII.
[b] The angles in this row derive from Eqs. (3a)–(3d) in Sec. III and produce Eq. (3g).
[c] The angles in this row derive from Eqs. (4a)–(4d) in Sec. IV and produce Eq. (4g).
[d] The angles in this row derive from Eqs. (5a)–(5d) in Sec. V and produce Eq. (5g).

TABLE II: In Eqs. (3g), (4g), and (5g) the quark matrix elements equal $1/3^{\mathrm{rd}}$ the leptonic elements, a result of Eqs. (2a)–(2d).

Identity	Quark matrix elements	Leptonic matrix elements	g_{12}	g_{13}
Eq. (3g)	0.05	0.15	1/10	0
Eq. (4g)	$1.895\,936 \times 10^{-8}$	$5.687\,808 \times 10^{-8}$	0	1/30 000
Eq. (5g)	0.049 963 56	0.149 8907	1/10	1/30 000

The above definitions, in turn, aid the definition of

$$\sin\theta_{12}^{L} = \sqrt{3g_{12}} \tag{2a}$$
$$\sin\theta_{13}^{Q} = \sqrt{g_{13}/3} \tag{2b}$$
$$\sin\theta_{12}^{Q} = \sqrt{g_{12}} \times \sin\theta_{23\ \mathrm{offset}}^{L} \tag{2c}$$

and

$$\sin\theta_{13}^{L} = \sqrt{g_{13}} \times \left(\sin\theta_{23\ \mathrm{offset}}^{Q}\right)^{+1} \tag{2d}$$
$$\sin\theta_{13\ \mathrm{large}}^{L} = \sqrt{g_{13}} \times \left(\sin\theta_{23\ \mathrm{offset}}^{Q}\right)^{-1} \quad . \tag{2e}$$

Note that Eqs. (2a)–(2d) follow the 2007 model, whereas Eq. (2e) is new, differing from Eq. (2d) in the sign of an exponent. Below, it will be $\theta_{13\ \mathrm{large}}^{L}$, rather than θ_{13}^{L}, that will fit recently-measured, smallest neutrino mixing angle [4]. (Also, observe that the subscripts used above were chosen because g_{12} helps define the "12" mixing angles, whereas g_{13} helps define the "13" mixing angles.) At this point the reader perhaps has noticed that to determine the four "12" and "13" angles, just defined, we need only know the "23" angles: $\theta_{23\ \mathrm{offset}}^{L}$ and $\theta_{23\ \mathrm{offset}}^{Q}$. Therefore, let us specify them at once:

$$\theta_{23\ \mathrm{offset}}^{L} = 45°$$
$$\theta_{23\ \mathrm{offset}}^{Q} = 2.367\,442° \quad .$$

But how to justify this *particular* value for $\theta_{23\ \mathrm{offset}}^{Q}$ and the *form* of Eqs. (2a)–(2d)?

In the next three sections it will be shown that a property possessed by the leptonic matrices derived from $\theta_{23\ \mathrm{offset}}^{L}$, θ_{13}^{L}, θ_{12}^{L} is also possessed by the quark matrices derived from $\theta_{23\ \mathrm{offset}}^{Q}$, θ_{13}^{Q}, θ_{12}^{Q}, where this property is threefold larger for the leptonic matrices than for the quark matrices. This closely mirrors the way that the sum of the charges of the leptons

$$-1 + 0 = -1$$

measures threefold larger than the sum of the charges of the quarks

$$-\frac{2}{3} + \frac{1}{3} = -\frac{1}{3} \quad .$$

Moreover, this property will be shown to be threefold larger *in three independent ways*, an intentional consequence of the particular value chosen for $\theta_{23\ \mathrm{offset}}^{Q}$ and the form chosen for Eqs. (2a)–(2d).

