Prime coordinates on a modulo map, and sine representation

Diego Alonso Cortez
diego_cortez@mckinsey.com

Abstract

We compute the primes up to 1 million by starting an arithmetic progression at every positive integer *2n,* that grows by *n,* where *n > 1.* All numbers not in the progression are prime.

1 Introduction

If we draw a modulo map for two integers x and y such that

$$f(x,y) = y \bmod x,\ \ x,y \geq 2, \tag{1}$$

we see that all zeroes of the map start at even y's and move in steps $(1,\ y/2)$. The leftmost column counts the number of divisors (excluding 1). The y-axis is inverted for practical purposes regarding MS Excel, where calculations were performed.

		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	2	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	3	1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	4	0	1	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
1	5	1	2	1	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
3	6	0	0	2	1	0	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
1	7	1	1	3	2	1	0	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
3	8	0	2	0	3	2	1	0	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
2	9	1	0	1	4	3	2	1	0	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
3	10	0	1	2	0	4	3	2	1	0	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
1	11	1	2	3	1	5	4	3	2	1	0	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
5	12	0	0	0	2	0	5	4	3	2	1	0	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
1	13	1	1	1	3	1	6	5	4	3	2	1	0	13	13	13	13	13	13	13	13	13	13	13	13	13	13
3	14	0	2	2	4	2	0	6	5	4	3	2	1	0	14	14	14	14	14	14	14	14	14	14	14	14	14
3	15	1	0	3	0	3	1	7	6	5	4	3	2	1	0	15	15	15	15	15	15	15	15	15	15	15	15
4	16	0	1	0	1	4	2	0	7	6	5	4	3	2	1	0	16	16	16	16	16	16	16	16	16	16	16
1	17	1	2	1	2	5	3	1	8	7	6	5	4	3	2	1	0	17	17	17	17	17	17	17	17	17	17
5	18	0	0	2	3	0	4	2	0	8	7	6	5	4	3	2	1	0	18	18	18	18	18	18	18	18	18
1	19	1	1	3	4	1	5	3	1	9	8	7	6	5	4	3	2	1	0	19	19	19	19	19	19	19	19
5	20	0	2	0	0	2	6	4	2	0	9	8	7	6	5	4	3	2	1	0	20	20	20	20	20	20	20
3	21	1	0	1	1	3	0	5	3	1	10	9	8	7	6	5	4	3	2	1	0	21	21	21	21	21	21
3	22	0	1	2	2	4	1	6	4	2	0	10	9	8	7	6	5	4	3	2	1	0	22	22	22	22	22
1	23	1	2	3	3	5	2	7	5	3	1	11	10	9	8	7	6	5	4	3	2	1	0	23	23	23	23
7	24	0	0	0	4	0	3	0	6	4	2	0	11	10	9	8	7	6	5	4	3	2	1	0	24	24	24
2	25	1	1	1	0	1	4	1	7	5	3	1	12	11	10	9	8	7	6	5	4	3	2	1	0	25	25
3	26	0	2	2	1	2	5	2	8	6	4	2	0	12	11	10	9	8	7	6	5	4	3	2	1	0	26
3	27	1	0	3	2	3	6	3	0	7	5	3	1	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 1: remainder of the numbers on the (inverted) y-axis divided by the x-axis

Up to *y = 264,* this is how the map looks.

