Q-FORMULÆ

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Last version: February 4, 2013

Abstract

This compilation of formula of quaternionic algebra and quaternionic differentials is for a significant part derived from Bo Thidé's book "Electromagnetic Field Theory"; http://www.plasma.uu.se/CED/Book. I have merely converted the vector formula into quaternionic format.

Two types of quaternionic differentiation exist.

- Flat differentiation uses the quaternionic nabla and ignores the curvature of the parameter space.
- Full differentiation uses the distance function $\wp(x)$ that defines the curvature of the parameter space.

The text focuses at applications in quantum mechanics, in electrodynamics and in fluid dynamics.

Note of the author

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Contents

1 Introduction

Let x be the position vector (radius vector, coordinate vector) from the origin of the Euclidean space $\mathbb{R}^3$ coordinate system to the coordinate point $(x_1;\ x_2;\ x_3)$ in the same system and let $|x|$ denote the magnitude ('length') of x. Let further $\alpha(x), \beta(x), \gamma(x)$, …, be arbitrary scalar fields, $\boldsymbol{a}(x), \boldsymbol{b}(x), \boldsymbol{c}(x)$, …, arbitrary vector fields, and $\boldsymbol{A}(x), \boldsymbol{B}(x), \boldsymbol{C}(x)$, …,arbitrary rank two tensor fields in this space.

Let q be the position relative to the origin of the space $\mathbb{H}$ that is spanned by the quaternions and that is given by the coordinate point $(q_0;\ q_1;\ q_2;\ q_3))$ and let $|q|$ denote the norm of q.

Let * denote complex or quaternionic conjugate and † denote Hermitian conjugate (transposition and, where applicable, complex or quaternionic conjugation).

1.1 Cayley-Dickson construction

The Cayley-Dickson construction formula enable the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \quad (1)$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \quad (2)$$

$$(a, b)\,(c, d) = (ac - db^*;\ a^*d + cb) \quad (3)$$

$$r = pq \quad (4)$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \quad (5)$$

$$r_\mathbf{k} = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \quad (6)$$

$$r_\mathbf{i} = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \quad (7)$$

$$r_j = -a_1 d_0 + a_0 d_1 + b_0 c_1 + b_1 c_0 \quad (8)$$

1.2 Warren Smith's numbers

All hyper-complex numbers are based on real numbers. Two main construction formulas for hyper-complex numbers exist. The Cayley-Dickson construction is the most widely known. The Warren-Smith construction gives best algorithmic properties at higher dimensions. Until the octonions both construction formulas deliver the same results.

The quaternions are the highest dimensional hyper-complex numbers that deliver a division ring.

1.2.1 2^n-on construction

The 2^n-ons use the following doubling formula

$$(a,b)(c,d) = (a\,c - (b\,d^*)^*, (b^*c^*)^* + (b^*(a^*((b^{-1})^*d^*)^*)^*)^*) \quad (1)$$

Up until the 16-ons the formula can be simplified to

$$(a,b)(c,d) = (a\,c - b\,d^*, c\,b + (a^*\,b^{-1})\,(b\,d)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the 2^n-on construction. From n>3 the 2^n-ons are 'nicer'.

1.2.1.1 2^n-ons

Table of properties of the 2^n-ons.

See scorevoting.net/WarrenSmithPages/homepage/nce2.ps.

Type	***name***	**Lose**
1-ons	*Reals.*	
2-ons	*Complex numbers*	$z^* = z$ (the * denotes conjugating); the ordering properties that both {z > 0, -z > 0, or z = 0} and {w > 0, z > 0 implies w + z > 0, wz > 0}.
4-ons	*Quaternions*	commutativity ab = ba; the algebraic closedness property that every univariate polynomial equation has a root.
8-ons	*Octonions*	associativity ab · c = a · bc.

16-ons	(*not Sedenions!*)	right-alternativity $x \cdot yy = xy \cdot y$; right-cancellation $x = xy \cdot y^{-1}$; flexibility $x \cdot yx = xy \cdot x$; left-linearity $(b + c)a = ba + ca$; anti-automorphism $ab = ba$, $(ab)^{-1} = b^{-1} a^{-1}$; left-linearity $(b + c)a = ba + ca$; continuity of the map $x \rightarrow xy$; Moufang and Bol identities; diassociativity
32-ons		generalized-smoothness of the map $x \rightarrow xy$; right-division properties that $xa = b$ has (generically) a solution x, and the uniqueness of such an x; the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula: $\lim_{n\to\infty}\left[e^{x/n}e^{y/n}\right]^n = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n = e^{x+y}$

Type	**Retain**
2^n-ons	Unique 2-sided multiplicative & additive identity elements 1 & 0; Norm-multiplicativity $\|xy\|^2 = \|x\|^2 \cdot \|y\|^2$; Norm-subadditivity $\|a + b\| \leq \|a\| + \|b\|$; 2-sided inverse $a^{-1} = a^*/\|a\|^2$ ($a \neq 0$); $a^{**} = a$; $(x \pm y)^* = x^* \pm y^*$; $(a^{-1})^{-1} = a$; $(a^*)^{-1} = (a^{-1})^*$; $\|a\|^2 = \|a\|^2 = a^*a$; Left-alternativity $yy \cdot x = y \cdot yx$; Left-cancellation $x = y^{-1} \cdot yx$; Right-linearity $a(b + c) = ab + ac$; r^{th} power-associativity $a^n a^m = a^{n+m}$; Scaling $s \cdot ab = sa \cdot b = as \cdot b = a \cdot sb = a \cdot bs = ab \cdot s$ (s real); Power-distributivity $(ra^n + sa^m)b = ra^n b + sa^m b$ (r, s real); Vector product properties of the imaginary part: **ab** - re(**ab**) of the product for pure-imaginary 2^n-ons **a**,**b** regarded as $(2^n - 1)$-vectors; $\langle xa,b\rangle = \langle a,x^*b\rangle$, $\langle xa,xb\rangle = \|x\|^2 \cdot \langle a,b\rangle$ and $\langle x,y\rangle = \langle x^*,y^*\rangle$ Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate ``niner'' versions of most of those properties; contains 2^{n-1}-ons as subalgebra.