III. THE MIXING MATRICES THAT DERIVE FROM $g_{12} = 1/10$ AND $g_{13} = 0$

Define the usual CKM mixing matrix [3], *but without its phase*, as

$$V = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} ,$$

where $s_{12} \equiv \sin\theta^Q_{12}$, $c_{12} \equiv \cos\theta^Q_{12}$, etc., and where θ^Q_{23}, θ^Q_{13}, and θ^Q_{12} are the CKM mixing angles. And define the usual leptonic mixing matrix [8], *also without its phase*, as

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} ,$$

where $s_{12} \equiv \sin\theta^L_{12}$, $c_{12} \equiv \cos\theta^L_{12}$, etc., and where θ^L_{23}, θ^L_{13}, and θ^L_{12} are the leptonic mixing angles. Now consider that the following matrix

$$\begin{array}{c} \\ u \\ t \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} \end{array}$$

results if the above CKM matrix *with its elements squared* has its angles determined by

$$g_{12} = 1/10 \tag{3a}$$

$$g_{13} = 0 \tag{3b}$$

$$\theta^L_{23} = \theta^L_{23\text{ offset}} + 90^\circ \tag{3c}$$

$$\theta^Q_{23} = 90^\circ \tag{3d}$$

and Eqs. (2a)–(2d). Observe that the above matrix's second and third rows (i.e., its c- and t-quarks) are interchanged relative to convention, a consequence of $\theta^Q_{23} = 90^\circ$. Subtracting this matrix from its transpose gives

$$\begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} - \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 \\ 0 & 1.00 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +0.05 & -0.05 \\ -0.05 & 0 & +0.05 \\ +0.05 & -0.05 & 0 \end{bmatrix} . \tag{3e}$$

Now consider that the following matrix

$$\begin{array}{c} \\ \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

results if the above leptonic matrix *also with its elements squared* has its angles also determined by Eqs. (3a)–(3d) and Eqs. (2a)–(2d). Observe that the above matrix's second and third rows (i.e., ν_μ and ν_τ) also are interchanged relative to convention, but this time it is a consequence of $\theta^L_{23} = \theta^L_{23\text{ offset}} + 90^\circ$. Subtracting the above matrix from its transpose gives

$$\begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} - \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.30 & 0.35 & 0.35 \\ 0 & 0.50 & 0.50 \end{bmatrix} = \begin{bmatrix} 0 & +0.15 & -0.15 \\ -0.15 & 0 & +0.15 \\ +0.15 & -0.15 & 0 \end{bmatrix} . \tag{3f}$$

Now the right hand sides of Eqs. (3e) and (3f) combine to form the identity

$$3 \times \begin{bmatrix} 0 & +0.05 & -0.05 \\ -0.05 & 0 & +0.05 \\ +0.05 & -0.05 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +0.15 & -0.15 \\ -0.15 & 0 & +0.15 \\ +0.15 & -0.15 & 0 \end{bmatrix} , \tag{3g}$$

the first of the three key constraints of the mixing model of 2007. The values associated with this property occupy row one of Tables I and II.

IV. THE MIXING MATRICES THAT DERIVE FROM $g_{12} = 0$ AND $g_{13} = 1/30\,000$

Consider that the following matrix

$$\begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 9.999\,889 \times 10^{-1} & 0 & 1.111\,111 \times 10^{-5} \\ 1.895\,936 \times 10^{-8} & 9.982\,937 \times 10^{-1} & 1.706\,323 \times 10^{-3} \\ 1.109\,215 \times 10^{-5} & 1.706\,342 \times 10^{-3} & 9.982\,826 \times 10^{-1} \end{bmatrix} \end{array}$$

results if the earlier CKM matrix *with its elements squared* has its angles determined by