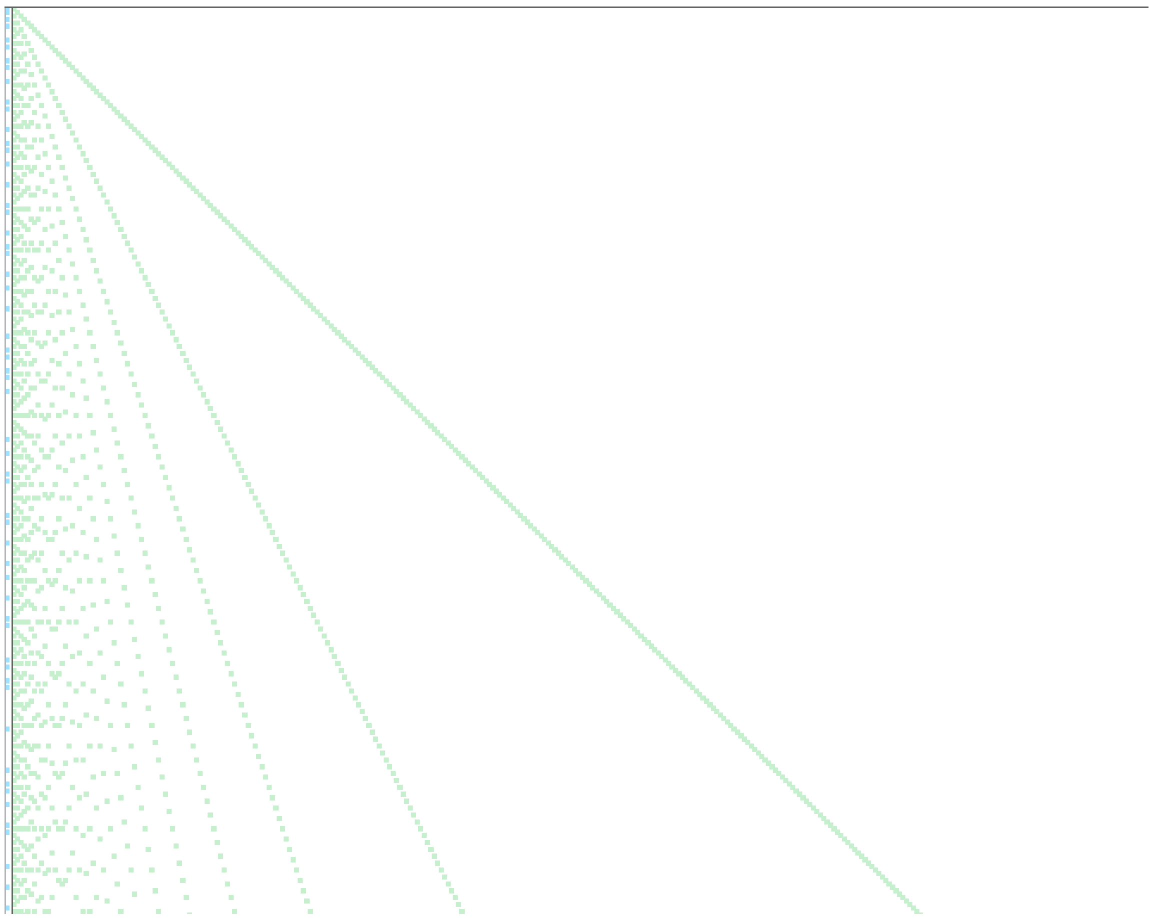

Figure 1: projection of f(x,y) = y mod x, 1< x,y <265, on two dimensions

Plotting the number of divisors we get the familiar divisor function.

