On the Fundamental Nature of the Quantum Mechanical Probability Function

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Abstract

The probability of occurrence of an event or that of the existence of a physical state has no relative existence in the sense that motion is strongly believed to only exist in the relative sense. If the probability of occurrence of an event or that of the existence of a physical state is known by one observer, this probability must be measured to have the same numerical value by any other observer anywhere in the Universe. If we accept this bare fact, then, the probability function can only be a scalar. Consequently, from this fact alone, we argue that the quantum mechanical wavefunction can not be a scalar function as is assumed for the Schrödinger and the Klein-Gordon wavefunctions. This has fundamental implications on the nature of the wavefunction insofar as translations from one reference system to the other is concerned.

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"My work always tried to unite the Truth with the Beautiful,
but when I had to choose one or the other,
I usually chose the Beautiful."

– **Hermann Klaus Hugo Weyl** $(1885 - 1955)$

1 Introduction

The probability of occurrence of an event or that of the existence of a physical state has no relative existence in the sense that motion is strongly believed to only exist in the relative sense where observers will in general no agree on the numerical value of the speed of an object. For example, if I have 6 similar balls that differ only in their color such that 2 are white and 4 are black and these are placed in a closed container such that one ball is drawn at random, then, the probability of picking a white ball first is distinctly $1/3$. It really does not matter the relative state of motion between me and any observer anywhere in the Universe – logic and physical reality dictates and compels that, they too will measure the same probability for picking a white ball first as $1/3$.

Generalizing the above thesis, it follows that if the probability of occurrence of an event or that of the existence of a physical state is known by one observer, this probability must be measured to have the same numerical value by any other observer anywhere in the Universe. If we accept this bare fact, then, any probability function can only be a scalar because only scalars have this property that what one observer measures at any given time and place, every other observer must measure and find the same numerical value

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for that quantity at the corresponding time and place for that observer. Consequently, from this fact alone, it is crystal clear that the quantum mechanical wavefunction can not be a scalar function as is assumed for the Schrödinger and the Klein-Gordon wavefunctions. This has fundamental implications on the nature of the wavefunction insofar as translations from one reference system to the other is concerned.

The synopsis of this reading is as follows: in the subsequent section, we present an unequivocal argument that clearly demonstrates that no quantum mechanical wavefunction can ever be a pure scalar. In the section that follows, we generalize the properties of the quantum probability function to include dynamic probabilities. In section (4) we show that the required transformation properties of the quantum mechanical wavefunction naturally complements (or is demanded by) the proposed *Unified Field Theory* proposed in the reading Nyambuya (2010). In section (5), we give a general discussion and the conclusion drawn thereof.

2 Scalar Quantum Mechanical Probability Function

If Ψ is the wavefunction of a particle, then, the probability that this particle will be found in the region $\boldsymbol{r}$ and $\boldsymbol{r}_0$ is:

$$P(\boldsymbol{r},\boldsymbol{r}_0)=\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\Psi^{\dagger}\Psi dx^3dx^2dx^1. \tag{1}$$

If we have two systems of reference (traditionally denoted the primed and the unprimed) that are in a state of relative motion, then, for the primed system of reference, we will have:

$$P'(\boldsymbol{r}',\boldsymbol{r}_0')=\int_{\boldsymbol{r}_0'}^{\boldsymbol{r}'}\Psi'^{\dagger}\Psi' dx'^3dx'^2dx'^1. \tag{2}$$

If Ψ is a scalar, then $\Psi'=\Psi$. For the differentials dx^j, they transform as:

$$dx^{j'}=\frac{\partial x^{j'}}{\partial x^j}dx^j, \tag{3}$$

so that:

$$P'(\boldsymbol{r}',\boldsymbol{r}_0')=\int_{t_0}^{t}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\Psi^{\dagger}\Psi\left(\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^1}{\partial x^1}\right)dx^3dx^2dx^1dx^0\neq P(\boldsymbol{r},\boldsymbol{r}_0). \tag{4}$$

