Octonionic non-Abelian Gauge Theory

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Abstract

We have made an attempt to describe the octonion formulation of Abelian and non-Abelian gauge theory of dyons in terms of 2×2 Zorn vector matrix realization. As such, we have discussed the $U(1)_e \times U(1)_m$ Abelian gauge theory and $U(1) \times SU(2)$ electroweak gauge theory and also the $SU(2)_e \times SU(2)_m$ non-Abelian gauge theory in term of 2×2 Zorn vector matrix realization of split octonions. It is shown that $SU(2)_e$ characterizes the usual theory of the Yang Mill's field (isospin or weak interactions) due to presence of electric charge while the gauge group $SU(2)_m$ may be related to the existence of t-Hooft-Polyakov monopole in non-Abelian Gauge theory. Accordingly, we have obtained the manifestly covariant field equations and equations of motion.

Key Words: octonions, Zorn vector matrix realization, monopole, dyons, Abelian and non-Abelian gauge theories.

1 Introduction

Physicists were fascinated about magnetic monopoles since its ingenious idea was put forward by Dirac [1] and also by Saha [2]. So many attempts [3, 4] were made for the experimental verification of conclusive existence of magnetic monopoles and after the failure of attempts, the literature [5, 6, 7] turned partially negative casting doubts on the existence of such particles. The work of the Schwinger [8] was the first exception to the argument against the existence of monopoles. At the same time so many paradoxes were related to the theory of pure Abelian monopoles, as Dirac's veto, wrong spin-statistics connection [9] and many others [10, 11, 12]. Several problems were soon resolved by the invention of dyons [10, 12, 13, 14] particle carrying simultaneous of electric and magnetic charges. Fresh interest in this subject was enhanced by the idea given by 't Hooft [15] and Polyakov [16] showing that monopoles are the intrinsic parts of grand unified theories (GUT). The Dirac monopole is an elementary particle but the 't Hooft-Polyakov monopole [15, 16] is a complicated extended object having a definite mass and finite size inside of which massive fields play a role in providing a smooth structure and outside it they vanish rapidly leaving the field configuration identical to Abelian Dirac monopole. Julia and Zee [17] have extended the idea of 't Hooft [15] and Polyakov [16] to construct the classical solutions for non-Abelian dyon. Prasad and Sommerfield [18, 19] have derived the analytic stable solutions for the non-Abelian monopoles and dyons of finite mass by keeping the symmetry of vacuum broken but letting the self-interaction of Higgs field approaching zero. Such solutions, satisfying Bogomolnys condition [20] are described as Bogomolny-Prasad-Sommerfield (BPS) monopoles. On the other hand, according to the celebrated Hurwitz Theorem [21] there exists four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathcal{O}$). All four Algebra's are alternative with totally anti symmetric associators. In 1961 Pais [22] pointed out a striking similarity between the algebra of interactions and split-octonion algebra. We [23, 24, 25, 26, 27] have studied octonion electrodynamics, split-octonion electrodynamics, dyonic field equation and octonion gauge analyticity of dyons consistently and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation. Keeping in mind the potential importance of monopoles and dyons along with the applications of octonions in mind, we have made an attempt to describe the octonion formulation of Abelian and non-Abelian gauge theory of dyons in terms of 2×2 Zorn vector matrix realization. As such, we have discussed the $U(1)_e \times U(1)_m$ Abelian gauge theory and $U(1) \times SU(2)$ electroweak gauge theory and also the $SU(2)_e \times SU(2)_m$ non-Abelian gauge theory in term of 2×2 Zorn vector matrix realization of split octonions. It is shown that $SU(2)_e$ characterizes the usual theory of the Yang Mill's field (isospin or weak interactions) due to presence of electric charge while the gauge group $SU(2)_m$ may be related to the existence of 't-Hooft-Polyakov monopole in non-Abelian Gauge theory. Accordingly, we have obtained the manifestly covariant field equations and equations of motion.

2 Octonion Definition

An octonion $X \in \mathcal{O}$ is expressed [23, 24, 25] as a set of eight real numbers

$$X = (X_0,\, X_1, ..., \, X_7) = X_0 e_0 + \sum_{A=1}^{7} X_A e_A \qquad (A = 1, 2, ..., 7) \tag{1}$$

where $e_A (A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules

$$e_0 = 1, \qquad e_0 e_A = e_A e_0 = e_A \qquad e_A e_B = -\delta_{AB} e_0 + f_{ABC}\, e_C. \quad (A, B, C = 1, 2, ...7) \tag{2}$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 i.e.

$$f_{ABC} = +1 = (123),\ (471),\ (257), (165),\ (624), (543),\ (736). \tag{3}$$

Here the octonion algebra $\mathcal{O}$ is described over the algebra of rational numbers having the vector space of dimension 8.