1.2.1.1.1 The most important properties of 2^n-ons

If a,b,x,y are 2^n-ons, $n \geq 0$, and s and t are scalars (i.e. all coordinates are 0 except the real coordinate) then

unit: A unique 2^n-on 1 exists, with $1 \cdot x = x \cdot 1 = x$.

zero: A unique 2^n-on 0 exists, with $0 + x = x + 0 = x$ and $0 \cdot x = x \cdot 0 = 0$.

additive properties: $x+y = y+x$, $(x+y)+z = x+(y+z)$;

$-x$ exists with $x + (-x) = x - x = 0$.

norm: $|x|^2 = xx^* = x^*x$.

norm-multiplicativity: $|x|^2 \cdot |y|^2 = |x \cdot y|^2$.

scaling: $s \cdot x \cdot y = s \cdot x \cdot y = x \cdot s \cdot y = x \cdot s \cdot y = x \cdot y \cdot s$.

weak-linearity: $(x + s) \cdot y = x \cdot y + s \cdot y$ and $x \cdot (y + s) = x \cdot y + x \cdot s$.

right-linearity: $x \cdot (y + z) = x \cdot y + x \cdot z$.

inversion: If $x \neq 0$ then a unique x^{-1} exists, obeying $x^{-1} \cdot x = x \cdot x^{-1} = 1$. It is $x^{-1} = x \cdot |x|^{-2}$.

left-alternativity: $x \cdot xy = x^2 \cdot y$.

left-cancellation: $x \cdot x^{-1} \cdot y = y$.

effect on inner products: $\langle x \cdot a, b \rangle = \langle a, x^* \cdot b \rangle$, $\langle x, y \rangle = \langle x^*, y^* \rangle$, $\langle x^* \cdot a, x^{-1} \cdot b \rangle = \langle a, b \rangle$,

and $\langle x \cdot a, x \cdot b \rangle = |x|^2 \cdot \langle a, b \rangle$.

Conjugate of inverse: $(x^{-1})^* = (x^*)^{-1}$.

Near-anticommutativity of unequal basis elements: $e_k^2 = -1$ and $e_k \cdot e_l^* = -e_l \cdot e_k^*$ if $k \neq l$.

(Note: the case k; $l > 0$ shows that unequal pure-imaginary basis elements anticommute.)

Alternative basis elements: $e_k \cdot e_l \cdot e_k = e_k \cdot e_l \cdot e_k$, $e_l \cdot e_k \cdot e_k = e_l \cdot e_k \cdot e_k$, and $e_k \cdot e_k \cdot e_l = e_k \cdot e_k \cdot e_l$. (However, when $n \geq 4$ the 2^n-ons are not flexible i.e. it is not generally true that $x \cdot y \cdot x = x \cdot y \cdot x$ if x and y are 16-ons that are not basis elements. They also are not right-alternative.)

Quadratic identity: If x is a 2^n-on (over any field F with charF $\neq 2$), then $x^2 + |x|^2 = 2 \cdot x$ re x

Squares of imaginaries: If x is a 2^n-on with re x = 0 ("pure imaginary") then $x^2 = -|x|^2$ is nonpositive pure-real.

Powering preserves im*x* direction

1.2.1.1.2 Niners

Niners are 2n-ons whose coordinates with index > 8 are zero. The index starts with 0.

9-**flexibility** $xp \cdot x = x \cdot px$, $px \cdot p = p \cdot xp$.

9-**similitude unambiguity** $xp \cdot x^{-1} = x \cdot px^{-1}$, $px \cdot p^{-1} = p \cdot xp^{-1}$.

9-**right-alternativity** $xp \cdot p = x \cdot p^2$, $px \cdot x = p \cdot x^2$.

9-**right-cancellation** $xp^{-1} \cdot p = x$, $px^{-1} \cdot x = p$.

9-**effect on inner products** $\langle x, yp\rangle = \langle xp, y\rangle$, $\langle xp, yp\rangle = |p|^2\langle x, y\rangle$.

9-**left-linearity** $(x + y)p = xp + yp$, $(p + q)x = px + qx$.

9-**Jordan-identity** $xp \cdot xx = x(p \cdot xx)$, $py \cdot pp = p(y \cdot pp)$.

9-**coordinate-distributivity** $([x + y]z)_{0;:::;8} = (xz + yz)_{0;:::;8}$.

9-**coordinate-Jordan-identity** $[xy \cdot xx]_{0;:::;8} = [x(y \cdot xx)]_{0;:::;8}$.

9-**anticommutativity for orthogonal imaginary 2^n-ons**

If $\langle p, x\rangle = re\ p = re\ x = 0$ then $px = -xp$.

9-**reflection** If $|a| = 1$ and the geometric reflection operator is defined below then $-(refl[a](y))_{0;:::;8} = (a \cdot y^* a)_{0;:::;8}$, and $-\{refl[a](y)\}^*_{0;:::;8} = (a^* y \cdot a^*)_{0;:::;8}$, and

if either a or y is a niner then $-refl[a](y) = a \cdot y^* a$ and $-refl[a](y) = a^* y \cdot a^*$.

$$\mathrm{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t}\rangle}{|\vec{x}|^2}\vec{x} \tag{1}$$

What holds for the niners, also holds for the octonions.

1.3 Waltz details

The 16-ons lose the continuity of the map $x \Rightarrow xy$. Also, in general holds $(x\ y)x \neq x\ (y\ x)$ for 16-ons. However, for all 2^n-ons the base numbers fulfill $(e_i\ e_j)\ e_i = e_i\ (e_j\ e_i)$. All 2^n-ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The 2^n-ons support the **number waltz**

$$c = a\ b/a. \tag{1}$$

Often the number waltz appears as a unitary number waltz

$$c = u^* b\ u \tag{2}$$

where u is a unit size number and u^* is its conjugate $u\,u^* = 1$.

In quaternion space the **quaternion waltz** $a\ b/a$ can be written as

$$a\ b\ /\ a = exp(2\,\pi\,\tilde{\imath}\,\varphi)\ b\ exp(-2\,\pi\,\tilde{\imath}\,\varphi) \tag{3}$$

$$= b - \boldsymbol{b}_\perp + exp(2\,\pi\,\tilde{\imath}\,\varphi)\ \boldsymbol{b}_\perp\ exp(-2\,\pi\,\tilde{\imath}\,\varphi)$$

$$= b - \boldsymbol{b}_\perp + exp(4\,\pi\,\tilde{\imath}\,\varphi)\boldsymbol{b}_\perp$$

$$\Delta b = (exp(4\,\pi\,\tilde{\imath}\,\varphi) - 1)\boldsymbol{b}_\perp \tag{4}$$

$$= (cos(4\,\pi\,\varphi) + \tilde{\imath}\,sin(4\,\pi\,\varphi) - 1)\ \boldsymbol{b}_\perp$$

$$= exp(2\,\pi\,\tilde{\imath}\,\varphi)\ 2\ \tilde{\imath}\,sin\,(2\,\pi\,\varphi)\ \boldsymbol{b}_\perp$$