$$g_{12} = 0 \tag{4a}$$

$$g_{13} = 1/30\,000 \tag{4b}$$

$$\theta_{23}^{L} = -90^{\circ} \tag{4c}$$

$$\theta_{23}^{Q} = \theta_{23\ \mathrm{offset}}^{Q} \tag{4d}$$

and Eqs. (2a)–(2d). Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 9.999\,889 \times 10^{-1} & 0 & 1.111\,111 \times 10^{-5} \\ 1.895\,936 \times 10^{-8} & 9.982\,937 \times 10^{-1} & 1.706\,323 \times 10^{-3} \\ 1.109\,215 \times 10^{-5} & 1.706\,342 \times 10^{-3} & 9.982\,826 \times 10^{-1} \end{bmatrix} \\ &- \begin{bmatrix} 9.999\,889 \times 10^{-1} & 1.895\,936 \times 10^{-8} & 1.109\,215 \times 10^{-5} \\ 0 & 9.982\,937 \times 10^{-1} & 1.706\,342 \times 10^{-3} \\ 1.111\,111 \times 10^{-5} & 1.706\,323 \times 10^{-3} & 9.982\,826 \times 10^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} \\ +1.895\,936 \times 10^{-8} & 0 & -1.895\,936 \times 10^{-8} \\ -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} & 0 \end{bmatrix} . \end{aligned} \tag{4e}$$

Now consider that the following matrix

$$\begin{array}{c} \\ \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 9.999\,999 \times 10^{-1} & 0 & 5.687\,808 \times 10^{-8} \\ 5.687\,808 \times 10^{-8} & 0 & 9.999\,999 \times 10^{-1} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

results if the earlier leptonic matrix *also with its elements squared* has its angles also determined by Eqs. (4a)–(4d) and Eqs. (2a)–(2d). Observe that the above matrix's second and third rows (i.e., ν_μ and ν_τ) are interchanged relative to convention, a consequence of $\theta_{23}^{L} = -90^{\circ}$. Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 9.999\,999 \times 10^{-1} & 0 & 5.687\,808 \times 10^{-8} \\ 5.687\,808 \times 10^{-8} & 0 & 9.999\,999 \times 10^{-1} \\ 0 & 1 & 0 \end{bmatrix} \\ &- \begin{bmatrix} 9.999\,999 \times 10^{-1} & 5.687\,808 \times 10^{-8} & 0 \\ 0 & 0 & 1 \\ 5.687\,808 \times 10^{-8} & 9.999\,999 \times 10^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} \\ +5.687\,808 \times 10^{-8} & 0 & -5.687\,808 \times 10^{-8} \\ -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} & 0 \end{bmatrix} . \end{aligned} \tag{4f}$$

Now the right hand sides of Eqs. (4e) and (4f) combine to form the identity

$$\begin{aligned} &3 \times \begin{bmatrix} 0 & -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} \\ +1.895\,936 \times 10^{-8} & 0 & -1.895\,936 \times 10^{-8} \\ -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} \\ +5.687\,808 \times 10^{-8} & 0 & -5.687\,808 \times 10^{-8} \\ -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} & 0 \end{bmatrix} , \end{aligned} \tag{4g}$$

the second of the three key constraints of the mixing model of 2007. The values associated with this property occupy row two of Tables I and II.

V. THE MIXING MATRICES THAT DERIVE FROM $g_{12} = 1/10$ AND $g_{13} = 1/30\,000$

Consider that the following matrix

$$\begin{array}{c} \\ u \\ t \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 9.499\,894 \times 10^{-1} & 4.999\,944 \times 10^{-2} & 1.111\,111 \times 10^{-5} \\ 3.588\,691 \times 10^{-5} & 1.681\,548 \times 10^{-3} & 9.982\,826 \times 10^{-1} \\ 4.997\,467 \times 10^{-2} & 9.483\,190 \times 10^{-1} & 1.706\,323 \times 10^{-3} \end{bmatrix} \end{array}$$

results if the earlier CKM matrix *with its elements squared* has its angles determined by