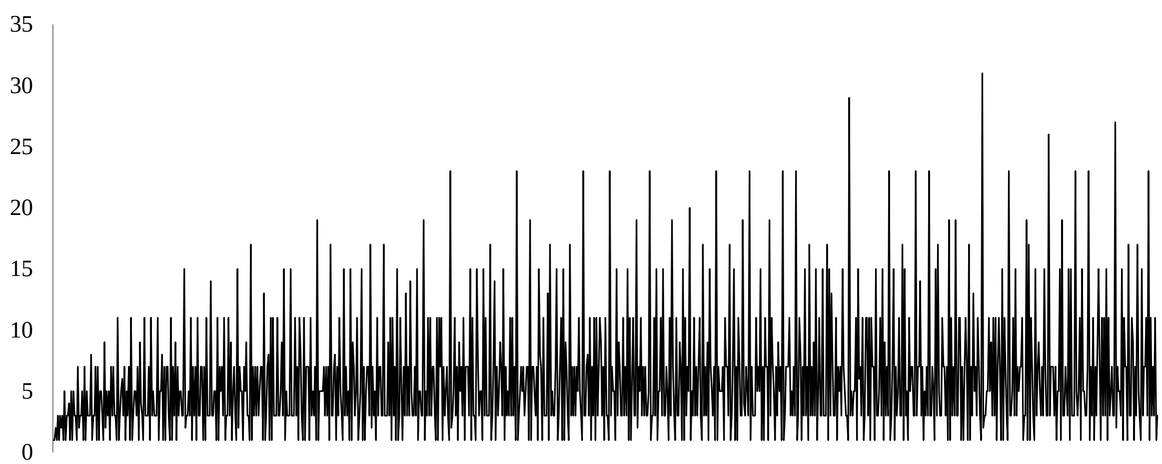

Figure 2: divisor function up to n = 1000.

2 Primes

Let P be the set of all prime numbers and y a natural number larger than 2. Then

$$y \in P \; iff$$
$$\nexists n,\; y \bmod n = 0 \rightarrow \forall n, n \in \mathbb{N} \land 2 \leq n \leq \lfloor y/2 \rfloor \,, \tag{2}$$

where $\lfloor a \rfloor$ is the floor function.

Remembering (1), we can map the set P as all the b values where the line $y = b$ doesn't intersect any of the *even chains*. An *even chain* number *2n* on the modulo map $f(x, y)$ is defined as

$$y = nx + 2n \tag{3}$$

for $x, y \in \mathbb{N} - \{1\},\; n > 1$. We start at *n = 2* because that excludes the chain starting at *2n* for *n = 1* that accounts for each number being divisible by itself. So, effectively, we start at $4 = 2 * 2$.

In an interval $x, y \in \mathbb{N} - \{1\}$, the amount *t* of *even chains* (excluding *n = 1*) is given by

$$t = \left\lfloor \frac{y}{2} \right\rfloor - 1, \tag{4}$$

where $\lfloor a \rfloor$ is the floor function.

And so we get the familiar result that a number y is prime if $y/n \in \mathbb{N}$ for at least one natural n in the interval $[2; \lfloor y/2 \rfloor]$. Looking for primes using an algorithm based on this knowledge leads to trial division.

We can also write this result as a Diophantine system of linear equations of the form

$$y = 2n + nx \tag{5}$$

for each natural *n* on the interval $[2, \lfloor y/2 \rfloor]$. The system would be:

$$\begin{gathered} y = 2 * 2 + 2x \\ y = 2 * 3 + 3x \\ y = 2 * 4 + 4x \\ \cdots \\ y = 2\lfloor y/2 \rfloor + \lfloor y \,/\, 2 \rfloor x. \end{gathered} \tag{6}$$

Y is prime if the system has no integer solutions for any x, and y is composite if at least one equation has an integer solution.

3 Sines

These progressions can also be expressed with sines of the form

$$\sin(\pi x/n), \; x \geq 2n. \tag{7}$$

In order to visualize the number of divisors of every positive integer larger than 2 up to 2n (without counting 1 and itself), we use this set of functions up to n.

f(x) = sin (πx/n)

