Zanaboni Theorem and Saint-Venant's Principle

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Abstract

Violating the law of energy conservation, Zanaboni Theorem is invalid and Zanaboni's proof is wrong. Zanaboni's mistake of " proof " is analyzed. Energy Theorem for Zanaboni Problem is suggested and proved. Equations and conditions are established in this paper for Zanaboni Problem, which are consistent with , equivalent or identical to each other. Zanaboni Theorem is, for its invalidity , not a mathematical formulation or proof of Saint-Venant's Principle.

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`Keywords` : Zanaboni Theorem, Saint-Venant's Principle, Energy Theorem, Proof, Disproof, Equations, Conditions

1 Introduction

Saint-Venant's Principle in elasticity has its over 100 year's history [1, 2]. Boussinesq and Love announced general statements of Saint-Venant's Principe [3, 4]. The early and important researches contributed to the principle are the articles [3-9]. Zanaboni [7] " proved " a theorem trying to concern Saint-Venant's Principle in terms of work and energy, but, in the present paper, we will prove that Zanaboni Theorem is false and its proof by Zanaboni is wrong. Zanaboni Theorem is seriously concerned because of its profound influence on the development of Saint-Venant's Principle.

2 Zanaboni Theorem

In 1937, Zanaboni published a theorem dealing with energy decay of bodies of general shape [7]. The result played an influential role in the history of research

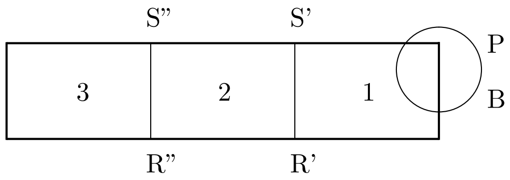

Fig.1

on Saint-Venant's Principle, restoring confidence in formulating the principle [10] .

The theorem is described as follows [7, 11]:

Let an elastic body of general shape be loaded in a small sphere B by P , an arbitrary system of self-equilibrated forces, otherwise the body is free. Let S' and S'' be two arbitrary nonintersecting cross sections outside of B and S'' be farther away from B than S' . Suppose that the body is cut into two parts at S' . The system of surface tractions acting on the section S' is R', and the total strain energy that would be induced by R' in the two parts is denoted by $U_{R'}$. Similarly, we use R'' and $U_{R''}$ for the case of the section S'' which would also imaginarily cut the body into two pieces (See Fig.1).

Then, according to Zanaboni,

$$0 < U_{R''} < U_{R'}. \tag{1}$$

3 Zanaboni Theorem is Invalid

3.1 Zanaboni's Proof

The proof of Zanaboni Theorem is (See [7] and pp 300-303 of [11]) :

Assume that the stresses in the enlarged body $C_1 + C_2$ are constructed by the following stages . First, C_1 is loaded by P. Second, each of the separate surfaces S_1 and S_2 is loaded by a system of surface traction R. Suppose that R is distributed in such a way that the deformed surfaces S_1 and S_2 fit each other precisely, so that displacements and stresses are continuous across the joint of S_1 and S_2. Then C_1 and C_2 are brought together and joined with S as an interface. The effect is the same if C_1 and C_2 were linked in the unloaded state and then the combined body $C_1 + C_2$ is loaded by P.(See Fig.2)

Thus

$$U_{1+2} = U_1 + U_{R1} + U_{R2} + U_{PR}, \tag{2}$$

where U_{1+2} is the strain energy stored in $C_1 + C_2$, U_1 is the work done by P in the first stage, U_{R2} is the work done by R on C_2 in the second stage, U_{R1} is the work done by R on C_1 if C_1 were loaded by R alone , U_{PR} is the work done by P on C_1 due to the deformation caused by R , in the second stage.