$$g_{12} = 1/10 \tag{5a}$$

$$g_{13} = 1/30\,000 \tag{5b}$$

$$\theta^L_{23} = \theta^L_{23\text{ offset}} \tag{5c}$$

$$\theta^Q_{23} = \theta^Q_{23\text{ offset}} + 90^\circ \tag{5d}$$

and Eqs. (2a)–(2d). Observe that the above matrix's second and third rows (i.e., its c- and t-quarks) are interchanged relative to convention, a consequence of $\theta^Q_{23} = \theta^Q_{23\text{ offset}} + 90^\circ$. Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 9.499\,894 \times 10^{-1} & 4.999\,944 \times 10^{-2} & 1.111\,111 \times 10^{-5} \\ 3.588\,691 \times 10^{-5} & 1.681\,548 \times 10^{-3} & 9.982\,826 \times 10^{-1} \\ 4.997\,467 \times 10^{-2} & 9.483\,190 \times 10^{-1} & 1.706\,323 \times 10^{-3} \end{bmatrix} \\ &- \begin{bmatrix} 9.499\,894 \times 10^{-1} & 3.588\,691 \times 10^{-5} & 4.997\,467 \times 10^{-2} \\ 4.999\,944 \times 10^{-2} & 1.681\,548 \times 10^{-3} & 9.483\,190 \times 10^{-1} \\ 1.111\,111 \times 10^{-5} & 9.982\,826 \times 10^{-1} & 1.706\,323 \times 10^{-3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} \\ -4.996\,356 \times 10^{-2} & 0 & +4.996\,356 \times 10^{-2} \\ +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} & 0 \end{bmatrix} . \end{aligned} \tag{5e}$$

Now consider that the following matrix

$$\begin{array}{c} \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 6.999\,999\,602 \times 10^{-1} & 2.999\,999\,829 \times 10^{-1} & 5.687\,808\,086 \times 10^{-8} \\ 1.501\,093\,103 \times 10^{-1} & 3.498\,907\,181 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \\ 1.498\,907\,295 \times 10^{-1} & 3.501\,092\,990 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \end{bmatrix} \end{array}$$

results if the earlier leptonic matrix *also with its elements squared* has its angles also determined by Eqs. (5a)–(5d) and Eqs. (2a)–(2d). Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 6.999\,999\,602 \times 10^{-1} & 2.999\,999\,829 \times 10^{-1} & 5.687\,808\,086 \times 10^{-8} \\ 1.501\,093\,103 \times 10^{-1} & 3.498\,907\,181 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \\ 1.498\,907\,295 \times 10^{-1} & 3.501\,092\,990 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \end{bmatrix} \\ &- \begin{bmatrix} 6.999\,999\,602 \times 10^{-1} & 1.501\,093\,103 \times 10^{-1} & 1.498\,907\,295 \times 10^{-1} \\ 2.999\,999\,829 \times 10^{-1} & 3.498\,907\,181 \times 10^{-1} & 3.501\,092\,990 \times 10^{-1} \\ 5.687\,808\,086 \times 10^{-8} & 4.999\,999\,716 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & +1.498\,906\,726 \times 10^{-1} & -1.498\,906\,726 \times 10^{-1} \\ -1.498\,906\,726 \times 10^{-1} & 0 & +1.498\,906\,726 \times 10^{-1} \\ +1.498\,906\,726 \times 10^{-1} & -1.498\,906\,726 \times 10^{-1} & 0 \end{bmatrix} . \end{aligned} \tag{5f}$$

Now the right hand sides of Eqs. (5e) and (5f) combine to form the identity

$$\begin{aligned} 3 \times &\begin{bmatrix} 0 & +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} \\ -4.996\,356 \times 10^{-2} & 0 & +4.996\,356 \times 10^{-2} \\ +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & +1.498\,907 \times 10^{-1} & -1.498\,907 \times 10^{-1} \\ -1.498\,907 \times 10^{-1} & 0 & +1.498\,907 \times 10^{-1} \\ +1.498\,907 \times 10^{-1} & -1.498\,907 \times 10^{-1} & 0 \end{bmatrix} , \end{aligned} \tag{5g}$$

the third of the three key constraints of the mixing model of 2007. The values associated with this property occupy row three of Tables I and II.

