Several Metrical Relations Regarding the Anti-Bisectrix, the Anti-Symmedian, the Anti-Height and their Isogonal

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We suppose known the definitions of the isogonal cevian and isometric cevian; we remind that the anti-bisectrix, the anti-symmedian, and the anti-height are the isometrics of the bisectrix, of the symmedian and of the height in a triangle.

It is also known the following Steiner (1828) relation for the isogonal cevians AA_1 and AA_1':

$$\frac{BA_1}{CA_1}\cdot\frac{BA_1'}{CA_1'}=\left(\frac{AB}{AC}\right)^2$$

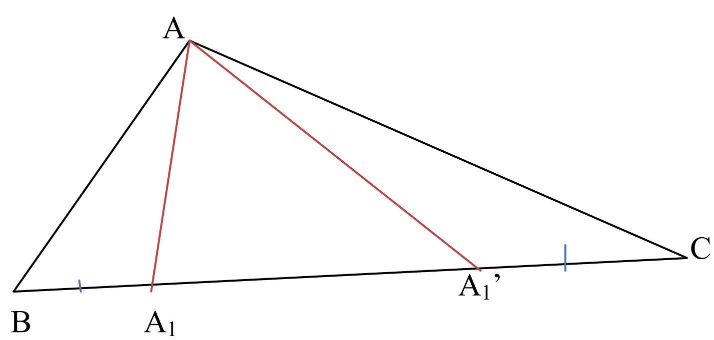
We'll prove now that there is a similar relation for the isometric cevians

Proposition

In the triangle ABC let consider AA_1 and AA_1' two isometric cevians, then there exists the following relation:

$$\frac{\sin\left(\widehat{BAA_1}\right)}{\sin\left(\widehat{CAA_1}\right)}\cdot\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)}=\left(\frac{\sin B}{\sin C}\right)^2 \qquad (*)$$

Proof



The sinus theorem applied in the triangles ABA_1, ACA_1 implies (see above figure)

$$\frac{\sin\left(\widehat{BAA_1}\right)}{BA_1}=\frac{\sin B}{AA_1} \qquad (1)$$

$$\frac{\sin\left(\widehat{CAA_1}\right)}{CA_1}=\frac{\sin C}{AA_1} \qquad (2)$$

From the relations (1) and (2) we retain

$$\frac{\sin\left(\widehat{BAA_1}\right)}{\sin\left(\widehat{CAA_1}\right)} = \frac{\sin B}{\sin C} \cdot \frac{BA_1}{CA_1} \qquad (3)$$

The sinus theorem applied in the triangles ACA_1', ABA_1' leads to

$$\frac{\sin\left(\widehat{CAA_1'}\right)}{A_1'C} = \frac{\sin C}{AA_1'} \qquad (4)$$

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{BA_1'} = \frac{\sin B}{AA_1'} \qquad (5)$$

From the relations (4) and (5) we obtain:

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{\sin B}{\sin C} \cdot \frac{BA_1'}{CA_1'} \qquad (6)$$

Because $BA_1 = CA_1'$ and $A_1C = BA_1'$) the cevians being isometric), from the relations (3) and (6) we obtain relation (*) from the proposition's enouncement.

Applications

1. If AA_1 is the bisectrix in the triangle ABC and AA_1' is its isometric, that is an anti-bisectrix, then from (*) we obtain

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{\sin B}{\sin C}\right)^2 \qquad (7)$$

Taking into account of the sinus theorem in the triangle ABC we obtain

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \qquad (8)$$

2. If AA_1 is symmedian and AA_1' is an anti-symmedian, from (*) we obtain

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^3$$

Indeed, AA_1 being symmedian it is the isogonal of the median AM and

$$\frac{\sin\left(\widehat{MAB}\right)}{\sin\left(\widehat{MAC}\right)} = \frac{\sin B}{\sin C}$$ and

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{\sin\left(\widehat{MAC}\right)}{\sin\left(\widehat{MAB}\right)} = \frac{\sin C}{\sin B} = \frac{AB}{AC}$$

3. If AA_1 is a height in the triangle ABC, $A_1 \in (BC)$ and AA_1' is its isometric (anti-height), the relation (*) becomes.

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

Indeed

$$\sin\left(\widehat{BAA_1'}\right) = \frac{BA_1}{AB};\ \sin\left(\widehat{CAA_1'}\right) = \frac{CA_1}{AC}$$

therefore

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{AC}{AB} \cdot \frac{BA_1}{CA_1}$$

From (*) it results

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \frac{AC}{AB} \cdot \frac{CA_1}{BA_1}$$

or

$$CA_1 = AC \cdot \cos C \text{ and } BA_1 = AB \cdot \cos B$$

therefore

$$\frac{\sin\left(\widehat{BAA_1'}\right)}{\sin\left(\widehat{CAA_1'}\right)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

4. If AA_1'' is the isogonal of the anti-bisectrix AA_1' then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^3 \quad \text{(Maurice D'Ocagne, 1883)}$$

Proof

The Steiner's relation for AA_1'' and AA_1' is

$$\frac{BA_1''}{A_1''C} \cdot \frac{BA_1'}{A_1'C} = \left(\frac{AB}{AC}\right)^2$$

But AA_1 is the bisectrix and according to the bisectrix theorem $\frac{BA_1}{CA_1} = \frac{AB}{AC}$ but $BA_1' = CA_1$ and $A_1'C = BA_1$ therefore

$$\frac{CA_1'}{BA_1'} = \frac{AB}{AC}$$

and we obtain the D'Ocagne relation

5. If in the triangle ABC the cevian AA_1'' is isogonal to the symmedian AA_1' then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^4$$

Proof

Because AA_1 is a symmedian, from the Steiner's relation we deduct that

$$\frac{BA_1}{CA_1} = \left(\frac{AB}{AC}\right)^2$$

The Steiner's relation for AA_1'', AA_1' gives us

$$\frac{BA_1''}{A_1''C} \cdot \frac{BA_1'}{CA_1'} = \left(\frac{AB}{AC}\right)^2$$

Taking into account the precedent relation, we obtain

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^4$$

6.

If AA_1'' is the isogonal of the anti-height AA_1' in the triangle ABC in which the height AA_1 has $A_1 \in (BC)$ then

$$\frac{BA_1''}{A_1''C} = \left(\frac{AB}{AC}\right)^3 \cdot \frac{\cos B}{\cos C}$$

Proof

If AA_1 is height in triangle ABC $A_1 \in (BC)$ then

$$\frac{BA_1}{A_1C} = \frac{AB}{AC} \cdot \frac{\cos B}{\cos C}$$

Because AA_1' is anti-median, we have $BA_1 = CA_1'$ and $A_1C = BA_1'$ then

$$\frac{BA_1''}{A_1''C} = \frac{AC}{AB} \cdot \frac{\cos C}{\cos B}$$

Observation

The precedent results can be generalized for the anti-cevians of rang k and for their isogonal.