The connection between the Riemann Hypothesis and model theory

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Abstract. In this paper, I present a connection between the Riemann Hypothesis and model theory, and this connection leads to a possible proof of the Riemann Hypothesis.

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Introduction.

The Riemann Hypothesis has been an open problem for a long time. This is an attempt to give a proof of the hypothesis based mainly on model theory. Also essential here are known results that were first given by Littlewood and Robin.

Section 1. Preliminary facts.

The following reformulations have been known for some time and the proofs of the statements in this section can be found in many references.

Robin's reformulation of RH [7]**.** The Riemann Hypothesis is true if and only if there is an n_0 (and in fact $n_0 = 5041$) such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ (here $\sigma(n)$ is the sum of divisors function).

Littlewood's reformulation of RH [1]**.** The Riemann Hypothesis is equivalent to the statement that for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$ (here M(x) is the Mertens' function).

We write (R) for the statement in Robin's reformulation (Robin inequalities). We also write (L) for the statement in Littlewood's reformulation (that is $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$).

We can conclude that the statement:

"there is an n_0 such that $\sigma(n)/n < e^{\gamma} \cdot \log(\log(n))$, for all $n > n_0$ "

is equivalent to the statement:

"for every $\varepsilon > 0$, we have $M(x) = O(x^{(1/2 + \varepsilon)})$, when $x \to \propto$".

We will write (R) $\Leftrightarrow$ (L) for this equivalence (which is a known result).

Section 2. Model Theory.

Some of the following theorems will be used in our results (for a brief introduction to model theory, see the appendix).

Definition. Let $\aleph_\alpha$ be an uncountable cardinal. A model M is said to be $\aleph_\alpha$ - saturated if for every set Φ (of fewer than $\aleph_\alpha$ formulas) of formulas $\varphi(x)$ in the diagram language of M, if for every finite subset of formulas $\varphi_1, \varphi_2, \varphi_3, \ldots.\varphi_n \in \Phi$ the sentence $\exists x\, (\varphi_1(x) \wedge \varphi_2(x) \wedge \varphi_3(x) \wedge \ldots. \wedge \varphi_n(x))$ is true in M, then the infinitely long sentence $\exists x\, (\bigwedge_{\varphi \in \Phi} \varphi(x))$ is also true in M.

We also note the following theorem on the existence of saturated models, due to Keisler.

Theorem (Keisler, 1963) [2]. Let I be a set of power $\aleph_\alpha$. There is an ultrafilter D over I such that every ultraproduct $\Pi_D M_i$ is $\aleph_{\alpha+1}$ - saturated.

Observation 1. We also note the following facts. Any model of Peano arithmetic which occurs as the integers in some model of non - standard analysis is recursively saturated (but this property will not be used in the following). We also know (see [2]) that if a model M is α - saturated and infinite, then $|A| \geq \alpha$. We will work in an enlargement R* of R, where N* is a definable subset of R*, also N* is a model of the Peano system of axioms.

Definitions. We write N for the natural numbers, and N* for the corresponding set in R*. We also write (R*) for the statement in Robin's reformulation in R*, in other words (R*) will be the statement:

"for any $n \in$ N and $n > 5041$ the relations $\sigma^*(n)/n < e^\gamma \cdot \log^*(\log^*(n))$ are all satisfied."*

We also write (L*) for the statement in Littlewood's reformulation in R*, in other words (L*) will be the statement:

"for every $\varepsilon > 0$, $\varepsilon \in R^*$ there is a $K > 0$, $K \in R^*$, such that $|M^*(x)| < K \cdot x^{(1/2 + \varepsilon)}$ for every x in N*"

As a notation, we must distinguish between (R*) - Robin's reformulation, and R* which is the enlargement of standard analysis on R. We also note that when we work in R*, all operations, relations, and functions are transferred to R* (but we will not always write <*, M*(x), σ*(n), and so on, it will be clear from the context).

Observation 2. We will briefly discuss ultrapowers. Let R be the set of real numbers and F any nonprincipal ultrafilter over ω (the natural numbers, we try to follow the notation from [4]. We could write $^{\omega}R$ or ^{N}R for the set of functions with domain N and values in R, in other words, the set of infinite sequences of real numbers). Then we can form the ultra power $^{\omega}R/F$ independent of any logical considerations. The map ur = {< r : i ∈ ω >} for all r ∈ R is a one - one map of R into $^{\omega}R/F$. There is then a set R* ⊇ R and a bijection t : $^{\omega}R/F$ → R* such that t ° u is the identity on R. Once F, u, R* and t are fixed, the construction follows. Given any relation S on R, we can take a suitable language for the structure < R, S > and again form the ultrapower $^{\omega}$< R, S >/F. Then there is a relation S* on R* such that t is an isomorphism from $^{\omega}$< R, S >/F onto < R*, S*>. Then u is an elementary embedding of < R, S > into $^{\omega}$< R, S >/F. Any other operations on R can be extended to operations on R*. We obtain structures on R* which are elementary extensions of the original structures on R, and we can carry over facts from R to R* and vive - versa. Members of R* are called nonstandard real numbers.