$$\|\Delta b\| = \|2\,sin(2\,\pi\,\varphi)\ \boldsymbol{b}_\perp\| \tag{5}$$

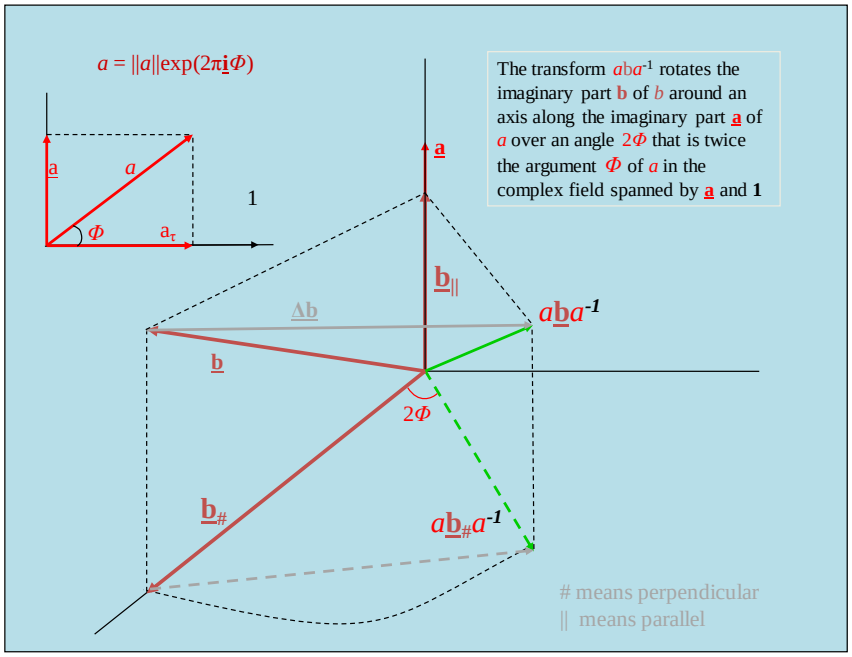

Figure 1. The rotation of a quaternion by a second quaternion.

Another way of specifying the difference is:

$$\Delta b \;=\; (a \cdot b - b \cdot a)/a \;=\; 2 \cdot (\boldsymbol{a} \times \boldsymbol{b})/a \tag{6}$$

$$\|\Delta b\| \;=\; 2\,\|\boldsymbol{a} \times \boldsymbol{b}\|/\,\|a\| \tag{7}$$

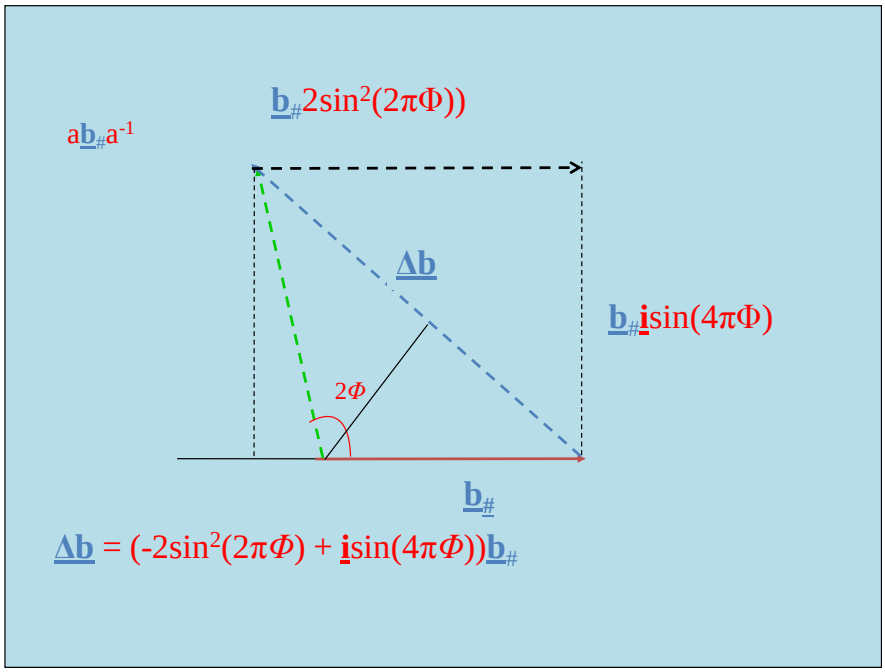

Figure 2: The difference after rotation

1.4 Spinors and matrices

In contemporary physics complex probability amplitude distributions (CPAD's) are used rather than QPAD's. Spinors and matrices are used to simulate QPAD behavior for CPAD's.

1.4.1 Symmetries

The quaternionic number system exists in sixteen discrete symmetry sets (sign flavors). When the real part is ignored, then eight different symmetry sets result. The values of a continuous distribution all belong to the same symmetry set. The parameter space of the distribution may belong to a different symmetry set.

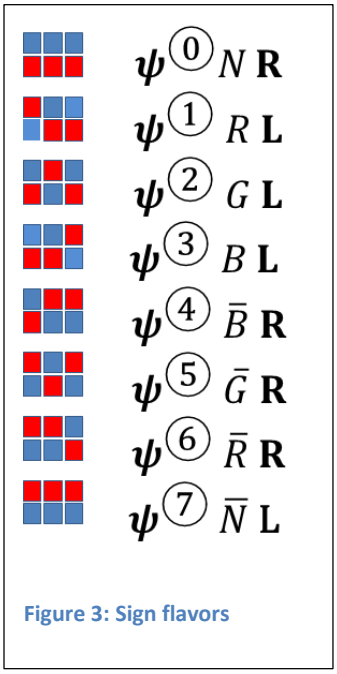

Figure 3: Sign flavors

Eight sign flavors

(discrete symmetries)

Colors N, R, G, B, $\overline{\text{R}}$, $\overline{\text{G}}$, $\overline{\text{B}}$, W

Right or Left handedness R,L

[1]The red block indicates sign up or down with respect to the base sign flavor. For quaternionic distributions the (quaternionic) parameter space acts as base sign flavor.

The 3D Kronecker delta tensor

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{1}$$

The fully antisymmetric Levi-Civita tensor

$$\in_{ijk} = \begin{cases} 1 \text{ if } i,j,k \text{ is an even permutation of } 1,2,3 \\ 0 \text{ if at least two of } i,j,k \text{ are equal} \\ -1 \text{ if } i,j,k \text{ is an even permutation of } 1,2,3 \end{cases} \tag{2}$$

[1] This picture has been changed!