The relations among octonion basis elements define as

$$[e_A,\ e_B] \quad = \quad 2f_{ABC}e_C; \quad \{e_A,\ e_B\} = -\delta_{AB}e_0; \quad e_A(e_Be_C) \neq (e_Ae_B)e_C; \tag{4}$$

where brackets [ ] and { } are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol. Octonion conjugate is thus defined as,

$$\overline{X} = X_0e_0 - \sum_{A=1}^{7} X_Ae_A \qquad (A = 1, 2, ..., 7). \tag{5}$$

An Octonion can be decomposed in terms of its scalar $(Sc(X))$ and vector $(Vec(X))$ parts as

$$Sc(X) \quad = \quad \frac{1}{2}(X + \overline{X}) = X_0; \quad Vec(X) = \frac{1}{2}(X - \overline{X}) = \sum_{A=1}^{7} X_Ae_A. \tag{6}$$

Conjugates of product of two octonions and its own are described as

$$(\overline{XY}) \quad = \quad \overline{Y}\,\overline{X}\ ; \quad \overline{(\bar{X})} \ = X. \tag{7}$$

The norm of the octonion $N(X)$ is defined as

$$N(X) = \overline{X}\,X \quad = X\,\bar{X} \quad = \sum_{\alpha=0}^{7} X_\alpha^2 e_0 \tag{8}$$

which is zero if $X = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(XY) \quad = N(X)N(Y) \quad = N(Y)N(X). \tag{9}$$

As such, for a nonzero octonion X , we define its inverse as

$$X^{-1} = \frac{\bar{X}}{N(X)}; \tag{10}$$

which shows that

$$X^{-1}X = XX^{-1} = 1.e_0; \quad (XY)^{-1} = Y^{-1}X^{-1}. \tag{11}$$

3 Split Octonion

The split octonions [25, 26] are a non associative extension of split quaternions. They differ from the octonion in the signature of quadratic form. The split octonion have a signature (4,4) whereas the octonions have positive signature (8,0). The Cayley algebra of octonion over the field of complex number $\mathbb{C}_{\mathbb{C}} = \mathbb{C} \otimes C$ is visualized as the algebra of split octonions with its following basis elements,

$$\begin{aligned} u_0 &= \frac{1}{2}(e_0 + ie_7), \qquad u_0^* = \frac{1}{2}(e_0 - ie_7); \\ u_j &= \frac{1}{2}(e_j + ie_{j+3}), \qquad u_j^* = \frac{1}{2}(e_j - ie_{j+3}); \qquad (\forall\, j = 1, 2, 3.) \end{aligned} \tag{12}$$

where $i = \sqrt{-1}$ is assumed to commute with $e_A (A = 1, 2, ..., 7)$ octonion units. The split octonion basis element satisfy the following multiplication rule;

$$\begin{aligned} u_i u_j &= \epsilon_{ijk} u_k^*; \qquad u_i^* u_j^* = -\epsilon_{ijk} u_k^* \qquad (\forall i, j, k = 1, 2, 3) \\ u_i u_j^* &= -\delta_{ij} u_0; \qquad u_i u_0 = 0; \qquad u_i^* u_0 = u_i^* \\ u_i^* u_j &= -\delta_{ij} u_0; \qquad u_i u_0^* = u_0; \qquad u_i^* u_0^* = 0 \\ u_0 u_i &= u_i; \qquad u_{i0}^* u_i = 0; \qquad u_0 u_i^* = 0 \\ u_0^* u_i^* &= u_i; \qquad u_0^2 = u_0; \qquad u_0^{*2} = u_0^*; \qquad u_0 u_0^* = u_0^* u_0 = 0. \end{aligned} \tag{13}$$

Günaydin and Gursey [28, 29] pointed out that the automorphism group of octonion is G_2 and its subgroup which leaves imaginary octonion unit e_7 invariant (or equivalently the idempotents u_0 and u_0^*) is $SU(3)$ where the units u_j and $u_j^* (j = 1, 2, 3)$ transform respectively like a triplet and anti triplet accordingly associated with colour and anti colour triplets of $SU(3)$ group. Let us introduce a convenient realization for the basis elements (u_0, u_j, u_0^*, u_j^*) in term of Pauli spin matrices as

$$u_0=\begin{bmatrix}0&0\\0&1\end{bmatrix};\qquad u_0^*=\begin{bmatrix}1&0\\0&0\end{bmatrix};$$
$$u_j=\begin{bmatrix}0&0\\e_j&0\end{bmatrix};\qquad u_j^*=\begin{bmatrix}0&-e_j\\0&0\end{bmatrix}.\qquad(\forall j=1,2,3)\tag{14}$$

The split Cayley (octonion) algebra is thus expressed in terms of 2×2 Zorn's vector matrices components of which are scalar and vector parts of a quaternion i.e.

$$\mathcal{O}=\{\begin{pmatrix}m&\vec{p}\\\vec{q}&n\end{pmatrix};\quad m,n\in Sc(H);\quad \&\,\vec{p},\vec{q}\in Vec(H)\,\}.\tag{15}$$

As such, we may write an arbitrary split octonion A in terms of following 2×2 Zorn's vector matrix realization as

$$A=au_0^*+bu_0+x_iu_i^*+y_iu_i=\begin{pmatrix}a&-\overrightarrow{x}\\\overrightarrow{y}&b\end{pmatrix},\tag{16}$$

where a and b are scalars and $\vec{x}$ and $\vec{y}$ are three vectors. Thus the product of two octonions in terms of following 2×2 Zorn's vector matrix realization is expressed as

$$\begin{pmatrix}a&\overrightarrow{x}\\\overrightarrow{y}&b\end{pmatrix}*\begin{pmatrix}c&\overrightarrow{u}\\\vec{v}&d\end{pmatrix}=\begin{pmatrix}ac+(\overrightarrow{x}.\overrightarrow{v})&a\vec{u}+d\vec{x}+(\overrightarrow{y}\times\vec{v})\\c\overrightarrow{y}+b\vec{v}-(\vec{x}\times\overrightarrow{u})&bd+(\overrightarrow{y}.\overrightarrow{u})\end{pmatrix},\tag{17}$$