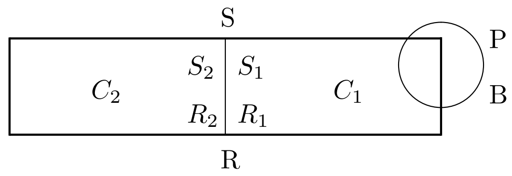

Fig.2

Now the minimum complementary energy theorem is used. All the actual forces R are considered as varied by the ratio $1 : (1+\varepsilon)$, then the work U_{R1} and U_{R2} will be varied to $(1+\varepsilon)^2 U_{R1}$ and $(1+\varepsilon)^2 U_{R2}$ respectively because the load and the deformation will be varied by a factor $(1+\varepsilon)$ respectively. U_{PR} will be varied to $(1+\varepsilon)U_{PR}$ because the load P is not varied and the deformation is varied by a factor $(1+\varepsilon)$. Hence, U_{1+2} will be changed to

$$U'_{1+2} = U_1 + (1+\varepsilon)^2 (U_{R1} + U_{R2}) + (1+\varepsilon)U_{PR}. \tag{3}$$

The virtual increment of U_{1+2} is

$$\Delta U_{1+2} = \varepsilon(2U_{R1} + 2U_{R2} + U_{PR}) + \varepsilon^2 (U_{R1} + U_{R2}). \tag{4}$$

For U_{1+2} to be a minimum, it is required from Eq.(4) that

$$2U_{R1} + 2U_{R2} + U_{PR} = 0. \tag{5}$$

Substituting Eq.(5) into Eq.(2), he obtains

$$U_{1+2} = U_1 - (U_{R1} + U_{R2}). \tag{6}$$

By repeated use of Eq.(6) for $U_{1+(2+3)}$ and $U_{(1+2)+3}$ (See Fig.1), then

$$U_{1+(2+3)} = U_1 - (U_{R'1} + U_{R'(2+3)}), \tag{7}$$

$$\begin{aligned} U_{(1+2)+3} &= U_{1+2} - (U_{R''(1+2)} + U_{R''3}) \\ &= U_1 - (U_{R1} + U_{R2}) - (U_{R''(1+2)} + U_{R''3}). \end{aligned} \tag{8}$$

Equating Eq.(7) with Eq.(8) he obtains

$$U_{R'1} + U_{R'(2+3)} = U_{R1} + U_{R2} + U_{R''(1+2)} + U_{R''3}. \tag{9}$$

It is from Eq.(9) that

$$U_{R'1} + U_{R'(2+3)} > U_{R''(1+2)} + U_{R''3} \tag{10}$$

because U_{R1} and U_{R2} are essentially positive quantities . Equation (10) is Eq.(1) , on writing $U_{R'}$ for $U_{R'1} + U_{R'(2+3)}$, etc. And Eq.(1) is “ proved ”.

3.2 Confusion in Zanaboni's Proof

In Zanaboni's proof (See [7] , pp 300-303 of [11]), Eq.(8) is deduced by confusing. The first is the confusion of the construction $(1+2)$ in Fig.1 , where its " far end " is loaded (by R'') , with the construction $C_1 + C_2$ in Fig.2, where its " far end " is free. The second confusion is that of work W and energy U, especially W_{1+2} and U_{1+2}. In fact, Eq.(2) should be revised to be

$$U_{1+2} = W_1 + W_{R1} + W_{R2} + W_{PR} \tag{11}$$

and Eq.(6) should be corrected to

$$U_{1+2} = W_1 - (W_{R1} + W_{R2}). \tag{12}$$

And then the use of Eq.(12) should result in (See Fig.1)

$$U_{1+(2+3)} = W_1 - (W_{R'1} + W_{R'(2+3)}), \tag{13}$$

$$U_{(1+2)+3} = W_{1+2} - (W_{R''(1+2)} + W_{R''3}). \tag{14}$$

Thus Eq.(8) , then Zanaboni Theorem, which would be equivalent to

$$0 < W_{R''} < W_{R'}, \tag{15}$$

(See Eq.(1)) , is not deducible from Eq.(12), Eq.(13) and Eq.(14) because of

$$W_{1+2} \neq U_{1+2}, \tag{16}$$

as is reviewed by Zhao [12].