1.4.2 Spinor

We use square brackets for indicating spinors. Spinors use real component functions ψ_i. . Complex component functions $\psi_i$would result in spinor representations of bi-quaternions. Bi-quaternions do not form a division ring[2].

A 2×2 spinor is defined by the row:

$$[\psi] \equiv \left[[\psi_0][\boldsymbol{\Psi}]\right] \tag{1}$$

$$[\psi]^{\ddagger} \equiv \left[[\boldsymbol{\Psi}][\psi_0]\right] \tag{2}$$

Where

$$[\psi_0] \equiv \begin{bmatrix} \Psi_0 & 0 \\ 0 & \Psi_0 \end{bmatrix} \tag{3}$$

$$[\boldsymbol{\Psi}] \equiv \begin{bmatrix} \Psi_3 & \Psi_1 - i\Psi_2 \\ \Psi_1 + i\Psi_2 & -\Psi_3 \end{bmatrix} \tag{4}$$

Spinors obey[3]

$$[\boldsymbol{\Psi}] + [\boldsymbol{\phi}] = 2[\langle \boldsymbol{\Psi}, \boldsymbol{\phi} \rangle] \tag{5}$$

$$[\boldsymbol{\Psi}] - [\boldsymbol{\phi}] = 2\, i[\boldsymbol{\Psi} \times \boldsymbol{\phi}] \tag{6}$$

1.4.2.1 Sign flavors

The relation with the sign flavors is

$$[\boldsymbol{\Psi}] = [\boldsymbol{\Psi}]^{⓪} = \left[\boldsymbol{\psi}^{⓪}\right] \tag{1}$$

[2] The author uses its own notation for spinors and sign flavors
[3] http://en.wikipedia.org/wiki/Spinors_in_three_dimensions

$$[\boldsymbol{\Psi}]^{①} = [\boldsymbol{\psi}^{①}] \tag{2}$$

$$[\boldsymbol{\Psi}]^{②} = [\boldsymbol{\psi}^{②}] \tag{3}$$

$$[\boldsymbol{\Psi}]^{③} = [\boldsymbol{\psi}^{③}] \tag{4}$$

$$[\boldsymbol{\Psi}^{*}] = [\boldsymbol{\Psi}^{*}]^{⓪} = [\boldsymbol{\psi}^{⑦}] \tag{5}$$

$$[\boldsymbol{\Psi}^{*}]^{①} = [\boldsymbol{\psi}^{⑥}] \tag{6}$$

$$[\boldsymbol{\Psi}^{*}]^{②} = [\boldsymbol{\psi}^{⑤}] \tag{7}$$

$$[\boldsymbol{\Psi}^{*}]^{③} = [\boldsymbol{\psi}^{④}] \tag{8}$$

1.4.3 Dirac spinors

The 4×4 spinors target the application in the Dirac equation.

A general 4×4 spinor is defined by the column:

$$\begin{bmatrix} [\psi] \\ [\phi^{*}]^{‡} \end{bmatrix} \equiv \begin{bmatrix} [\Psi_0] & [\boldsymbol{\Psi}] \\ [-\boldsymbol{\phi}] & [\phi_0] \end{bmatrix} \tag{1}$$

A compacted spinor $]\Psi[$ is a 1×4 matrix consisting of real functions that represent all sixteen sign flavors of a QPAD.

$$]\Psi[\equiv \begin{bmatrix} [\psi] \\ [\psi^{*}]^{‡} \end{bmatrix} = \begin{bmatrix} [\Psi_0] & [\boldsymbol{\Psi}] \\ [-\boldsymbol{\Psi}] & [\Psi_0] \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \Psi_0 & 0 & \Psi_3 & \Psi_1 - i\Psi_2 \\ 0 & \Psi_0 & \Psi_1 + i\Psi_2 & -\Psi_3 \\ -\Psi_3 & -\Psi_1 + i\Psi_2 & \Psi_0 & 0 \\ -\Psi_1 - i\Psi_2 & +\Psi_3 & 0 & \Psi_0 \end{bmatrix}$$

1.4.4 Spinor base

The $\boldsymbol{\alpha}$ and β matrices form the base of spinor $]\Psi[$ and its elements

$$\alpha_1 \equiv \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{1}$$

$$\alpha_2 \equiv \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_3 \equiv \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{3}$$

$$\beta \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ represent imaginary base vectors of the simulated quaternion. β represents the conjugation action for the spinor.

A relation exist between $\alpha_1, \alpha_2, \alpha_3$ and the Pauli[4] matrices $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

$$1 \mapsto I, \qquad \boldsymbol{i} \mapsto \sigma_1, \qquad \boldsymbol{j} \mapsto \sigma_2, \qquad \boldsymbol{k} \mapsto \sigma_3 \tag{6}$$

1.4.5 Gamma matrices

This combination is usually represented in the form of gamma matrices.

In Dirac representation, the four contravariant gamma matrices are

$$\gamma^0 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \gamma^1 \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

$$\gamma^2 \equiv \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma^3 \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

[4] http://en.wikipedia.org/wiki/Pauli_matrices

It is useful to define the product of the four gamma matrices as follows:

$$\gamma^5 \equiv i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2}$$

The gamma matrices as specified here are appropriate for acting on Dirac spinors written in the Dirac basis; in fact, the Dirac basis is defined by these matrices. In the Dirac basis[5]:

$$\gamma^0 \equiv \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{3}$$

This corresponds with $\alpha_k = \gamma^k, \beta = \gamma^5$.

Apart from the Dirac basis, a Weyl basis exists

$$\gamma^0 = \gamma^\beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tag{4}$$

The Weyl basis has the advantage that its chiral projections[6] take a simple form:

$$\psi_L = ½\,(1 - \gamma^5)[\psi] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}[\psi] \tag{5}$$

$$\psi_R = ½\,(1 + \gamma^5)[\psi] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}[\psi] \tag{6}$$

[5] http://en.wikipedia.org/wiki/Gamma_matrices#Dirac_basis

[6] http://en.wikipedia.org/wiki/Chirality_(physics)

$$[\psi^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [\psi] \tag{7}$$

1.5 Differentiation in flat space

We treat quaternionic distributions as if they possess a continuous parameter space. The differential vector operator $\boldsymbol{\nabla}$ is in Cartesian coordinates given by

$$\boldsymbol{\nabla} = \sum_{i=1}^{3} \boldsymbol{e}_i \frac{\partial}{\partial x_i} \tag{1}$$

The flat quaternionic differential operator ∇ is in Cartesian coordinates given by

$$\nabla = \sum_{i=0}^{3} e_i \nabla_i = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}; \quad e = (1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}) \tag{2}$$

$$\nabla f = \sum_{i=0}^{3} \sum_{j=0}^{3} e_i e_j \frac{\partial f_j}{\partial x_i} \tag{3}$$