where $(\times)$ denotes the usual vector product, e_j $(j=1,2,3)$ with $e_j\times e_k=\epsilon_{jkl}e_l$ and $e_je_k=-\delta_{jk}$. Octonion conjugate of equation (16) in terms of 2×2 Zorn's vector matrix realization is now defined as

$$\overline{A}\;=au_0+bu_0^\star-x_iu_i^\star-y_iu_i\;=\begin{pmatrix}b&\overrightarrow{x}\\-\overrightarrow{y}&a\end{pmatrix}.\tag{18}$$

The norm of A is defined as

$$N(A)=\overline{A}A=A\overline{A}=(ab+\overrightarrow{x}.\overrightarrow{y})\hat{1}=n(A)\vec{1},\tag{19}$$

where $\hat{1}$ is the identity elements of matrix order 2×2, and the expression $n(A)=(ab+\overrightarrow{x}.\overrightarrow{y})$ defines the quadratic form which admits the composition as

$$n(\vec{A}\cdot\vec{B}) = n(\vec{A})\,n(\vec{B}), \quad (\forall\,\vec{A},\vec{B}\in\mathcal{O}) \tag{20}$$

As such, we may easily express the Euclidean or Minkowski four vector in split octonion formulation in terms of 2×2 Zorn's vector matrix realizations. So, any four - vector A_μ (complex or real) can equivalently be written in terms of the following Zorn matrix realization as

$$Z(A) \;=\; \begin{pmatrix} x_4 & -\overrightarrow{x} \\ \overrightarrow{y} & y_4 \end{pmatrix}; \;\; Z(\overline{A}) = \begin{pmatrix} x_4 & \overrightarrow{x} \\ -\overrightarrow{y} & y_4 \end{pmatrix}. \tag{21}$$

4 U(1) × U(1) Octonion Gauge Formulation

Let us write [30] the split octonion valued space time vector $\mathbb{Z}^\mu$ $(\mu = 0,1,2,3)$ in terms of the 4×4 (space time vector-valued) Zorn matrix $\mathbb{Z}^\mu_{ab}$ as

$$\mathbb{Z}^\mu = x_0^\mu u_0^* + y_0^\mu u_0 + x_j^\mu u_j^* + y_j^\mu u_j \cong \begin{pmatrix} x_0^\mu e_0 & -x_j^\mu e_j \\ y_j^\mu e_j & y_0^\mu e_0 \end{pmatrix}, \quad (\forall\, j = 1,2,3) \tag{22}$$

where $(\mu = 0,1,2,3)$ represent the internal four dimensional space with $(\mu = 0)$ representing $U(1)$ abelian gauge structure while $\mu = j$ $(\forall\, j = 1,2,3)$ may be used for non-Abelian gauge structure. Here x_0^μ, x_j^μ, y_0^μ, y_j^μ are real valued variables for abelian and non-Abelian gauge fields. When the space time metric [30] is $\eta_{\mu\nu}\hat{1}_{4\times 4}$, the bi linear term

$$\begin{aligned}\frac{1}{4}Trace[\eta_{\mu\nu}\mathbb{Z}^\mu.\mathbb{Z}^\nu] =& \frac{1}{4}\eta_{\mu\nu}[x_0^\mu x_0^\nu + y_0^\mu y_0^\nu + x_j^\mu x_j^\nu + y_j^\mu y_j^\nu]\,Trace[\hat{1}_{2\times 2}] \\ =& \frac{1}{2}\eta_{\mu\nu}[x_0^\mu x_0^\nu + y_0^\mu y_0^\nu + x_j^\mu x_j^\nu + y_j^\mu y_j^\nu]\end{aligned} \tag{23}$$

describe the inner product. The octonion conjugation is accordingly defined as

$$\overline{\mathbb{Z}}^\mu = x_0^\mu u_0 + y_0^\mu u_0^* - x_j^\mu u_j^* - y_j^\mu u_j \cong \begin{pmatrix} y_0^\mu e_0 & x_j^\mu e_j \\ -y_j^\mu e_j & x_0^\mu e_0 \end{pmatrix}, \tag{24}$$

while the Hermitian conjugation is described [30] as

$$(\mathbb{Z}^\mu)^\dagger = (x_0^\mu)^* u_0 + (y_0^\mu)^* u_0^* - (x_j^\mu)^* u_j^* - (y_j^\mu)^* u_j \cong \begin{pmatrix} (y_0^\mu)^* e_0 & (x_j^\mu)^* e_j \\ -(y_j^\mu)^* e_j & (x_0^\mu)^* e_0 \end{pmatrix}. \tag{25}$$

So, we may write $x_j^\mu = y_j^\mu = (x_j^\mu)^* = (y_j^\mu)^* = 0$ for Abelian gauge fields. Thus, the split octonion differential operator [25] is written as,

$$\boxdot = u_0^* \partial_\mu + u_0 \partial_\mu \longmapsto \partial_\mu e_0 \hat{1}, \tag{26}$$

where $u_0 = \frac{1}{2}(e_0 + ie_7)\,,\; u_0^* = \frac{1}{2}(e_0 - ie_7)$ are split octonion basis (12). Thus, the equation (26) in term of 2×2 Zorn matrix may be written as

$$\boxdot = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} \approx \partial_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \approx \partial_\mu \hat{1}_{2\times 2}. \tag{27}$$