n	x ≥
2	4
3	6
4	8
5	10
6	12
7	14
8	16
9	18
…	…

If we plot this for $n = 21$ we get

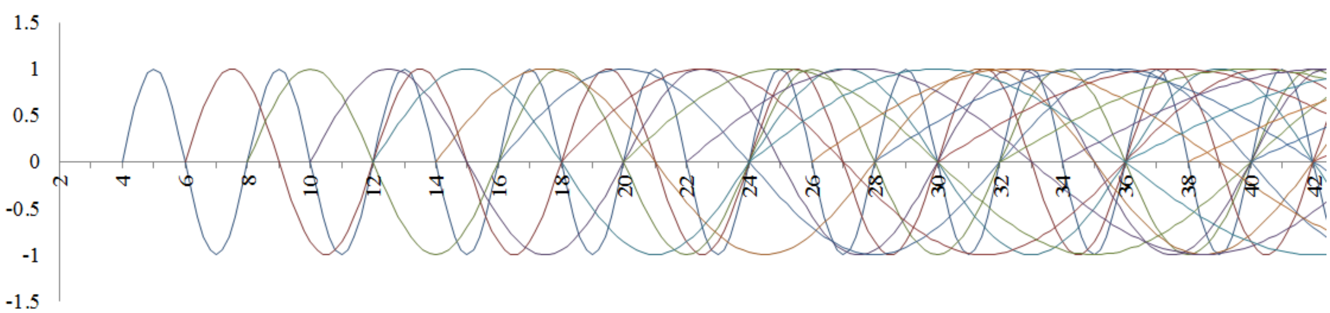

Figure 3: sin (πx/2), x ≥ 4; sin (πx/3), x ≥ 6; sin (πx/4) x ≥ 8; …; sin (πx/21), x ≥ 42

which maps the primes up to 42.

Alternatively we, if we want to find the number of divisors up to a, where a is an integer larger than 3, we use

$$\sin\left(\pi x/\lfloor a/2 \rfloor\right), \; x \geq a, \tag{8}$$

where $\lfloor a \rfloor$ is the floor function.

We present the sine equations up to n = 100, which map the primes up to 200.