1.6 Differentiation in curved space

The distance function $\wp(x)$ has a flat parameter space that is spanned by the rational or the real quaternions[7]. However, in this section we treat $\wp(x)$ as if it has a continuous parameter space. Thus we, treat the A-type $\wp(x)$ as if it is a B-type $\wp(x)$. That makes it possible to use regular differential calculus. The full quaternionic difference operator $\mathrm{d}\wp$ is given by

$$\mathrm{d}\wp = \sum_{\mu=0}^{3} q^{\mu} \, dx_{\mu} = \sum_{\mu=0}^{3} \frac{\partial \wp}{\partial x_{\mu}} dx_{\mu} = \sum_{\nu=0}^{3} e_{\mu} \sum_{\mu=0}^{3} \frac{\partial \wp_{\nu}}{\partial x_{\mu}} dx_{\mu} \tag{1}$$

Here the coefficients q^{μ} are quaternionic coefficients, which are determined by the quaternionic distance function $\wp(x)$.
$\wp(x)$ defines a curved target space. This curved space can act as parameter space to other quaternionic distributions.

$$q^{\mu} = \frac{\partial \wp}{\partial x_{\mu}}; \quad \wp = \sum_{\nu=0}^{3} e_{\nu} \wp_{\nu} \tag{2}$$

The distance function $\wp(x)$ may include an isotropic scaling function $a(\tau)$ that only depends on progression τ. It defines the expansion/compression of the curved space.

[7] http://en.wikipedia.org/wiki/Quaternion_algebra#Quaternion_algebras_over_the_rational_numbers

The quaternionic infinitesimal interval $d\wp$ defines the quaternionic metric of the curved space that is defined by $\wp(x)$.

In this way, the quaternionic function g(ζ), which has a curved parameter space defined by $\zeta = \wp(x)$ corresponds to a new function h(x)= g($\wp$(x)), which has a flat parameter space. The flattened nabla $\bar{\nabla}$ is defined as:

$$\bar{\nabla} g = \sum_{\nu=0}^{3} e_\nu \frac{\partial g(\zeta)}{\partial x_\nu} = \sum_{\nu=0}^{3} e_\nu \sum_{\lambda=0}^{3} e_\lambda \frac{\partial g_\lambda}{\partial x_\nu} = \sum_{\nu=0}^{3} e_\nu \sum_{\lambda=0}^{3} e_\lambda \sum_{\mu=0}^{3} \frac{\partial g_\lambda}{\partial \zeta_\mu} e_\mu \frac{\partial \zeta_\mu}{\partial x_\nu} \tag{3}$$

$$= \sum_{\nu=0}^{3} \sum_{\lambda=0}^{3} \sum_{\mu=0}^{3} e_\nu e_\lambda e_\mu \frac{\partial g_\lambda}{\partial \zeta_\mu} \frac{\partial \wp_\mu}{\partial x_\nu}$$

2 Coordinate systems

2.1 Cylindrical circular coordinates

2.1.1 Base vectors

2.1.1.1 Cartesian to cylindrical circular

$$\rho = x_1 \cos(\theta) + x_2 \sin(\theta) \tag{1}$$

$$\varphi = -x_1 \sin(\theta) + x_2 \cos(\theta) \tag{2}$$

$$z = x_3 \tag{3}$$

2.1.1.2 Cylindrical circular to Cartesian

$$x_1 = \boldsymbol{\rho} \cos(\theta) - \boldsymbol{\varphi} \sin(\theta) \tag{1}$$

$$x_2 = \rho \sin(\theta) + \boldsymbol{\varphi} \cos(\theta) \tag{2}$$

$$x_3 = z \tag{3}$$

2.1.1.3 Directed line element

$$dl = dx \frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \boldsymbol{e}_\rho d\rho + \boldsymbol{e}_\varphi \rho d\varphi + \boldsymbol{e}_z dz \tag{1}$$

2.1.1.4 Solid angle element

$$d\Omega = \sin(\theta)\, d\theta\, d\varphi \tag{1}$$

2.1.1.5 Directed area element

$$d\boldsymbol{S} = \boldsymbol{e_r}\, r^2\, d\Omega + \boldsymbol{e_\theta}\, r\, sin(\theta)\, dr\, d\varphi + \boldsymbol{e_\varphi}\; r\, dr\, d\theta \tag{1}$$

2.1.1.6 Volume element

$$dV = dx^3 = dr\, r^2\, d\Omega \tag{1}$$

2.1.1.7 Spatial differential operators

$$\alpha \;=\; \alpha(r,\theta,\varphi) \tag{1}$$

$$\boldsymbol{a} = \boldsymbol{a}(r,\theta,\varphi) \tag{2}$$

Gradient

$$\nabla\alpha = \boldsymbol{e_r}\,\frac{\partial\alpha}{\partial r} + \boldsymbol{e_\theta}\,\frac{1}{r}\frac{\partial\alpha}{\partial\theta} + \mathbf{e}_{\boldsymbol{\varphi}}\,\frac{1}{r\, sin(\theta)}\frac{\partial\alpha}{\partial\varphi} \tag{3}$$

Divergence

$$\langle\boldsymbol{\nabla}, \boldsymbol{a}\rangle = \frac{1}{r^2}\frac{\partial(r^2\alpha_r)}{\partial r} + \frac{1}{r\, sin(\theta)}\frac{\partial(a_\theta\, sin(\theta))}{\partial\theta} + \frac{1}{r\, sin(\theta)}\frac{\partial a_\varphi}{\partial\varphi} \tag{4}$$

Curl

$$\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{e_r}\,\frac{1}{r\, sin(\theta)}\left(\frac{\partial\left(a_\varphi\, sin(\theta)\right)}{\partial\theta} - \frac{\partial a_\varphi}{\partial\varphi}\right) + \boldsymbol{e_\theta}\,\frac{1}{r}\left(\frac{1}{sin(\theta)}\frac{\partial\alpha_r}{\partial\varphi} - \frac{\partial a_\varphi}{\partial r}\right) \tag{5}$$

$$+\boldsymbol{e_\varphi}\,\frac{1}{r}\left(\frac{\partial r\, a_\varphi}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)$$

The Laplacian

$$\nabla^2\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) + \frac{1}{r^2\, sin(\theta)}\frac{\partial}{\partial\theta}\left(sin(\theta)\frac{\partial\alpha}{\partial\theta}\right) + \frac{1}{r^2\, sin^2(\theta)}\frac{\partial^2\alpha}{\partial\varphi^2} \tag{6}$$