Hence, the covariant derivative for $U(1) \times U(1)$ gauge theory [27] of dyons may be written as split octonion valued in terms of 2×2 Zorn vector matrix realization as

$$D_\mu = \begin{pmatrix} \partial_\mu + A_\mu & 0 \\ 0 & \partial_\mu + B_\mu \end{pmatrix}; \tag{28}$$

which yields

$$D_\mu D_\nu = \begin{pmatrix} \partial_\mu \partial_\nu + \partial_\mu A_\nu + A_\mu \partial_\nu + A_\mu A_\nu & 0 \\ 0 & \partial_\mu \partial_\nu + \partial_\mu B_\nu + B_\mu \partial_\nu + B_\mu B_\nu \end{pmatrix}, \tag{29}$$

The commutator$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu$, may be expressed as

$$[D_\mu, D_\nu] = \begin{pmatrix} \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu & 0 \\ 0 & \partial_\mu B_\nu - \partial_\nu B_\mu + B_\mu B_\nu - B_\nu B_\mu \end{pmatrix}, \tag{30}$$

which reproduces

$$[D_\mu, D_\nu] = \begin{pmatrix} F_{\mu\nu} & 0 \\ 0 & \mathcal{F}_{\mu\nu} \end{pmatrix} \longmapsto \mathbb{F}_{\mu\nu}; \tag{31}$$

where

$$
\begin{aligned}
F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu \longmapsto E_{\mu\nu}; \\
\mathcal{F}_{\mu\nu} =& \partial_\mu B_\nu - \partial_\nu B_\mu + B_\mu B_\nu - B_\nu B_\mu \longmapsto H_{\mu\nu};
\end{aligned} \tag{32}
$$

Here $E_{\mu\nu}$ and $H_{\mu\nu}$ represent the octonionic forms of generalized field tensors of electromagnetic fields of dyons. Now operating D_μ given by the equation (28) to the generalized four electromagnetic fields $\mathbb{F}_{\mu\nu}$ (31), we get

$$
D_\mu \mathbb{F}_{\mu\nu} = \begin{pmatrix} \partial_\mu E_{\mu\nu} & 0 \\ 0 & \partial_\mu H_{\mu\nu} \end{pmatrix} \longmapsto \begin{pmatrix} j_\nu & 0 \\ 0 & k_\nu \end{pmatrix} \Longrightarrow \mathbb{J}_\nu; \tag{33}
$$

where

$$
\begin{aligned}
j_\nu =& \partial_\mu E_{\mu\nu}; \\
k_\nu =& \partial_\mu H_{\mu\nu};
\end{aligned} \tag{34}
$$

are respectively the four currents associated with electric charge and magnetic monopole (i.e. the constituents of dyons) in the case of $U(1) \times U(1)$ Octonion Gauge formalism. Thus, we have obtained the justification of $U(1) \times U(1)$ gauge theory of dyons in terms of split octonions and their correspondence with 2×2 Zorn vector matrix realization. Here we may infer that the $U(1) \times U(1)$ gauge theory is described well by split octonion formulation where the spinor and iso spinor take part together.

5 U(1) × SU(2) Octonion Gauge Formulation

So, by virtue of split octonion formulation we may extend $U(1)$ gauge theory to the $U(1)\times SU(2)$ gauge theory. Accordingly, we write an octonion as the combination of two gauge fields expanded in terms of quaternions i.e.

$$
\begin{aligned}
A_\mu \mapsto & A_\mu^0 + A_\mu^a e_a, \\
B_\mu \mapsto & B_\mu^0 + B_\mu^a e_a, \qquad (\forall a = 1, 2, 3.)
\end{aligned} \tag{35}
$$

So, the co-variant derivative in case of $U(1) \times SU(2)$ octonion gauge field in the split octonion form may be expressed as

$$
D_\mu = \begin{pmatrix} \partial_\mu + A_\mu^0 + A_\mu^a e_a & 0 \\ 0 & \partial_\mu + B_\mu^0 + B_\mu^a e_a \end{pmatrix}, \tag{36}
$$

where the components of electric A_{μ}^{0} and magnetic B_{μ}^{0} are the four potentials of dyons in case of $U(1)$ while A_{μ}^{a} and B_{μ}^{a} describe of the $SU(2)$ gauge field theory. The commutator of two derivatives is

$$[D_{\mu}, D_{\nu}] = \begin{pmatrix} G_{\mu\nu}^{0} + G_{\mu\nu}^{a} e_{a} & 0 \\ 0 & \mathsf{G}_{\mu\nu}^{0} + \mathsf{G}_{\mu\nu}^{\mathsf{a}} e_{a} \end{pmatrix} \longmapsto \mathbb{G}_{\mu\nu}; \tag{37}$$

which is $U(1) \times SU(2)$ octonion gauge field strength for dyons in 2×2 Zorn matrix realization.
where