VI. MIXING MODEL PREDICTIONS

As promised earlier, and as summarized in Tables I and II, the mixing angles $\theta^L_{23\text{ offset}}$, θ^L_{13}, θ^L_{12} and $\theta^Q_{23\text{ offset}}$, θ^Q_{13}, θ^Q_{12} have been shown to possess the same property in three related, but still independent, ways. This explains this article's choice of value for $\theta^Q_{23\text{ offset}}$ and choice of form for Eqs. (2a)–(2d). It only remains to calculate $\theta^L_{13\text{ large}}$ to complete the list of *predicted* mixing angles. Equation (2e) gives

$$\sin^2\theta^L_{13\text{ large}} = \frac{g_{13}}{\sin^2\theta^Q_{23\text{ offset}}} \approx \frac{1}{30\,000 \times \sin^2 2.367\,442^\circ} \approx 0.019\,53 \quad , \tag{6a}$$

so that

$$\theta^L_{13\text{ large}} \approx 8.034\,394^\circ \quad . \tag{6b}$$

See Tables III and IV for a summary of how the predictions for all six mixing angles have fared. (Note: Those interested in how the 2011 model handles phase should consult [2].)

VII. THE FINE STRUCTURE CONSTANT INVERSE AND $\sin^{\pm 2}\theta^Q_{23\text{ offset}}$

At this point the reader may object that the earlier definition of $\theta^L_{13\text{ large}}$ was artificially designed to fit the recent $\sim 9^\circ$ measurement of the smallest leptonic mixing angle [4]. But $\theta^L_{13\text{ large}}$ is interesting in its own right, as it neatly combines with the other mixing angles to produce this FSC inverse approximation

$$\left(\frac{1}{\sin^2\theta^L_{12}} - \sin^2\theta^Q_{13}\right)^3 + \left(\frac{1}{\sin^2\theta^Q_{12}} \times \sin^2\theta^L_{23\text{ offset}} - \sin^2\theta^L_{13\text{ large}} \times \sin^{+2}\theta^Q_{23\text{ offset}}\right)^2 = 137.036\,000\,0023\ldots \quad . \tag{7a}$$

In this way the new mixing model retains the original model's ability to produce the FSC inverse from the sines squared of the model angles, where the 2007 method was

$$\left(\frac{1}{\sin^2\theta^L_{12}} - \sin^2\theta^Q_{13}\right)^3 + \left(\frac{1}{\sin^2\theta^Q_{12}} \times \sin^2\theta^L_{23\text{ offset}} - \sin^2\theta^L_{13} \times \sin^{-2}\theta^Q_{23\text{ offset}}\right)^2 = 137.036\,000\,0023\ldots \quad . \tag{7b}$$

Observe, firstly, that it is the differing exponents of the terms in light blue that cause $\sin^2\theta^L_{13\text{ large}}$ and $\sin^2\theta^L_{13}$ to differ by

$$\frac{\sin^{-2}\theta^Q_{23\text{ offset}}}{\sin^{+2}\theta^Q_{23\text{ offset}}} = \sim 343\,453 \quad ; \tag{7c}$$

and, secondly, that both FSC equations are mere variants of Eq. (1). The new mixing model is, therefore, only a slightly modified version of the 2007 model, retaining five of six of its predictions, while constraining $\theta^L_{13\text{ large}}$ to a value that mirrors θ^L_{13} in its ability to reproduce the FSC inverse. It follows that the term $\theta^L_{13\text{ large}}$ is not freely adjusted to fit the new mixing data, but is better characterized as a twin of the term θ^L_{13}. (Note: there is also the side issue of whether θ^L_{13} in combination with θ^L_{23} and θ^L_{12} of Table I accurately model mixing for an as-yet-unobserved set of particles, or if instead θ^L_{13} is entirely nonphysical.)

TABLE III: Model predictions from 2007 compared against CKM mixing data.