4 Energy Theorem for Zanaboni Problem

4.1 Understanding $U_{R'}$ and $U_{R''}$

From Eq. (2) we know that U_{R1} is the work consisting of the work done by R on the displacement induced by R itself and the work done by R on the displacement induced by P, regardless of the claim in the proof that U_{R1} is the work done by R on C_1 if C_1 were loaded by R alone . Therefore, $U_{R1} + U_{R2}$ is the total work done by R on the displacement of the section S, then $U_{R'}$ and $U_{R''}$ are the total work done by R' and R'' on the displacement of sections S' and S'' respectively.

4.2 Energy Theorem for Zanaboni Problem

If energy decay has to be discussed for Zanaboni Problem, we have, from the understanding of $U_{R'}$ and $U_{R''}$ in the last subsection, that

$$U_{R''} = U_{R'} = 0. \tag{17}$$

where $U_{R'}$ and $U_{R''}$ are the " total " strain energies induced respectively in the related parts. We will prove Eq.(17) in the following subsections.

4.3 Proof of Energy Theorem, Equation of Continuity of Stress and Displacement, First Disproof of Zanaboni Theorem

We consider the section S, which is outside B and cuts the body into two pieces C_1 and C_2 and where R_1 and R_2 are the tractions on the opposite sides of the section respectively (See Fig.2).

We suppose that Cartesian coordinates are established for defining stresses and displacements of the body. Then continuity, across the section, of stresses and displacements results in Eq. (17). In fact, for linear elasticity, the work done by the traction R_1 on the right side of the section, S_1, is

$$W_{R_1} = \frac{1}{2} \int\limits_S \int (\sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij} n_j u_i) \, \mathrm{d}s \tag{18}$$

where τ_{ij} are the stress components at the face S_1, n_j are the direction cosines of the normal to the right face S_1 and u_i are the displacement components of the face S_1.

The work done by the traction R_2 on the left side of the section, S_2, is

$$W_{R_2} = \frac{1}{2} \int\limits_S \int [\sum_{i=1}^{3} \sum_{j}^{3} \tau_{ij} (-n_j) u_i] \, \mathrm{d}s \tag{19}$$

where τ_{ij} are the stress components at the face S_2 because of continuity of stress, $(-n_j)$ are the direction cosines of the normal to the left face S_2 and u_i are the displacement components of the face S_2 because of continuity of displacement. From Eq.(18) and Eq.(19) we have the total work done by R on the section as

$$W_R = W_{R_1} + W_{R_2} = 0. \tag{20}$$

Equation (20) is defined to be the equation of continuity of stress and displacement for Zanaboni's problem.

Using Eq.(20) repeatedly for R' and R'' (or S' and S'') in Fig.1 , it is obtained that

$$W_{R''} = W_{R'} = 0. \tag{21}$$

The total work $W_{R''}$ and $W_{R'}$ are equal to the total induced strain energy $U_{R''}$ and $U_{R'}$ respectively , and so Eq.(17) is deduced from Eq.(21), Zanaboni Theorem Eq.(1) is disproved.

4.4 Another Proof of Energy Theorem, Equation of Energy Conservation, Second Disproof of Zanaboni Theorem

The energy of the body without sectioning (See Fig.2) is

$$U = W_P \tag{22}$$

where W_P is the work done by the load P .

The energy of the imaginarily-sectioned body (See Fig.2) is

$$U_{(C_1+C_2)} = W_P + W_{R_1} + W_{R_2} \tag{23}$$

where W_P is the work done by the load P , W_{R_1} and W_{R_2} are the work done by R_1 and R_2 respectively. It is obtained from Eq.(22) and Eq.(23) that

$$W_R = W_{R_1} + W_{R_2} = 0 \tag{24}$$

because

$$U = U_{(C_1+C_2)}. \tag{25}$$

Equation (24) is defined to be the equation of energy conservation for Zanaboni's problem because of the argument put forward in the next subsection.