$$\begin{aligned} G_{\mu\nu}^{0} =& \partial_{\mu} A_{\nu}^{0} - \partial_{\nu} A_{\mu}^{0} + \left[A_{\mu}^{0}, A_{\nu}^{0}\right] \longmapsto E_{\mu\nu}^{0}, \\ G_{\mu\nu}^{a} =& \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a} + e_{a}\left[A_{\mu}^{a}, A_{\nu}^{a}\right] \longmapsto E_{\mu\nu}^{a}, \end{aligned} \tag{38}$$

are respectively abelian and non-Abelian $U(1)_{e} \times SU(2)_{e}$ gauge structures in presence of electric charge and

$$\begin{aligned} \mathsf{G}_{\mu\nu}^{0} =& \partial_{\mu} B_{\nu}^{0} - \partial_{\nu} B_{\mu}^{0} + \left[B_{\mu}^{0}, B_{\nu}^{0}\right] \longmapsto H_{\mu\nu}^{0}, \\ \mathsf{G}_{\mu\nu}^{\mathsf{a}} =& \partial_{\mu} B_{\nu}^{a} - \partial_{\nu} B_{\mu}^{a} + e_{a}\left[B_{\mu}^{a}, B_{\nu}^{a}\right] \longmapsto H_{\mu\nu}^{a}, \end{aligned} \tag{39}$$

is $U(1)_{m} \times SU(2)_{m}$ gauge structure associated with the presence of magnetic monopole. Operating D_{μ} given by the equation (36) to the $U(1) \times SU(2)$ octonion gauge field strength $\mathbb{G}_{\mu\nu}$ (37), we get

$$D_{\mu} \mathbb{G}_{\mu\nu} = \begin{pmatrix} \partial_{\mu} G_{\mu\nu}^{0} + \partial_{\mu} G_{\mu\nu}^{a} e_{a} & 0 \\ 0 & \partial_{\mu} \mathsf{G}_{\mu\nu}^{0} + \partial_{\mu} \mathsf{G}_{\mu\nu}^{\mathsf{a}} e_{a} \end{pmatrix}, \tag{40}$$

which may further be reduced in terms of compect notation of split octonion formulation i.e.

$$D_{\mu} \mathbb{G}_{\mu\nu} = \mathbb{J}_{\nu} \Longrightarrow \begin{pmatrix} j_{\nu}^{0} + j_{\nu}^{a} e_{a} & 0 \\ 0 & k_{\nu}^{0} + k_{\nu}^{a} e_{a} \end{pmatrix}. \tag{41}$$

Here $\mathbb{J}_{\nu}$ is $U(1) \times SU(2)$ form of octonion gauge current for dyons. From equation (41), we may write the following field equations

$$\begin{aligned} j_\nu^0 =& \partial_\mu G_{\mu\nu}^0; \\ j_\nu^a =& \partial_\mu G_{\mu\nu}^a; \\ k_\nu^0 =& \partial_\mu G_{\mu\nu}^0; \\ k_\nu^a =& \partial_\mu G_{\mu\nu}^a; \end{aligned} \tag{42}$$

where j_ν^0 and j_ν^a are generalized octonion current for $U(1)_e \times SU(2)_e$(electric case) and k_ν^0 and k_ν^a for $U(1)_m \times SU(2)_m$(magnetic case). The analogous continuity equation then changes to be

$$D_\mu \mathbb{J}_\mu = \begin{pmatrix} \partial_\mu j_\mu^0 + \partial_\mu j_\mu^a e_a & 0 \\ 0 & \partial_\mu k_\nu^0 + \partial_\mu k_\nu^a e_a \end{pmatrix} =0. \tag{43}$$

6 Non-Abelian $SU(2)_e \times SU(2)_m$ Gauge Formulation

In order to describe $SU(2)_e \times SU(2)_m$ gauge formulation, let us write the covariant derivative D_μ as

$$D_\mu =\partial_\mu + \mathbb{V}_\mu; \tag{44}$$

where $\mathbb{V}_\mu$ is the octonion form of generalized four potential [31] expressed as

$$\mathbb{V}_\mu = e_0 \left(A_\mu^\tau e_\tau \right) + i e_7 \left(B_\mu^\tau e_\tau \right). \tag{45}$$

Here $\tau \mapsto 1, 2, 3$ denotes $SU(2)$ generator. Thus, the covariant derivative (46) may be expressed as

$$\begin{aligned} D_\mu =& \partial_\mu + e_0 \left(A_\mu^\tau e_\tau \right) + i e_7 \left(B_\mu^\tau e_\tau \right) \\ =& u_0^* \left(\partial_\mu + A_\mu^\tau e_\tau + B_\mu^\tau e_\tau \right) + u_0 \left(\partial_\mu + A_\mu^\tau e_\tau - B_\mu^\tau e_\tau \right). \end{aligned} \tag{46}$$

The split octonion equivalent of equation (46) in term of 2×2 Zorn's vector matrix realization may be expressed as

$$D_{\mu}=\left(\begin{array}{cc}\partial_{\mu}+\left(A_{\mu}^{\tau}+B_{\mu}^{\tau}\right)e_{\tau} & 0\\ 0 & \partial_{\mu}+\left(A_{\mu}^{\tau}-B_{\mu}^{\tau}\right)e_{\tau}\end{array}\right). \tag{47}$$