2.1.2 Polar coordinates

The equivalent to rectangular coordinates in quaternion space is (a_τ, a_x, a_y, a_z)

$$a = a_\tau + \boldsymbol{i}\, a_x + \boldsymbol{j}\, a_y \pm \boldsymbol{i}\boldsymbol{j}\, a_z \quad (1)$$

The equivalent to polar coordinates in quaternion space is

$$\mathrm{a}_\tau = \|\mathrm{a}\| \cos(\psi) \quad (2)$$

$$\mathrm{a}_\mathrm{x} = \|\mathrm{a}\| \sin(\psi) \sin(\theta) \cos(\varphi) \quad (3)$$

$$\mathrm{a}_\mathrm{y} = \|\mathrm{a}\| \sin(\psi) \sin(\theta) \sin(\varphi) \quad (4)$$

$$a_z = \|a\|\, sin(\psi)\, cos(\theta) \quad (5)$$

$sin(\psi)$, where $\psi = (0, \pi)$, is known as the (imaginary) amplitude of the quaternion.

Angle $\theta = (0, \pi)$ is the (co-)latitude and angle $\varphi = (0,2\pi)$ is the longitude.

For any fixed value of ψ, θ and φ parameterize a 2-sphere of radius $sin(\psi)$, except for the degenerate cases, when ψ equals 0 or π, in which case they describe a point.

This suggests the following structure of the argument ***Λ***

$$a = \|a\|\, exp(\tilde{\imath} \cdot \psi) \quad (6)$$

$$= \|a\|\, (cos(\psi) + \tilde{\imath}\, sin(\psi)) \quad (7)$$

$$= a_\tau + \|a\|\, \tilde{\imath}\, sin(\psi) = a_\tau + \boldsymbol{a} \quad (8)$$

The imaginary number ĩ may take any direction.

2.1.3 3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously shrunk to a point without leaving the 3-sphere. The Poincaré conjecture[8] proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism)[9].

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + sin^2(\psi)\,(d\theta^2 + sin^2(\theta)d\varphi^2) \quad (1)$$

The volume form is given by

$$dV = sin^2(\psi)\,sin(\theta)\,d\psi \wedge d\theta \wedge d\varphi \quad (2)$$

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius *r* is

$$2\,\pi^2\,r^3 \quad (3)$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

$$½\,\pi^2\,r^4 \quad (4)$$

The 3-sphere has constant positive sectional curvature equal to $1/r^2$.

The 3-sphere has a natural Lie group structure SU(2) given by quaternion multiplication.

The 3-sphere admits non-vanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and non-vanishing vector fields. These may be taken to be any left-invariant vector

[8] http://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture
[9] http://en.wikipedia.org/wiki/3-sphere

fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group $\mathbb{T}$ on $\mathbb{S}^3$ giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of $\mathbb{S}^3$ as a subset of $\boldsymbol{C}^2$, the action is given by

$$(z_1, z_2)\,\lambda \;=\; (z_1\,\lambda, z_2\,\lambda)\;\forall_{\lambda\,\in\,\mathbb{T}}. \tag{5}$$

The orbit space of this action is homeomorphic to the two-sphere $\mathbb{S}^2$. Since $\mathbb{S}^3$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, the Hopf bundle is nontrivial.

2.1.4 Hopf coordinates

Another choice of hyperspherical coordinates, (η, ξ_1, ξ_2), makes use of the embedding of $\mathbb{S}^3$ in $\boldsymbol{C}^2$. In complex coordinates $(z_1, z_2) \in \boldsymbol{C}^2$ we write

$$z_1 = exp(\tilde{\imath}\,\xi_1)\,sin(\eta) \tag{1}$$

$$z_2 \;=\; exp(\tilde{\imath}\,\xi_2)\,cos(\eta) \tag{2}$$

Here η runs over the range 0 to $\pi/2$, and ξ_1 and ξ_2 can take any values between 0 and 2π. These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \tag{3}$$

For any fixed value of η between 0 and $\pi/2$, the coordinates (ξ_1, ξ_2) parameterize a 2-dimensional torus. In the degenerate cases, when η equals 0 or $\pi/2$, these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + sin^2(\eta)\,(d\zeta_1^2 + cos^2(\eta)\,d\,\zeta_2^2) \tag{4}$$

and the volume form by

$$dV = sin(\eta)\, cos(\eta)\, d\eta \hat{} d\zeta_1 \hat{} d\zeta_2 \quad (5)$$

2.1.5 Group structure

Because the set of unit quaternions is closed under multiplication, $\mathbb{S}^3$ takes on the structure of a group. Moreover, since quaternionic multiplication is smooth, $\mathbb{S}^3$ can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group $\mathbb{S}^3$ is often denoted $Sp(1)$ or $\mathrm{U}(1, \mathbb{H})$.

It turns out that the only spheres which admit a Lie group structure are $\mathbb{S}^1$, thought of as the set of unit complex numbers, and $\mathbb{S}^3$, the set of unit quaternions. One might think that $\mathbb{S}^7$, the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-associative. The octonionic structure does give $\mathbb{S}^7$ one important property: *parallelizability*[10]. It turns out that the only spheres which are parallelizable are $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$.

By using a matrix representation of the quaternions, $\mathbb{H}$, one obtains a matrix representation of $\mathbb{S}^3$. One convenient choice is given by the Pauli matrices:

$$\left(a_\tau + a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}\right) = \begin{bmatrix} a_\tau + \tilde{\imath} \cdot a_x & a_y + \tilde{\imath} \cdot a_z \\ -a_y + \tilde{\imath} \cdot a_z & a_\tau - \tilde{\imath} \cdot a_x \end{bmatrix} \quad (1)$$

This map gives an injective algebra homomorphism from **H** to the set of 2×2 complex matrices. It has the property that the absolute value of a quaternion *q* is equal to the square root of the determinant of the matrix image of *q*.

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group SU(2). Thus, $\mathbb{S}^3$ as a Lie group is isomorphic to SU(2).

Using our hyperspherical coordinates (η, ξ_1, ξ_2) we can then write any element of SU(2) in the form

$$\begin{bmatrix} \exp(\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) & \exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) \\ -\exp(\tilde{\imath} \cdot \xi_2) \cdot \cos(\eta) & \exp(-\tilde{\imath} \cdot \xi_1) \cdot \sin(\eta) \end{bmatrix} \quad (2)$$

Another way to state this result is if we express the matrix representation of an element of SU(2) as a linear combination of the Pauli matrices. It is seen that an arbitrary element U ∈ SU(2) can be written as

[10] http://en.wikipedia.org/wiki/Parallelizability

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n \boldsymbol{I_n} \quad (3)$$

The condition that the determinant of U is +1 implies that the coefficients α_n are constrained to lie on a 3-sphere.