Using Eq.(24) repeatedly for R' and R'' (or S' and S'') in Fig.1 , Eq.(21) and then Eq.(17)) are proved, Zanaboni Theorem Eq.(1) is disproved again.

4.5 Absurdity of Zanaboni Theorem Violating the Law of Conservation of Energy

If Zanaboni Theorem Eq.(1) were true , it would be required that

$$W_R = W_{R_1} + W_{R_2} > 0. \tag{26}$$

Then it would be deduced from Eq.(22) , Eq.(23) and Eq.(26) that

$$U_{(C_1+C_2)} - U = W_{R_1} + W_{R_2} > 0, \tag{27}$$

which means energy growth of the body by imaginary sectioning. Then one could accumulate strain energy simply by increasing the "imaginary" cuts sectioning the elastic body. That violates the law of energy conservation because energy would be created from nothing only by imagination, as is reviewed by Zhao [12].

5 Variational Theorems for Zanaboni's Problem, Conditions of Joining

5.1 Variational Theorem of Potential Energy , Condition of Joining, Third Disproof of Zanaboni Theorem

From the construction of the body $C_1 + C_2$ in Section 3.1 (See Fig.2) we know that S_1 and S_2 are the parts of the boundaries of C_1 and C_2 for joining, or the opposite sides of the interface S inside the body $C_1 + C_2$. In the proof of Zanaboni (See [7] and pp 300-303 of [11]), he treats S_1 and S_2 in the latter way because stress-strain relation which is established inside elastic bodies has been used for the argument , that is, when R is considered as varied by the ratio $1:(1+\varepsilon)$, the deformation is considered as varied by a factor $(1+\varepsilon)$. However, to establish the variational theorem of potential energy for Zanaboni's problem , we deal with the structure of the body in the former way, that is :

Considering S_1 and S_2 are the joint boundaries of C_1 and C_2 , the potential energy or the strain energy in the combined body is

$$U^p_{(C_1+C_2)} = W_P + W_{R_1} + W_{R_2}, \tag{28}$$

where $U^p_{(C_1+C_2)}$ is the strain energy stored in $C_1 + C_2$; W_P is the work done by P ; W_{R_1} and W_{R_2} are the work done by R_1 and R_2 respectively.

Suppose that the displacements on S_1 and S_2 are varied by the ratio $1:(1+\varepsilon)$ respectively and the loads R_1 and R_2 remain unchanged, then it is easy to find, for linear elasticity, from Eq.(28), that

$$\delta U^p_{(C_1+C_2)} = \varepsilon(W_{R_1} + W_{R_2}). \tag{29}$$

And the condition of stationarity of $U^p_{(C_1+C_2)}$, according to Eq.(29), is

$$W_R = W_{R_1} + W_{R_2} = 0. \tag{30}$$

Therefore, the variational theorem of potential energy for Zanaboni's problem is:

The potential energy $U^p_{(C_1+C_2)}$ stored in the combined body $C_1 + C_2$ is stationary as the total work W_R done by the load R on the joint surface S equals zero.

Equation (30) is the condition of joining C_1 and C_2 to construct the body $C_1 + C_2$ for Zanaboni's problem, which leads to Eq.(17) because W_R is equal to U_R.

5.2 Variational Theorem of Complementary Energy , Identical Condition of Joining, Fourth Disproof of Zanaboni Theorem

Considering S_1 and S_2 are the joint boundaries of C_1 and C_2 , the complementary energy in the combined body , which is equal to the potential energy in the combined body for linear elasticity, is

$$U^c_{(C_1+C_2)} = W_P + W_{R_1} + W_{R_2}, \tag{31}$$

where $U^c_{(C_1+C_2)}$ is the complementary energy in $C_1 + C_2$; W_P is the work done by P ; W_{R_1} and W_{R_2} are the work done by R_1 and R_2 respectively.