Section 3. The main theorems.

Theorem 1. In the enlargement R* of standard analysis on R described above (observation 1, 3 and appendix 1), Littlewood's reformulation (L*) is a true statement.

Proof. Now we consider the statement (L*):

"for every ε > 0, ε ∈ R there is a K > 0, K ∈ R*, such that* $/M^*(x)\ / < K \cdot x^{(1/2+\varepsilon)})$ *for every x in* N*".

The elements of N* will be linearly ordered. We have an initial segment that is isomorphic with the series of standard natural numbers, followed by a sequence of segments, each of which is isomorphic with the sequence of integers. There is neither an earliest nor a latest member of this sequence, and between any two segments in it there is a third. The proof of this theorem depends on the construction (using ultrapowers) of a nonstandard model in which (L*) is true. In appendix 1 we will give a proof of this theorem conditioned by the validity of a lemma related to Littlewood sequences. **QED**.

Theorem 2. The equivalence (R*) ⇔ (L*) is true in R*.

Proof. Every mathematical statement which is meaningful and true for the system of analysis is meaningful and true also for R*, provided that we interpret any reference to entities of any given type (sets, relations, functions, etc.) in R* in terms of internal entities of that type. In the standard model it is known that (R) ⇔ (L). We can transfer this known result to the enlargement R*. This means that (R*) ⇔ (L*) is true in R*. The proof that (R) ⇔ (L) is true in standard analysis is long and it requires the introduction of several notions (for example, the extension ζ*(z)), but we work in exactly the same system of axioms, and everything can be transferred in R*, and we will have (L*) ⇔ (RH*) ⇔ (R*). **QED**.

Theorem 3. Riemann's Hypothesis (RH) is true in the standard model.

Proof. We note that any counterexample to (R) in the standard model can also be considered a counterexample to (R*) in R*. This means that (R*) $\Rightarrow$ (R). From theorem 1 we know that (L*) is true in R*. From theorem 2 we know that (R*) $\Leftrightarrow$ (L*) is true in R*. In other words we have (L*) $\Leftrightarrow$ (R*) $\Rightarrow$ (R). That means that the statement from Robin's reformulation (R) is true in the standard model. It is known (in the standard model) that (L) $\Leftrightarrow$ (RH) $\Leftrightarrow$ (R). That means that the Riemann's Hypothesis is true in the standard model. **QED**.

Observation 3. I am grateful to Professor Feferman, Professor Haskell, Professor Scanlon and specially to Professor Keisler for their observations and suggestions (and the correction of many errors in the first versions of the article). Any other errors still present in this article (if any) belong to the author (Cristian Dumitrescu), but the observations of the model theory experts above corrected many errors present in the first versions of this article. I also emphasize that the current version of this proof has not yet passed the expert analysis, at this point. The main error that I made in the previous versions of this proof is when applying the $\aleph_{\alpha+1}$ - saturation property in a model that was not suitable for the purpose of this proof.

Conclusion. The proof of the Riemann Hypothesis can be based on model theory, and on the results of Littlewood and Robin.

Appendix 1. In this appendix, we will give the conditional proof of theorem 1. The construction will be based on Frechet filters and free ultrafilters.

Definition. We can extend the Mertens function to the real numbers by defining $M_R(x)$ = M([x]), where x is a real number, [x] is the largest integer less or equal to x and M([x]) is the value of the Mertens function at [x].

Definition. Let's consider an infinite sequence of real numbers $x = (x_0, x_1, x_2, \ldots\ldots\ldots)$. We will call the sequence x a **Littlewood sequence** iff there is a positive real number K such that $|M_R(x_i)| \leq K \cdot x_i^{1/2}$ for any natural number i. In other words, each term of the infinite sequence satisfies the inequality above, for some positive real number K.

Lemma. The set L of Littlewood sequences is closed under addition and multiplication, each defined component wise. In other words, if $x = (x_0, x_1, x_2, \ldots\ldots\ldots)$ and $y = (y_0, y_1, y_2, \ldots\ldots\ldots)$ are two Littlewood sequences, then $x + y = x = (x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots\ldots\ldots\ldots)$ and $x \cdot y = (x_0 \cdot y_0, x_1 \cdot y_1, x_2 \cdot y_2, \ldots\ldots\ldots\ldots)$ are also Littlewood sequences.

Proof. At this moment, I do not have a proof of this lemma, which I will consider valid in the following.

We consider the ideal D in the ring (assuming the lemma above valid) L of Littlewood sequences for which all but a finite number of terms are zero.

There exists a maximal ideal M having D as a subset (from Zorn‘s lemma).

The relation $a \cong b \Leftrightarrow a - b \in M$ is an equivalence relation. We denote the set of equivalence classes by R*. In other words, we have R* = L/M.

The maximality of M ensures that L/M is a field.

For $x = (x_0, x_1, x_2, \ldots\ldots\ldots) \in L$, we define:

$Z(x) = \{ n \in N : \quad x_n = 0\}$.

We form $U = U_M = \{ Z(x) : x \in M\}$.

We note that U is a filter on N. If M is maximal, then U_M is an ultrafilter. If M included D, then U_M is a non - trivial ultrafilter.