Year	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$	$\lvert V_{cb}\rvert$
2007 Prediction	0.2236	0.003 333	0.041 31
2012 [a]	$0.225\,34^{+0.000\,65}_{-0.000\,65}$	$0.003\,51^{+0.000\,15}_{-0.000\,14}$	$0.0412^{+0.0011}_{-0.0005}$
Error in SD	2.7	1.3	0.2
2010 [b]	$0.2253^{+0.0007}_{-0.0007}$	$0.00347^{+0.000\,16}_{-0.000\,12}$	$0.0410^{+0.0011}_{-0.0007}$
Error in SD	2.4	1.1	0.3
2008 [c]	$0.2257^{+0.0010}_{-0.0010}$	$0.00359^{+0.000\,16}_{-0.000\,16}$	$0.0415^{+0.0010}_{-0.0011}$
Error in SD	2.1	1.6	0.2
2006 [d]	$0.2272^{+0.0010}_{-0.0010}$	$0.00396^{+0.000\,09}_{-0.000\,09}$	$0.04221^{+0.0001}_{-0.0008}$
Error in SD	3.6	7.0	1.1

[a]Ref. [3]. Particle Data Group 1σ global fit.
[b]Ref. [9]. Particle Data Group 1σ global fit.
[c]Ref. [10]. Particle Data Group 1σ global fit.
[d]Ref. [11]. Particle Data Group 1σ global fit.

TABLE IV: Model predictions from 2007 (and 2012) compared against leptonic mixing data. Normal hierarchy.

Year	$\sin^2\theta^L_{23\text{ offset}}$	$\sin^2\theta^L_{13\text{ large}}$	$\sin^2\theta^L_{12}$
2007 (2012) Prediction	0.5	(0.019 53)	0.3
2012 (Aug.)[a]	$0.427^{+0.034\,b}_{-0.027}$	$0.0246^{+0.0029}_{-0.0028}$	$0.320^{+0.016}_{-0.017}$
Error in SD	2.1	1.8	1.2
2010 [c]	$0.50^{+0.07}_{-0.06}$	$0.013^{+0.013}_{-0.009}$	$0.318^{+0.019}_{-0.016}$
Error in SD	0	0.5	1.1
2008 [d]	$0.50^{+0.07}_{-0.06}$	$0.010^{+0.016}_{-0.011}$	$0.304^{+0.022}_{-0.016}$
Error in SD	0	0.6	0.25
2006 [e]	$0.50^{+0.08}_{-0.07}$	≤ 0.025 [f]	$0.300^{+0.020}_{-0.030}$
Error in SD	0		0

[a]Ref. [4]. A 1σ global fit.
[b]Ref. [4]. One of two minima, the other being $0.613^{+0.022}_{-0.040}$. Maximal mixing (i.e., $\theta^L_{23\text{ offset}} = 45°$) is excluded at $\sim$90% C.L.
[c]Ref. [12]. A 1σ global fit. This source includes an update containing 2010 data.
[d]Ref. [12]. A 1σ global fit.
[e]Ref. [13]. A 1σ global fit.
[f]Ref. [13]. A 2σ global fit.

VIII. SUMMARY AND CONCLUSION

In order to compare the mixing model predictions against experiment it is useful to know that its angles produce the following CKM matrix elements [3]:

$$\begin{array}{lclcccl} V_{us} & \approx & \sin 12.920\,966° & \times & \cos 0.190\,986° & \approx & 0.2236 \\ V_{ub} & \approx & & & \sin 0.190\,986° & \approx & 0.003\,333 \\ V_{cb} & \approx & \sin 2.367\,442° & \times & \cos 0.190\,986° & \approx & 0.041\,31 \quad . \end{array}$$

Tables III and IV aid the comparison of these predictions against experiment:

- In 2007 the model's value for $|V_{us}|$ had a 3.6σ disagreement with experiment. This value is now off by 2.7σ, its absolute error having been reduced by 52%.
- In 2007 the model's value for $|V_{ub}|$ had an improbable 7.0σ disagreement with experiment. This value is now off by 1.3σ, its absolute error having been reduced by 72% (though the accuracy of the measurement of $|V_{ub}|$ has *reduced* since 2006).
- In 2007 the model's value for $\sin^2\theta^L_{12}$ matched experiment exactly. Its error now equals 1.2σ.
- In 2007 the model's value for $\sin^2\theta^L_{23}$ matched experiment exactly. Its error now equals 2.1σ with maximal mixing (i.e., $\theta^L_{23\text{ offset}} = 45°$) excluded at $\sim$90%.

- In 2007 the model predicted that the sines squared of the experimental mixing angles should combine to approximate closely the fine structure constant inverse. This claim remains valid, but new experimental results have required toggling the sign of the exponent of $\sin^{-2}\theta^{Q}_{23\text{ offset}}$, thereby replacing Eq. (7b) with Eq. (7a). This makes $\sin^2\theta^{L}_{13\text{ large}}$ larger than $\sin^2\theta^{L}_{13}$ by a factor of $\sin^{-4}\theta^{Q}_{23}$, or $\sim$343 453. The value $\sin^2\theta^{L}_{13\text{ large}}$, equaling $\sim$0.019 53, is within 1.8σ of its experimental counterpart.

As noted as the outset, it is shown by [6][7] that the value 137.036, in the form of an equation very similar to Eq. (1), arises independently in pure mathematics. Given such spontaneous agreement in quantity and form between pure mathematics and this article's mixing model it is particularly unlikely that the above fit occurs solely by chance.

[1] J. S. Markovitch, "Underlying symmetry of the quark and lepton mixing angles" (2007) `http://lss.fnal.gov/archive/other/jm-ph-2007-72.pdf`.

[2] J. S. Markovitch, "A mathematical model of the quark and lepton mixing angles (2011 Update)" (2011) `http://www.vixra.org/abs/1102.0021`.

[3] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix" in J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).

[4] D. V. Forero, M. Tortola and J. W. F. Valle, "Global status of neutrino oscillation parameters after Neutrino-2012," arXiv:1205.4018 [hep-ph].

[5] P. J. Mohr, B. N. Taylor, and D. B. Newell (2011), "The 2010 CODATA Recommended Values of the Fundamental Physical Constants" (Web Version 6.0). This database was developed by J. Baker, M. Douma, and S. Kotochigova. Available: http://physics.nist.gov/constants [Friday, 22-Jul-2011 10:04:27 EDT]. National Institute of Standards and Technology, Gaithersburg, MD 20899.

[6] J. S. Markovitch, "A rewriting system applied to the simplest algebraic identities" (2012) `http://www.vixra.org/abs/1211.0029`.

[7] J. S. Markovitch, "A theorem producing the fine structure constant inverse and the quark and lepton mixing angles" (2012) `http://www.vixra.org/abs/1203.0106`.

[8] B. Kayser, "Neutrino Mass, Mixing, and Flavor Change," in C. Amsler *et al.* (Particle Data Group), Phys. Lett. B **667**, 1 (2008).

[9] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix" in K. Nakamura *et al.* (Particle Data Group), J. Phys. **G37**:075021 (2010).

[10] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix," in C. Amsler, *et al.* (Particle Data Group), Phys, Lett. B **667**, 1 (2008).

[11] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix," in W.-M. Yao, *et al.* (Particle Data Group), J. Phys. G **33**, 1 (2006).

[12] T. Schwetz, M. A. Tortola, and J. W. F. Valle, "Three-flavour neutrino oscillation update," New J. Phys. **10**, 113011 (2010) [arXiv:0808.2016 [hep-ph]].

[13] T. Schwetz, "Global fits to neutrino oscillation data," Phys. Scripta **T127**, 1 (2006) [arXiv:hep-ph/0606060].