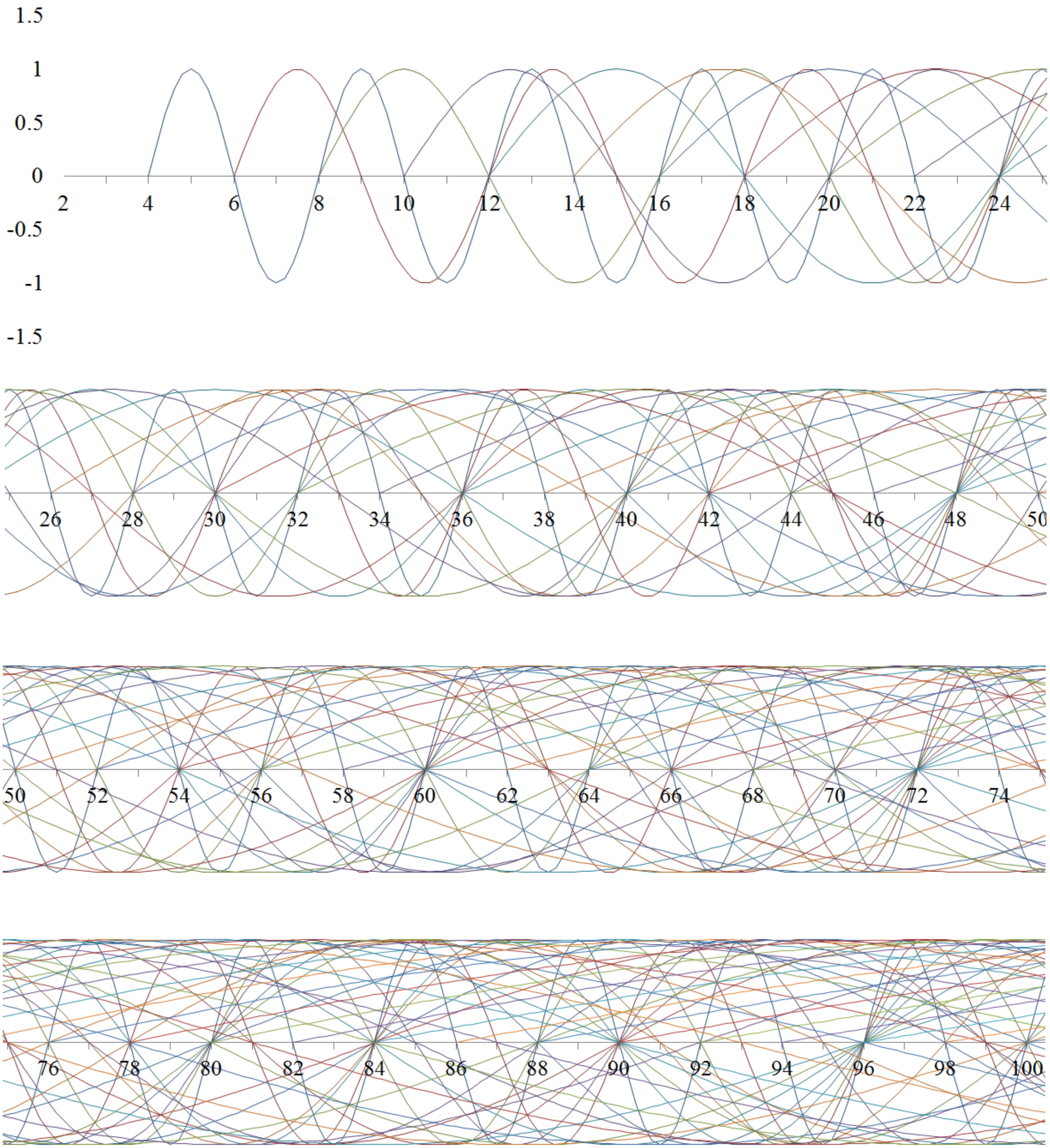

Figure 4: $\sin(\pi x/2), x \geq 4$; $\sin(\pi x/3), x \geq 6$; $\sin(\pi x/4)\ x \geq 8$; …; $\sin(\pi x/50), x \geq 100$