We note that in R* = L/M we have Littlewood statement (L*) satisfied because we can take K in R* defined by an increasing (to infinity) sequence of real numbers, in other words, we can take K an infinite nonstandard number. In this case, all the required inequalities will be eventually satisfied.

The proof of lemma 1 does not seem to be easy. Most likely, analytical number theory is required to give a proof of the lemma. If the lemma above is not true (and a counterexample can be found), then there might still exist other ways to prove that there exists a nonstandard model R* in which (L*) is true (by construction).

In any case, we can then prove the following conditional theorem:

Theorem 4. If the set L of Littlewood sequences is closed under addition and multiplication, each defined component wise, then the Riemann Hypothesis is true.

The proof follows immediately from the discussion in this appendix.

Appendix 2. In this appendix, we will briefly present some facts about model theory. Model theory is a combination of universal algebra and logic. We have a set L of symbols for operations, constants and relations, called a language.

Example. $L = \{+, \cdot, 0, 1, <\}$. The language L can be finite or countable. A model M for the language L is an object of the form $M = < A, +_M, \cdot_M, 0_M, 1_M, <_M \quad >$. A is a non - empty set, called the set of elements of M, and $+_M$ and $\cdot_M$,are binary operations on $A \times A$ into A, 0_M and 1_M are elements of A, and $<_M$ is a binary relation on A. Examples. The field of rationals $< Q, +, \cdot, 0, 1, >$ is a model for the language $\{+, \cdot, 0, 1\}$.

The ordered field $< Q, +, \cdot, 0, 1, <, >$ is a model for the language $\{+, \cdot, 0, 1, <\}$.

Many facts about models can be expressed in first order logic. In addition to the operation, relation, and constant symbols of L, first order logic has an infinite list of variables, the equality symbol =, the connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), and the quantifiers $\forall$ (for all), $\exists$ (there exists). Certain finite sequences of symbols are counted as terms, formulas, sentences. Every variable or constant is a term. If t, u are terms, so are $t + u$, $t \cdot u$. If t and u are terms, then $t = u$, $t < u$, are formulas. If φ, ψ are formulas and v is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall v\, \varphi$, $\exists v\, \varphi$ are formulas. A sentence is a formula all of whose variables are bound by quantifiers. For example, the sentence $\forall x\, (x = 0 \vee \exists y\, (x \cdot y) = 1)$ states that every non - zero element has a right inverse. The central notion in model theory is that of a sentence φ being true in a model M. This relation between models and sentences is defined by induction on the subformulas of φ. For example, the sentence $\forall x\, (x = 0 \vee \exists y\, (x \cdot y) = 1)$ is true in the field of rationals, but not in the ring of integers. A set of sentences is called a theory. M is a model of a theory T, if for every sentence $\varphi \in T$ is true in M.

Examples. The theory of rings is the familiar finite list of ring axioms. The theory of real closed fields is a set of sentences, consisting of axioms for ordered fields, the axiom stating that every positive element has a square root, and for each odd n an axiom stating that every polynomial of degree n has a root. For each model M, the theory Th(M) is the set of all sentences true in M. Two classical theorems in model theory are the compactness theorem and the Lowenheim - Skolem - Tarski theorem.

The Compactness Theorem.[2][3] If every finite subset of a set of sentences has a model, then T has a model.

Lowenheim - Skolem - Tarski Theorem. [2] If T has at least one infinite model, then T has a model of every infinite cardinality.

Almost all the deeper results in model theory depend on the construction of a model.

The diagram of a language for M is obtained by adding to L a new constant symbol for each element of A. The elementary diagram of M, written as Diag(M), is the set of all sentences in the diagram language of M which are true in M. The difference between Th(M) and Diag(M) is that Diag(M) has new symbols for the elements of M, while Th(M) does not. There are many other concepts that are fundamental in model theory, like elementary chains, ultraproducts, saturation, but we will stop here with this brief introduction (saturation and the theorem on the existence of a saturated model is presented in section 2).

References:

1. H. M. Edwards, *“Riemann’s Zeta Function“*, Dover Publications, Inc., 2001.

2. H. J. Keisler, C. C. Chang, *"Model Theory"*, Dover Publications, Inc., 2012.

3. H. R. Lewis, C. H. Papadimitriou, *"Elements of the Theory of Computation"*, Prentice Hall Software Series, 1981.

4. J. Donald Monk, *"Mathematical Logic"*, Springer - Verlag, 1976.

5. S. Naranong, *"A Model Theoretic Introduction to Nonstandard Analysis"*, 2010 http://www.math.tamu.edu/~saichu/ModelTheoreticNonstandard.pdf

6. Charles C. Pinter, *"Set Theory"*, Addison - Wesley, 1971.

7. G. Robin, *"Sur L'Ordre Maximum de la Fonction Somme des Diviseurs", Seminaire Delange-Pisot-Poitou, Theorie des nombres (1981 - 1982), Progress in Mathematics 38 (1983), 233 - 244.*

8. A. Robinson, *"Non-standard Analysis"*, Princeton University Press, 1996.

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