Suppose that R_1 and R_2, the loads on S_1 and S_2, are varied by the ratio $1:(1+\varepsilon)$ respectively and the displacements on S_1 and S_2 remain fixed without variation, then it is easy to find, from Eq.(31), that

$$\delta U^c_{(C_1+C_2)} = \varepsilon(W_{R_1} + W_{R_2}). \tag{32}$$

And the condition of stationarity of $U^c_{(C_1+C_2)}$, according to Eq.(32), is

$$W_R = W_{R_1} + W_{R_2} = 0. \tag{33}$$

Therefore, the variational theorem of complementary energy for Zanaboni's problem is:

The complementary energy $U^c_{(C_1+C_2)}$ in the combined body $C_1 + C_2$ is stationary as the total work W_R done by the load R on the joint surface S equals zero.

Equation (33) is the condition of joining identical to Eq.(30) for Zanaboni's problem, which leads to Eq.(17) because W_R is equal to U_R.

We emphasize the consistency, equivalence or identity of the equation of continuity of stress and displacement , Eq.(20), the equation of energy conservation, Eq.(24) and the condition of joining, Eq.(30) or Eq.(33), and each of them results in Eq.(17), instead of Eq.(1).

6 Zanaboni Theorem and Saint-Venant's Principle

Boussinesq, Mises and Sternberg try to express Saint-Venant's Principle in terms of stress or dilatation [3,5-6] , but Zanaboni Theorem tries to express Saint-Venant's Principle mathematically in terms of work and energy [7, 11]. This " pioneer " work has profound influence on study of the principle.

Fung translates and includes Zanaboni Theorem and its proof in his " textbook " as " one possible way to formulate Saint-Venant's principle with mathematical precision ", declaring " the principle is proved ". [11]

Toupin, however, does not evaluate Zanaboni Theorem with high opinion. He remarks at first that

" While the theorems of Boussinesq, von Mises , Sternberg and Zanaboni have independent interest, I have been unable to perceive an easy relationship between these theorems and the Saint-Venant Principle " [13] , then comments in another way in Ref.[10]:

" In 1937, O. Zanaboni proved an important theorem for bodies of general shape which begins to restore confidence in Saint-Venant's and our own intuition about the qualitative behavior of stress fields. " He continues his remark by saying that

" It is possible to sharpen Zanaboni's qualitative result and to derive a quantitative estimate for the rate at which the elastic energy diminishes with distance from the loaded part of the surface of an elastic body." Toupin's results are cited and explained afterwards.

It seems that the establishment of the well-known Toupin Theorem of energy decay should be the achievement of sharpening Zanaboni's " qualitative " result. [10, 13] However, Horgan and Knowles review Zanaboni's work, saying

“ The notion of examining the distribution of strain energy in an elastic body apparently first appeared in papers concerned with Saint-Venant's principle by Zanaboni (1937a,b,c); Zanaboni did not, however, estimate the rate of decay of energy away from the loaded portion of the boundary , and his results do not appear to be directly related to those of Toupin (1965a) or Knowles (1966). ” [14]

It seems that Horgan and Knowles do not qualify mathematically Zanaboni's results for formulation of Saint-Venant's Principle. Zanaboni's results are not valued highly. [14, 15]

Considering its influence on the history of Saint-Venant's Principle, further academic survey of Zanaboni's results is inevitable. Our result of mathematical analysis in this paper tells that Zanaboni Theorem is invalid, and so the argument, put forward by Fung [11], that Zanaboni Theorem is a mathematical formulation or proof of Saint-Venant's Principle is unreasonable.

7 Conclusion

A. Zanaboni Theorem is not true, violating the law of energy conservation. Four disproofs of the theorem are given in this paper.

B. Zanaboni's “ proof ” of Zanaboni Theorem is wrong for its confusion of work and energy.

C. Energy Theorem for Zanaboni Problem is suggested and proved.

D. The equations and conditions established in this paper for disproof of Zanaboni Theorem, the equation of continuity of stress and displacement, the equation of energy conservation and the condition of joining, are consistent with , equivalent or identical to each other.

E. Zanaboni Theorem is, for its invalidity, not a mathematical formulation or proof of Saint-Venant's Principle.

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