2.1.6 Versor

Any **unit quaternion** q can be written as a **versor**:

$$u = exp(\tilde{\imath}\ \psi) = cos(\psi) + \tilde{\imath}\ sin(\psi) \quad (1)$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in Im $\mathbb{H}$ so any such $\tilde{\imath}$ can be written:

$$\tilde{\imath} = \boldsymbol{i}\, cos(\varphi)\, sin(\theta) + \boldsymbol{j}\, sin(\varphi)\, sin(\theta) + \boldsymbol{k}\, cos(\theta) \quad (2)$$

2.1.7 Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number k with unit length.

$q = a + b\boldsymbol{j}$; where $a = w + x\boldsymbol{i}$; and $b = y + z\boldsymbol{i}$

$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

2.1.8 Quaternionic algebra

2.1.8.1 Quaternions

$$a = (a_0, a_1, a_2, a_3) = \sum_{\mu=0}^{3} e_\mu \mathrm{a}_\mu = a_0 + \boldsymbol{i}\, a_1 + \boldsymbol{j}\, a_2 + \boldsymbol{k}\, a_3 = \; a_0 + \boldsymbol{a} \tag{1}$$

$$a^* = a_0 - \boldsymbol{a} \tag{2}$$

$$a^* a = a\, a^* = |a|^2 \tag{3}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{\mu=1}^{3} a_\mu\, b_\mu = \delta_{\mu\nu} a_\mu b_\nu = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta) \tag{4}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = \pm\left(\in_{\boldsymbol{ijk}} \; \boldsymbol{e_i} a_j b_k\right) \tag{5}$$

$$a\, b = \; a_0 \boldsymbol{b} + \; b_0\, \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{6}$$

The colored $\pm$ indicates the handedness of the vector cross product.

$$\boldsymbol{a}\, \boldsymbol{b} = \; -\langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{7}$$

$$a\,(b + c) = a\, b + a\, c \tag{8}$$

$$(a \; + \; b)\, c \; = \; a\, c \; + \; b\, c \tag{9}$$

$$(a\, b) c \; = \; a(b\, c) \tag{10}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \times \boldsymbol{c} \rangle = \; \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \tag{11}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{a}, \boldsymbol{c} \rangle - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle \tag{12}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b} \langle \boldsymbol{a}, \boldsymbol{c} \rangle - \boldsymbol{a} \langle \boldsymbol{b}, \boldsymbol{c} \rangle \tag{13}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0 \tag{14}$$

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \times \boldsymbol{d} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d}) \rangle = \langle \boldsymbol{a}, \boldsymbol{c} \rangle \langle \boldsymbol{b}. \boldsymbol{d} \rangle - \langle \boldsymbol{a}, \boldsymbol{d} \rangle \langle \boldsymbol{b}. \boldsymbol{c} \rangle \tag{15}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{d} \rangle \boldsymbol{c} - \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{d} \tag{16}$$

3 Quaternionic distributions

We consider three kinds of quaternionic distributions

A. Quaternionic distributions with a discrete parameter space. That parameter space must be flat and it is spanned by the rational quaternions[11].
B. Quaternionic distributions with a continuous parameter space. That parameter space is flat and it is spanned by the real quaternions.
C. Quaternionic distributions with a continuous parameter space. That parameter space may be curved. The curvature is defined by a category A type quaternionic distribution. The C-type quaternionic distribution has a countable set of values. It inherits the sign flavor of the quaternionic distribution that defines the curvature of its parameter space.
D. Quaternionic distributions with a continuous parameter space. That parameter space may be curved. The curvature is defined by a category B type quaternionic distribution. The C-type quaternionic distributions inherit the sign flavor of the quaternionic distribution that defines the curvature of their parameter space.

3.1.1 Basic properties

A continuous quaternionic distribution contains a scalar field in its real part and a vector field in its imaginary part.

$$f(x) = f_0(x) + \boldsymbol{f}(x) \tag{3}$$

$$a\, f(x) = a_0 \boldsymbol{f}(x) + f_0(x)\, \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{f}(x) \rangle \pm \boldsymbol{a} \times \boldsymbol{f}(x) \tag{2}$$

$$f(x)\, b = f_0(x) \boldsymbol{b} + b_0\, \boldsymbol{f}(x) - \langle \boldsymbol{f}(x), \boldsymbol{b} \rangle \pm \boldsymbol{f}(x) \times \boldsymbol{b} \tag{3}$$

The distributions follow the rules for the quaternion algebra.

$$a\,(f(x) + g(x)) = a\, f(x) + a\, g(x) \tag{4}$$

$$(a + b) f(x) = a\, f(x) + b\, f(x) \tag{5}$$

$$f(x)\, g(x) = f_0(x) \boldsymbol{g}(x) + g_0(x) \boldsymbol{f}(x) - \langle \boldsymbol{f}(x), \boldsymbol{g}(x) \rangle \pm \boldsymbol{f}(x) \times \boldsymbol{g}(x) \tag{6}$$

$$\big(f(x) g(x)\big) h(x) = f(x)(g(x)\, h(x)) \tag{7}$$

3.1.2 Symmetries

Continuous quaternionic distributions keep the same discrete symmetries (sign flavor) throughout their domain. The sign flavor of the parameter space acts as reference sign flavor.

3.1.3 Differentials

The quaternionic nabla acts similarly as a normal quaternion

$$\nabla\,(f(x) + g(x)) = \nabla\, f(x) + \nabla\, g(x) \tag{1}$$

[11] http://en.wikipedia.org/wiki/Quaternion_algebra#Quaternion_algebras_over_the_rational_numbers

$$\nabla f(x) = \nabla_0 \boldsymbol{f}(x) + \boldsymbol{\nabla} f_0(x) - \langle \boldsymbol{\nabla}, \boldsymbol{f}(x) \rangle \pm \boldsymbol{\nabla} \times \boldsymbol{f}(x) \quad (2)$$