If $P(\boldsymbol{r},\boldsymbol{r}_0)$ is to be a scalar is argued, we must have $P'(\boldsymbol{r}',\boldsymbol{r}_0')=P(\boldsymbol{r},\boldsymbol{r}_0)$. If we are to have $P'(\boldsymbol{r}',\boldsymbol{r}_0')=P(\boldsymbol{r},\boldsymbol{r}_0)$ as logic and physical reality compels, then $\Psi'=S\Psi$, such that:

$$S^{\dagger}S\left(\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^1}{\partial x^1}\right)=\lambda\left(\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^1}{\partial x^1}\right)\mathcal{I}=1, \tag{5}$$

where $S^{\dagger}S=\lambda\mathcal{I}$. In this way, we achieve the desired result $P'(\boldsymbol{r}',\boldsymbol{r}_0')=P(\boldsymbol{r},\boldsymbol{r}_0)$. Clearly, the probability amplitude Ψ, nor the probability density function $\rho=\Psi^{\dagger}\Psi$, can not be a scalar.

3 Dynamic Quantum Mechanical Probability Function

According to Heisenberg's quantum mechanical energy-time uncertainty principle $\Delta E\Delta t=\hbar$ (where ΔE and Δt are the energy and time uncertainity respectively and $\hbar$ is Planck's normalised constant), it is impossible to make a measurement in a zero interval time. The probability $P(\boldsymbol{r},\boldsymbol{r}_0)$ and given in (1) is the probability of occurrence of a quantum mechanical event at a given instant in time *i.e.* in zero time interval. As just stated, this is not permitted by Heisenberg's uncertainty principle. All measurements must be made in a finite time interval. To take this into account, the probability function must in-cooperate in it, this finite time interval in which this event can or will occur, that is, we must have:

$$P'(\boldsymbol{r},\boldsymbol{r}_0;t,t_0)=\int_{t_0}^{t}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\Psi^{\dagger}\Psi dx^3dx^2dx^1dx^0. \tag{6}$$

In most if not all cases considered in QM, the probability density function $\Psi^{\dagger}\Psi$ is a static function because $\Psi \propto e^{iEt/\hbar}$. Because of this, the time dependence of the measurement process is usually ignored completely. For the sake of completeness and thoroughness in our effort to stay within the permissible physical bounds, the correction made in (6) is just and valid.

Now, under a translation of the reference system, we must have $P'(\boldsymbol{r}',\boldsymbol{r}_0';t',t_0')=P(\boldsymbol{r},\boldsymbol{r}_0;t,t_0)$, if the probability function is to be a scalar as argued. As before, the wavefunction will transform as $\Psi'=S\Psi$ and the coordinates as:

$$dx^{\mu'}=\frac{\partial x^{\mu'}}{\partial x^{\mu}}dx^{\mu}. \tag{7}$$

Again, S is such that $S^{\dagger}S=\lambda\mathcal{I}$. For $P'(\boldsymbol{r}',\boldsymbol{r}_0';t',t_0')=P(\boldsymbol{r},\boldsymbol{r}_0;t,t_0)$, we must have:

$$S^{\dagger}S\left(\frac{\partial x'^3}{\partial x^3}\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^1}{\partial x^1}\frac{\partial x'^0}{\partial x^0}\right)=\lambda\left(\frac{\partial x'^3}{\partial x^3}\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^1}{\partial x^1}\frac{\partial x'^0}{\partial x^0}\right)=1. \tag{8}$$

In this way, the dynamic probability function is a scalar function as required.

4 Link to the Proposed Unified Field Theory

What really motivated us to take a closer look into the transformational nature of the quantum mechanical is the *Unified Field Theory* (UFT) presented in Nyambuya (2010). It is important to note that the thesis put forward in §(1), (2) & (3) is independent of the what we shall say in the present section about the work Nyambuya (2010). This work (Nyambuya 2010) is actually what got us thinking on the nature of the QM wavefunction.

For the interested reader, we shall try and clearly elaborate on this effort (Nyambuya 2010) so as to give a clear perspective of where we are coming and where we are going with all this. With all having been said and done – at the end of it all; the proposed UFT (Nyambuya 2010) is actually an attempt at improving on Weyl (1918, 1927*a*,7)'s failed unified theory of gravitation and electricity. In a nutshell, what Weyl (1918) did was to supplement the usual metric of spacetime $g_{\mu\nu}$ with a scalar function $e^{-2\phi}$, so that the metric usual metric of Riemann spacetime $g_{\mu\nu}$ is transformed to a new metric $\hat{g}_{\mu\nu}$, that is:

$$\hat{g}_{\mu\nu}=e^{-2\phi}g_{\mu\nu}, \tag{9}$$

so that the corresponding line element of the emergent spacetime is $d\hat{s}^2=e^{-2\phi}g_{\mu\nu}dx^{\mu}dx^{\nu}$. With such a line element, Weyl obtained that the new affine connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$ belonging to this kind of spacetime is such that:

$$\hat{\Gamma}^{\alpha}_{\mu\nu}=-\Gamma^{\alpha}_{\mu\nu}+W^{\alpha}_{\mu\nu}, \tag{10}$$

where $W^{\alpha}_{\mu\nu}$ is the Weyl tensor and $\Gamma^{\alpha}_{\mu\nu}$ is the usual Christophel symbol of Riemann geometry. Weyl's geometry tends to Riemann geometry as $W^{\alpha}_{\mu\nu}\longrightarrow 0$. To try and find a link between electricity and gravitation, Weyl carefully choose the function ϕ so that $\phi=A_{\mu}x^{\mu}$ where A_{μ} is some four vector. If $\phi=A_{\mu}x^{\mu}$, and just as in pure Riemann geometry, the covariant derivative of the metric $\hat{g}_{\mu\nu}$ is upheld, *i.e.* $\hat{g}_{\mu\nu;\alpha}=0$, then:

$$W^{\alpha}_{\mu\nu}=\delta^{\alpha}_{\mu}A_{\nu}+\delta^{\alpha}_{\nu}A_{\mu}-g_{\mu\nu}g^{\alpha\lambda}A_{\lambda}. \tag{11}$$

What deeply intrigued Weyl and many others (including ourself) that came to admire the theory, is the '*seemingly divine and heaven sent*' fact that the Weyl connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$, is invariant under the following transformation:

$$
\begin{array}{lcl}
g_{\mu\nu} & \longmapsto & e^{\chi} g_{\mu\nu}, \\
A_{\mu} & \longmapsto & A_{\mu} + \partial_{\mu}\chi,
\end{array}
\tag{12}
$$

where $\chi = \chi(x)$ is an arbitrary scalar function. If $\hat{\Gamma}^{\alpha}_{\mu\nu}$ is invariant *i.e.* $\delta\hat{\Gamma}^{\alpha}_{\mu\nu} \equiv 0$, the curvature tensor $\hat{R}^{\alpha}_{\mu\lambda\nu}$ is also invariant *i.e.* $\delta\hat{R}^{\alpha}_{\mu\lambda\nu} \equiv 0$. Given that Weyl knew very well that Maxwellian electrodynamics is described by a four vector such that the entire Maxwellian electrodynamics is invariant under the transformation (12 b) – without wasting much time, Weyl seized the golden moment and identified A_{μ} with the Maxwellian four vector potential of electrodynamics. At this point, one can not help but endlessly and deeply admire Weyl's brilliantly convinced theory, and on the other hand, one can only be irretrievably and deeply sad to know that this theory, despite its esoteric grandeur and exquisite beauty, it does not have any correspondence with physical experience as we know it.

From a 'safe distance', the great Albert Einstein $(1879 - 1955)$ was the first to publicly exhibit his passionate *albeit* backhanded admiration of Weyl's theory, he said of it:

> "*... apart from the agreement with reality,*
> *it is at any rate a grandiose achievement of the mind ...*
> *a first class genius.*"

(Abraham Pais 2005, *Subtle is the Lord*, p. 341)

With equal passion, he made the one all-enduring 'aerial bombardment' to it, a bombardment from which Weyl's theory would never recover to this day. That is, the agile Einstein was quick and to point out that in Weyl's geometry, the frequency of the spectral lines of atomic clocks from different portions of the distant heavenly spaces would depend on the location and pre-histories of the atoms. This is in fragment disagreement with experience. The spectral lines are well-defined and sharp; they very strongly appear to be independent of an atom's pre-history. Atomic clocks define units of time, and experience shows they are integrally transported from one portion of the heavens to the other. So, with this criticism alone, Einstein gave Weyl's theory a stillbirth with his backhanded compliant. Weyl's brilliant and beautiful theory was hopelessly thwarted and, to no avail, he made last ditch effort to save his theory in latter year (Weyl 1927*a*,7). Einstein's criticism lay deep in the nimbus of the foundation stone of Weyl's theory, which is that the length of a vector varied from one point of spacetime to another. In wrapping-up his criticism, he [Einstein] said:

> "*... I do not believe that his theory will hold its ground in relation to reality.*"

(Einstein 1952, *Sidelights of Relativity*, p. 23)

Much for the great Hermann Weyl and his *all-brilliant, beautiful but 'failed' theory*. Be that it may, the theory presented in Nyambuya (2010) is a series of radical, modest yet subtly ambitious improvements on Weyl (1918)'s theory. The first and most important of all the improvements is that the role of the conformal object ϕ added by Weyl now (in Nyambuya 2010) takes a new role. It is now required of it that the Weyl affine connection $W^{\alpha}_{\mu\nu}$ must be constrained such that, at the end of the day when all is said and done, $\hat{\Gamma}^{\alpha}_{\mu\nu}$ is a tensor. In Weyl (1918)'s theory, $\hat{\Gamma}^{\alpha}_{\mu\nu}$ transforms much the way as the three Christophel symbol $\Gamma^{\alpha}_{\mu\nu}$ which transforms as:

$$
\Gamma^{\alpha'}_{\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \Gamma^{\alpha}_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'} \partial x^{\nu'}}. \tag{13}
$$

If $\hat{\Gamma}^{\alpha}_{\mu\nu}$ is to be a tensor, then $W^{\alpha}_{\mu\nu}$ must transform as:

$$
W^{\alpha'}_{\mu'\nu'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} W^{\alpha}_{\mu\nu} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'} \partial x^{\nu'}}. \tag{14}
$$

If $W^{\alpha}_{\mu\nu}$ is to transform as suggested above, then, the Weyl's four vector A_{μ} must seize to be a vector, it must transform as:

$$A_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} A_{\mu} + \frac{\partial^2 x^{\alpha}}{\partial x^{\mu} \partial x^{\alpha'}}. \qquad (15)$$

In Weyl's conformal connection (9), if we set $\rho = \Psi^{\dagger}\Psi = e^{2\phi}$, then $\hat{g}_{\mu\nu} = \rho g_{\mu\nu}$, so that:

$$A_{\mu} = \frac{1}{\rho} \frac{\partial \rho}{\partial x^{\mu}}. \qquad (16)$$

Now, if ρ transforms as $\rho' = \lambda\rho$, then:

$$A_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} A_{\mu} + \frac{1}{\lambda} \frac{\partial \lambda}{\partial x^{\mu'}}. \qquad (17)$$

It is not difficult to see that if λ is defined as it is defined in (8), then the object A_{μ} clearly transforms as desired, that is, it transforms as required in (15), thus leading us naturally to our desired tensorial affine connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$.

In Nyambuya (2010), the object ρ has been identified with the quantum mechanical probability density function. To be more specific, it has been identified with the Dirac probability density function $\rho = \psi^{\dagger}\psi$ where ψ is the Dirac four component. What was not clear at the time of writing down this theory is that the transformation properties required of this object would lead to the quantum probability function $[P(\boldsymbol{r}, \boldsymbol{r}_0; t, t_0) = \int_{t_0}^{t} \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \psi^{\dagger}\psi dx^3 dx^2 dx^1 dx^0]$ having its required scalar properties as argued in $\S(1)$, (2) and (3). The fact that this is so – in our modest view, this points to the ideas presented in Nyambuya (2010) as containing in them an element of truth in them.

5 Discussion and Conclusion

We have argued here that the QM wavefunction can not be a scalar as we have long assumed. So doing (*i.e.* assuming Ψ to be a scalar) leads to a probability function that has not the desired scalar properties for a probability function, which is that it must be a scalar. Simple as the arguments presented herein may be, they have profound implications on our foremost understanding of the QM. The main result of this rather brief study is that Ψ can not be a scalar.

5.1 Conclusion

Assuming the correctness or the plausibility of the ideas presented herein, we hereby make the following conclusions:

1. *If the quantum mechanical probability measure is defined as it is defined in quantum mechanics, then, it is not possible to have scalar wavefunctions. The quantum mechanical probability measure is what must be a scalar and not the wavefunction.*

2. *The required transform properties of the wavefunction for a scalar quantum mechanical probability measure is what is naturally required to obtain tensorial connections in the proposed UFT presented in Nyambuya (*2010*). We strongly believe that this fact certainly gives credence to this UFT.*

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