The commutator of two derivatives is

$$\left[D_{\mu},D_{\nu}\right]=\left(\begin{array}{cc}(G_{\mu\nu}^{\tau}+\mathsf{G}_{\mu\nu}^{\tau})e_{\tau} & 0\\ 0 & (G_{\mu\nu}^{\tau}-\mathsf{G}_{\mu\nu}^{\tau})e_{\tau}\end{array}\right)\longmapsto\mathbb{G}_{\mu\nu}^{\tau}; \tag{48}$$

where

$$\begin{aligned}G_{\mu\nu}^{\tau}= & \partial_{\mu}A_{\nu}^{\tau}-\partial_{\nu}A_{\mu}^{\tau}+e_{\tau}\left[A_{\mu}^{\tau},A_{\nu}^{\tau}\right]\longmapsto E_{\mu\nu}^{\tau};\\ \mathsf{G}_{\mu\nu}^{\tau}= & \partial_{\mu}B_{\nu}^{\tau}-\partial_{\nu}B_{\mu}^{\tau}+e_{\tau}\left[B_{\mu}^{\tau},B_{\nu}^{\tau}\right]\longmapsto H_{\mu\nu}^{\tau};\end{aligned} \tag{49}$$

respectively represent $SU(2)$ non-Abelian gauge structure associated with electric charge and magnetic monopole. Operating D_{μ} given by the equation (47) to the equation (48) i.e. $D_{\mu}\left[D_{\mu},D_{\nu}\right]$, we get

$$D_{\mu}\mathbb{G}^{\tau}{}_{\mu\nu}=\left(\begin{array}{cc}\left(\partial_{\mu}G_{\mu\nu}^{\tau}+\partial_{\mu}\mathsf{G}_{\mu\nu}^{\tau}\right)e_{\tau} & 0\\ 0 & \left(\partial_{\mu}G_{\mu\nu}^{\tau}-\partial_{\mu}\mathsf{G}_{\mu\nu}^{\tau}\right)e_{\tau}\end{array}\right), \tag{50}$$

which may further be reduced to the following compact notation of an octonion formulation as

$$D_{\mu}\mathbb{G}^{\tau}{}_{\mu\nu}=\mathbb{J}_{\nu}^{\tau}; \tag{51}$$

where $\mathbb{J}_{\nu}^{\tau}$ $(\forall\,\tau=1,2,3)$, the octonion gauge current in terms of 2×2 Zorn's matrix realization of $SU(2)_{e}\times SU(2)_{m}$, may be expressed as

$$\mathbb{J}_{\nu}^{\tau}=\left(\begin{array}{cc}\left(j_{\nu}^{\tau}+k_{\nu}^{\tau}\right)e_{\tau} & 0\\ 0 & \left(j_{\nu}^{\tau}-k_{\nu}^{\tau}\right)e_{\tau}\end{array}\right), \tag{52}$$

from which we may write following field equations for non-Abelian gauge fields of dyons

$$j_\nu^\tau = \partial_\mu G_{\mu\nu}^\tau;$$
$$k_\nu^\tau = \partial_\mu \mathsf{G}_{\mu\nu}^\tau; \tag{53}$$

Here j_ν^τ and k_ν^τ are octonion non-Abelian currents respectively used for electric charge and magnetic monopole for the case of $SU(2)_e \times SU(2)_m$ gauge field theory. Accordingly, the continuity equation generalizes as

$$D_\nu \mathbb{J}_\nu^\tau = \begin{pmatrix} (\partial_\nu j_\nu^\tau + \partial_\nu k_\nu^\tau)\, e_\tau & 0 \\ 0 & (\partial_\nu j_\nu^\tau - \partial_\nu k_\nu^\tau)\, e_\tau \end{pmatrix} = 0. \tag{54}$$

7 Condition of 't Hooft Polyakov Monopole

From the fore going analysis, we may easily obtain the case of 't Hooft Polyakov [15, 16] theory of magnetic monopoles. Let us write the complex conjugate of differential operator (46) as

$$D_\mu^* = \partial_\mu + e_0 \left(A_\mu^\tau e_\tau\right) - i e_7 \left(B_\mu^\tau e_\tau\right), \tag{55}$$

which may further be reduced to 2×2 vector matrix realization of split octonion as

$$D_\mu^* = \begin{pmatrix} \partial_\mu + \left(A_\mu^\tau - B_\mu^\tau\right) e_\tau & 0 \\ 0 & \partial_\mu + \left(A_\mu^\tau + B_\mu^\tau\right) e_\tau \end{pmatrix}. \tag{56}$$

Equation (47) and (56) gives rise to

$$\mathscr{D}_\mu = \frac{1}{2}\left(D_\mu + D_\mu^*\right) = \begin{pmatrix} \partial_\mu + A_\mu^\tau e_\tau & 0 \\ 0 & \partial_\mu + A_\mu^\tau e_\tau \end{pmatrix} \approx \left(\partial_\mu + A_\mu^\tau e_\tau\right) \hat{1}, \tag{57}$$

which describes the covariant derivative of Yang-Mill's field of $SU(2)$ gauge theory.