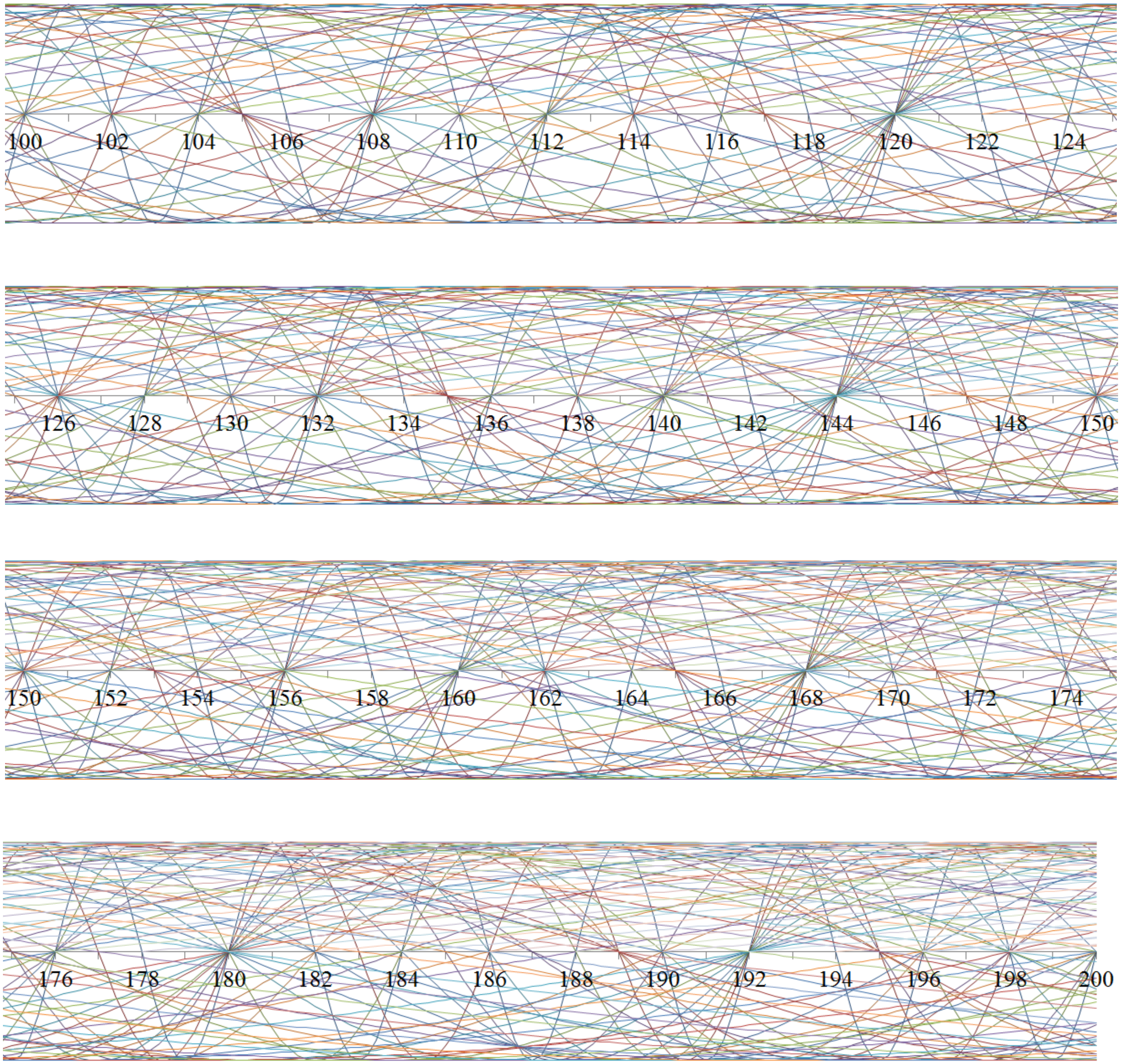

Figure 5 (continuation to figure 4): sin (πx/2), x ≥ 4; sin (πx/3), x ≥ 6; sin (πx/4) x ≥ 8;…; sin (πx/50), x ≥ 100;…; sin (πx/100), x ≥ 200

2	17	41	67	97	127	157	191
3	19	43	71	101	131	163	193
5	23	47	73	103	137	167	197
7	29	53	79	107	139	173	199
11	31	59	83	109	149	179	
13	37	61	89	113	151	181	