We may now use the covariant derivative (57) in order to discuss Lagrangian [32] used in the Georgi-Glashow model for the description of the $SO(3) \sim SU(2)$ Yang Mills field theory coupled to a Higgs field as

$$\mathscr{L} = -\frac{1}{4} G_{\mu\nu}.G^{\mu\nu} + \frac{1}{2}\left(\mathscr{D}_\mu \vec{\phi}\right)\cdot\left(\mathscr{D}^\mu \vec{\phi}\right) - V(\vec{\phi}), \tag{58}$$

Here

✤ the gauge field-strength $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} =\partial_\mu W_\nu - \partial_\nu W_\mu - eW_\mu \times W_\nu \tag{59}$$

$$W_\mu =A^\tau_\mu e_\tau; \quad G_{\mu\nu} = G^\tau_{\mu\nu} e_\tau; \tag{60}$$

where W_μ are gauge potentials associated with the Lie algebra of $SO(3)$, and

$$G^\tau_{\mu\nu} =\partial_\mu A^\tau_\nu - \partial_\nu A^\tau_\mu + e_\tau \left[A^\tau_\mu, A^\tau_\nu\right]. \tag{61}$$

✤ the Higgs field $\overrightarrow{\phi}$ is a vector in the (three-dimensional) adjoint representation of $SO(3)$, with components $\phi_a = (\phi_1, \phi_2, \phi_3)$ which is minimally coupled to the gauge field; the gauge-covariant derivative [32] is defined as

$$\mathscr{D}_\mu \vec{\phi} =\partial_\mu \overrightarrow{\phi} - eW_\mu \times \overrightarrow{\phi}; \tag{62}$$

✤ the Higgs potential $V\left(\overrightarrow{\phi}\right)$ is given by

$$V\left(\overrightarrow{\phi}\right) =\frac{\lambda}{4}\left(\phi^2 - a^2\right)^2; \tag{63}$$

where $\phi^2 = \overrightarrow{\phi} \cdot \overrightarrow{\phi}$ and λ is assumed to be non-negative.

The Lagrangian density (58) is invariant under the following $SO(3)$ gauge transformations [32],

$$\begin{aligned} \overrightarrow{\phi} \mapsto \overrightarrow{\phi}' &=g(x)\overrightarrow{\phi}; \\ W_\mu \mapsto W'_\mu &=g(x)W_\mu g(x)^{-1} + \frac{1}{e}\partial_\mu g(x) g(x)^{-1}; \end{aligned} \tag{64}$$

where $g(x)$ is a possibly x- dependent 3×3 orthogonal matrix with unit determinant.

The classical dynamics of the fields W_μ and $\overrightarrow{\phi}$ are determined from the following equations of motion

$$\begin{aligned} \mathscr{D}_\nu G^{\mu\nu} &= -e\overrightarrow{\phi} \times \mathscr{D}^\mu \overrightarrow{\phi}; \\ \mathscr{D}^\mu \mathscr{D}_\mu \overrightarrow{\phi} &= -\lambda\left(\phi^2 - a^2\right)\overrightarrow{\phi}; \end{aligned} \tag{65}$$

and by the Bianchi identity

$$\mathscr{D}_\mu G^{d\mu\nu} =0; \tag{66}$$

where $G^{d\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}G_{\lambda\rho}$.

The canonically conjugate momenta to the gauge field W_μ and the Higgs field $\overrightarrow{\phi}$ are described [32] as

$$\vec{E}^i = -\vec{G}^{0i}, \qquad \vec{\Pi} = \mathscr{D}_0\overrightarrow{\phi}; \tag{67}$$

$$\vec{G}_{ij} = -\epsilon_{ijk}\vec{B}^k; \tag{68}$$

The energy density may be written as

$$\mathcal{H} = \frac{1}{2}\vec{E}_i.\vec{E}_i + \frac{1}{2}\vec{\Pi}.\vec{\Pi} + \frac{1}{2}\vec{B}_i \cdot \vec{B}_i + \frac{1}{2}\mathscr{D}_i\overrightarrow{\phi} \cdot \mathscr{D}_i\overrightarrow{\phi} + V\left(\overrightarrow{\phi}\right). \tag{69}$$

Here, one can define a vacuum configuration to be one for which the energy density vanishes [32], i.e.

$$\vec{G}_{\mu\nu} = 0, \quad \mathscr{D}^\mu\overrightarrow{\phi} = 0, \quad V\left(\overrightarrow{\phi}\right) = 0. \tag{70}$$

It should be noticed that the Higgs field obeys $\phi^2 = (\phi_1^2 + \phi_2^2 + \phi_3^2) = a^2$ in the Higgs vacuum. Such vacuum configuration is no more invariant under the transformations of $SO(3)$, but only under an $SO(2) \cong U(1)$ subgroup. Thus, this model exhibits spontaneous symmetry breaking mechanism. On the other hand, Olive [33] obtained the resultant masses of gauge particles as,

$$M(e,0) = a \mid e \mid; \tag{71}$$

where e is the eigen value of electric charge of a massive eigenstate and a specifies the the magnitude of the vacuum expectation value of scalar Higgs field. In 't Hooft-Polyakov model, after symmetry breaking, we have the $U(1)$ gauge theory which has all the characteristics of Maxwell's electromagnetic theory. The 't Hooft - Polyakov monopole carries one Dirac unit of magnetic charge. These monopoles are not elementary particles like Dirac's monopoles but complicated extended objects having a definite size inside of which massive fields play a role in providing a smooth structure and outside they rapidly vanish leaving the field configuration identical to Dirac's monopoles. The 't Hooft -Polyakov monopole was known numerically but there is simplified model introduced by Prasad and Sommerfield [19] which has an explicit stable monopole solution. Such solution satisfying Bogomonly condition [20] are named as Bogomonly - Prasad - Sommerfield (BPS) monopoles. These static monopoles in R^3 - space have been extensively studied in recent years and it became clear that they have remarkable properties which are best understood as a special case of self-duality equations in four space for solutions independent of one of the variables. The mass of monopole solution with a smooth internal structure is calculable to have the following lower limit of Prasad and Sommerfield,

$$M(0,g) \geq a \mid g \mid . \tag{72}$$

Which is possible in Prasad-Sommerfield limit [19] , where the the 't Hooft- Polyakov monopole solutions are generalized to vanish the self interaction of Higgs field.