Table 2: primes up to 200

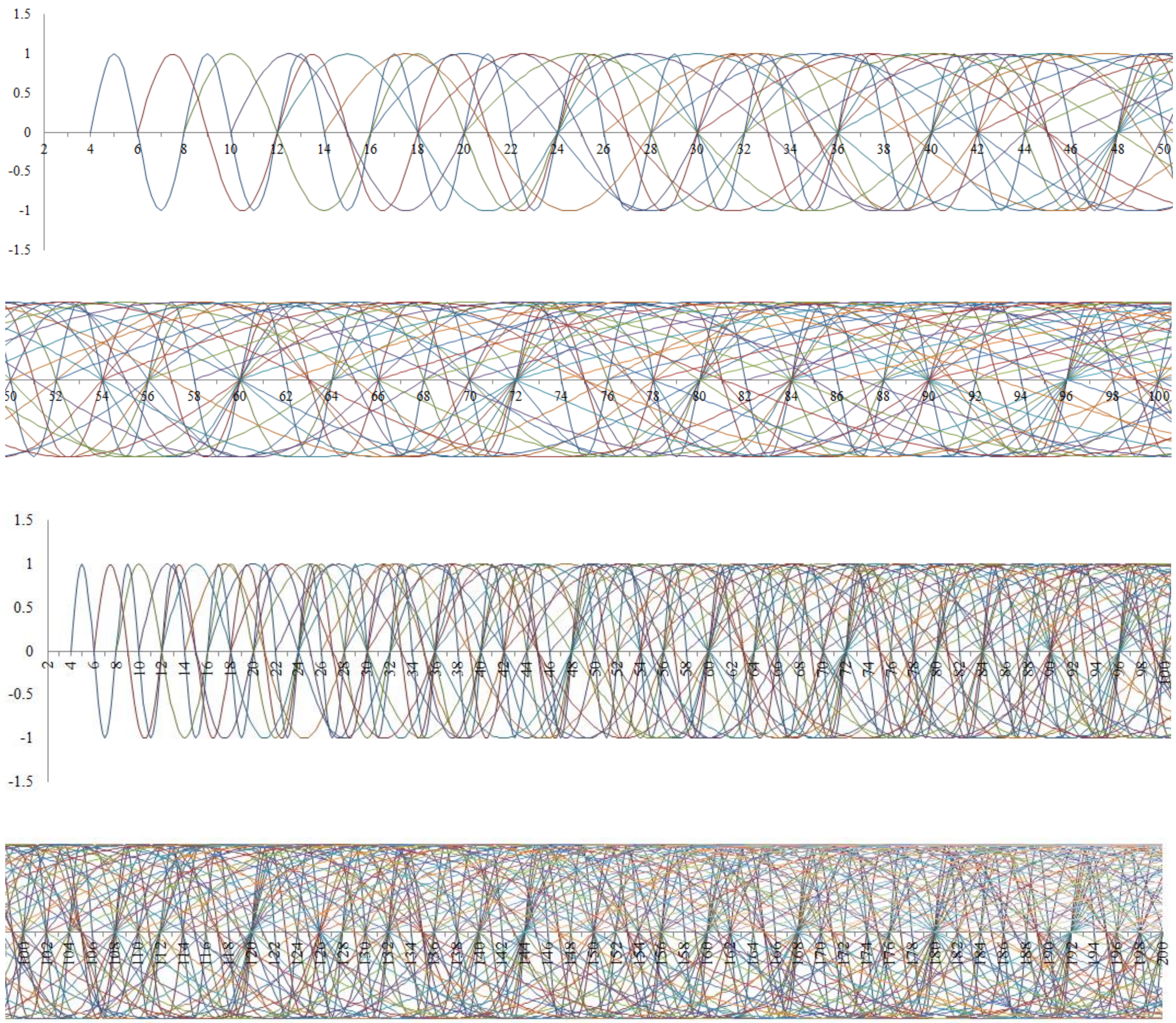

Figure 6: sine equations up to n = 50 and n = 100, respectively, in a more condensed fashion

Table 2 can be used to check if *Figure 4* and *Figure 5* have correctly mapped all primes up to $2n$.

If we start at $n = 1$ we're mapping the divisors including the number itself and excluding 1, so the primes would be the ones where only one sine function has a root. (7) is not defined at $n = 0$, so we cannot effectively map all divisors including 1. Perhaps the limit as $n \to 0$, with the resulting shorter and shorter oscillations, can be interpreted as $x \bmod 1 = x,\ \forall x \in \mathbb{R}, x < 1$. Considering only the real line (x-axis), each oscillation of a sine function is a discrete step of width n starting at $2n$. Then, as n -> 0, those discrete steps shrink to width 0, and thus appear like a continuous line, also subject to be interpreted that $x \bmod 1 = x,\ \forall x \in \mathbb{R}, x < 1$.

4 Computing primes

A VBA macro using these series was computationally cheaper than trial division for calculating the primes up to variable q. It was tested to q just over 1,000,000. The following is functional code for VBA. It requires a spreadsheet where column A has all natural numbers up to variable q, starting with 1 in cell A1. Arrays could be used for better results. In this form of the algorithm, q is limited to the number of rows in Excel.

```
Sub prime_zero_walk()
Dim t, w, y, q As Long
q = 10000 'test up to number q
y = 4 'number where the first n even chain must start
For t = 2 To q 't is the leaps in y that each link of even chain n takes
   For w = 4 To q
   If y > q Then Exit For 'this ensures that each even chain before y just runs up to y, because even chains are
the divisor generators
   If Cells(y, 2).Value = 1 Then
   y = y + t
   Else
   Cells(y, 2).Value = 1
   End If
   Next w
y = 2
y = y + 2 * t
Next t
Dim k As Long
k = 2
For y = 2 To q 'y = 2 means that 1 is not counted, even if it doesn't have a "1" tag
If Cells(y, 2).Value <> 1 Then
Cells(k, 5).Value = Cells(y, 1).Value
k = k + 1
End If
Next y
End Sub
```

5 Conclusion

Trying to interpret the primes as emerging from some general pattern of the divisor function can be difficult. We can try to visualize them as emerging from arithmetic progressions starting at each natural number $2n$, where the difference between terms is n and $n > 1$. Specifically, we've tried to identify linear functions defined for the positive integers on a modulo map. These linear discrete functions can also be interpreted as continuous periodic functions, where each sine function having a root represents a divisor, and the absence of roots represents a prime.

References

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