References

[1] P. A. M. Dirac, **"Quantized singularities in the electromagnetic field"**, Proc. Roy. Soc. London, **A133**, 60 (1931).

[2] M. N. Saha, "**Note on Dirac's theory of magnetic poles**", Phys. Rev., **75,** 1968 (1949).

[3] W. Malkus, **"The interaction of the Dirac magnetic monopole with matter",** Phys. Rev. **83,** 899 (1951).

[4] E. Goto, "**Expected behaviour of the Dirac monopole in the cosmic space**", Prog. Theor. Phys. **30,** 700 (1963).

[5] D. Rosenbaum, **"Direct action and magnetic monopoles"**, Phys. Rev., **147,** 891 (1966).

[6] D. Zwanziger, "**A Regular Theory of Magnetic Monopoles**", Phys. Rev., **B137,** 647 (1965).

[7] A. S. Goldhaber, **"On the monopole-dyon system in non-abelian Gauge theories",** Phys. Rev., **B140,** 1407 (1965).

[8] J. Schwinger, **"A Magnetic Model of Matter",** Phys. Rev. **144,** (1966) 1087; **151,** 1055 (1966).

[9] A. S. Goldhaber, **"Spin and statistics connection for charge monopole composites",** Phys. Rev. Lett., **36,** 1122 (1976).

[10] D. Zwanzinger, **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176,** 1489 (1968).

[11] A. Soper, **"Multi - monopoles close together",** Nucl. Phys., **B199,** 290 (1982).

[12] T. T. Wu and C. N. Yang, "**Dirac monopole without strings: monopole harmonics**", Nucl. Phys., **B107,** 365 (1976).

[13] P. B. Price **"Evidence for detection of a moving magnetic monopole"**, Phys. Rev. Lett., **35,** 487 (1975).

[14] N. Cabibbo and E. Ferrari, **"Quantum electrodynamics with Dirac monopoles",** Nuovo Cim., **23,** 1147 (1962).

[15] G. ' t Hooft, **"Magnetic monopoles in unified gauge theories"**, Nucl. Phys., **B79,** 276 (1974).

[16] A. M. Polyakov, **"Particle spectrum in quantum field theory"**, JETP Lett., **20**, 194 (1974).

[17] B. Julia and A. Zee , **"Poles with both magnetic and electric charges in non-Abelian gauge theory"**, Phys. Rev., **D11,** 2227_(1975).

[18] M. K. Prasad, **"Yang - Mills - Higgs monopole solutions of arbitrary topological charge"**, Comm. Math. Phys., **80,** 137 (1981).

[19] M. K. Prasad and C. M. Sommerfield, "**Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon**", Phys. Rev. Lett., **35**, 760 (1975).

[20] E. B. Bogomolny, "**Stability of Classical Solutions**", Sov. J. Nucl. Phys., **24**, 449 (1976).

[21] L. E. Dickson, **"On Quaternions and Their Generalization and the History of the Eight Square Theorem"**, Ann. Math., **20**, 155 (1919).

[22] A. Pais, **"Remark on the Algebra of Interactions"**, Phys. Rev. Lett., **7**, 291 (1961).

[23] P. S. Bisht, B. Pandey and O. P. S. Negi, **"Interpretations of octonion wave equations"**, FIZIKA (Zagreb), **B17,** 405 (2008).

[24] B. C. Chanyal, P. S. Bisht and O. P. S. Negi , **"Generalized Octonion Electrodynamics"**, Int. J. Theor. Phys., **49,** 1333 (2010).

[25] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"**, Int. J. Theor. Phys., **50,** 1919 (2011).

[26] S. Dangwal, P. S. Bisht and O. P. S. Negi, **"Octonionic Gauge Formulation for Dyonic fields",** arXiv: hep-th/ 0608061 (2006).

[27] O. P. S. Negi and H. Dehnen , **"Gauge Formulation for Two potential Theory of Dyons"**, Int. J. Theor. Phys., **50,** 2446 (2011).

[28] M. Günaydin and Feza Gürsey; **"Quark structure and octonions"**, J. Math. Phys., **14,** 1651 (1973).

[29] M. Günaydin, **"Octonionic Hilbert spaces, the Poincaré group and SU(3)"**, J. Math. Phys.,_**17,** 1875 (1976).

[30] S. Catto, **"Exceptional Projective Geometries and Internal Symmetries"**, hep-th/0302079 v1 (2003).

[31] B. C. Chanyal, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Octonion Quantum Chromodynamics",** Int. J. Theor. Phys., **51**, 3410 (2012).

[32] J. M. Figueroa and O' Farrill, **"Electromagnetic Duality for Children",** www.maths.ed.ac.uk, Teaching-Lectures (1998).

[33] D. I. Olive, **"Exact Electromagnetic Duality"**, hep-th/9508089, (1995).