However

$$\nabla(b\, c) \neq (\nabla\, b)c \quad (3)$$

and

$$\nabla(b\, c) \neq (\nabla\, b)c + b\, \nabla\, c \quad (4)$$

Further

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \alpha \equiv \boldsymbol{\nabla^2} \alpha \quad (5)$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \mathbf{a} \rangle = 0 \quad (6)$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = 0 \quad (7)$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \alpha = \mathbf{0} \quad (8)$$

$$\boldsymbol{\nabla}\, \boldsymbol{b} = -\langle \boldsymbol{\nabla}, \boldsymbol{b} \rangle \pm \boldsymbol{\nabla} \times \boldsymbol{b} \quad (9)$$

$$\boldsymbol{\nabla}\, (\alpha\, \beta) = \alpha \boldsymbol{\nabla}\, \beta + \beta \boldsymbol{\nabla}\, \alpha \quad (10)$$

$$\boldsymbol{\nabla}\, (\alpha\, \boldsymbol{a}) = \alpha \boldsymbol{\nabla} \times \boldsymbol{a} - \alpha \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle + (\boldsymbol{\nabla}\, \alpha) \boldsymbol{a} \quad (11)$$

$$\langle \boldsymbol{\nabla}, \alpha\, \boldsymbol{a} \rangle = \boldsymbol{a} \boldsymbol{\nabla} \alpha + \alpha \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle \quad (12)$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \times \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle - \langle \boldsymbol{a}, \boldsymbol{\nabla} \times \boldsymbol{b} \rangle \quad (13)$$

$$\langle \boldsymbol{\nabla}\, \alpha, \boldsymbol{\nabla}\, \beta \rangle = \langle \boldsymbol{\nabla}, \alpha \boldsymbol{\nabla}\, \beta \rangle - \boldsymbol{\alpha} \boldsymbol{\nabla^2} \beta \quad (14)$$

$$\langle \boldsymbol{\nabla}\, \alpha, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = -\boldsymbol{\nabla}\, , \boldsymbol{a} \times \boldsymbol{\nabla} \alpha \quad (15)$$

$$\langle \boldsymbol{\nabla} \times\, \boldsymbol{a}, \boldsymbol{\nabla} \times\, \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) \rangle - \langle \boldsymbol{a}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle \quad (16)$$

$$\boldsymbol{\nabla} \times (\alpha \mathbf{a}) = \alpha \boldsymbol{\nabla} \times \boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{\nabla} \alpha \quad (17)$$

$$\boldsymbol{\nabla} \times (\alpha \boldsymbol{\nabla} \beta) = (\boldsymbol{\nabla} \alpha) \times \nabla \beta \quad (18)$$

4 The separable Hilbert space Ң

We will specify the characteristics of a generalized quaternionic infinite dimensional separable Hilbert space. The adjective “quaternionic” indicates that the inner products of vectors and the eigenvalues of operators are taken from the number system of the quaternions. Separable Hilbert spaces can be using real numbers, complex numbers or quaternions. These three number systems are division rings.

4.1 Notations and naming conventions

$\{f_x\}_x$ means ordered set of f_x . It is a way to define functions.

The use of bras and kets differs slightly from the way Dirac uses them.

|f> is a ket vector, f> is the same ket

<f| is a bra vector, <f is the same bra

A is an operator. |A is the same operator

A† is the adjoint operator of operator A. A| is the same operator as A†

| on its own, is a nil operator

|A| is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$ means: sum over all items with index k.

$\int_x$ means: integral over all items with parameter x.

4.2 Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements |f>, |g> and |h> and numbers *a* and *b*:

4.2.1 Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \quad (1)$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \quad (2)$$

$$|(a + b)\, f\rangle = |f\rangle \cdot a + |f\rangle \cdot b \quad (3)$$

$$(|f\rangle + |g\rangle) \cdot a = |f\rangle \cdot a + |g\rangle \cdot a \quad (4)$$

$$|f\rangle \cdot 0 = |0\rangle \quad (5)$$

$$|f\rangle \cdot 1 = |f\rangle \quad (6)$$

Depending on the number field that the Hilbert space supports, *a* and *b* can be real numbers, complex numbers or (real) quaternions.

4.2.2 Bra vectors

The **bra** vectors form the dual Hilbert space $\mathbf{H}_1^{\dagger}$ of $\mathbf{H}_1$.

$$\langle f| + \langle g| = \langle g| + \langle f| = |g + f\rangle \quad (1)$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \quad (2)$$

$$\langle f\,(a + b)| = \langle f| \cdot a + \langle f| \cdot b = a^* \cdot \langle f| + b^* \cdot \langle f| \quad (3)$$

$$(\langle f| + \langle g|) \cdot a = \langle f| \cdot a + \langle g| \cdot a = a^* \cdot \langle f| + a^* \cdot \langle g| \quad (4)$$

$$0 \cdot \langle f| = \langle 0| \quad (5)$$

$$1 \cdot \langle f| = \langle f| \tag{6}$$

4.2.3 Scalar product

The Hilbert space contains a **scalar product,** also called **inner product,** $\langle f|g\rangle$ that combines $\mathbf{H}_t$ and $\mathbf{H}_t^\dagger$ in a direct product that we also indicate with $\mathbf{H}_t$.

The scalar product $\langle f|g\rangle$ satisfies:

$$\langle f|g + h\rangle = \langle f|g\rangle + \langle f|h\rangle \tag{1}$$

$$\langle f|\{|g\rangle \cdot a\}_g = \{\langle f|g\rangle\}_g \cdot a \tag{2}$$

With each ket vector $|g\rangle$ in $\mathbf{H}_t$ belongs a bra vector $\langle g|$ in $\mathbf{H}_t^\dagger$ such that for all bra vectors $\langle f|$ in $\mathbf{H}_t^\dagger$

$$\langle f|g\rangle = \langle g|f\rangle^* \tag{3}$$

$$\langle f|f\rangle = 0 \text{ when } |f\rangle = |0\rangle \tag{4}$$

$$\langle f|a\, g\rangle = \langle f|g\rangle \cdot a = \langle g|f\rangle^* \cdot a = \langle g\, a|f\rangle^* = (a^* \cdot \langle g|f\rangle)^* = \langle f|g\rangle \cdot a \tag{5}$$

In general is $\langle f|a\, g\rangle \neq \langle f\, a|g\rangle$. However for real numbers r holds $\langle f|r\, g\rangle = \langle f\, r|g\rangle$

Remember that when the number field consists of quaternions, then also $\langle f|g\rangle$ is a quaternion and a quaternion q and $\langle f|g\rangle$ do in general not commute.

The scalar product defines a **norm**:

(6)

$$||f|| = \sqrt{(<f|f>)}$$

And a **distance**:

$$D(f,g) = ||f - g|| \quad (7)$$

The Hilbert space $\mathbf{H}_t$ is closed under its norm. Each converging row of elements of converges to an element of this space.

4.2.4 Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Every continuous function on the separable space $\mathbf{H}_t$ is determined by its values on this countable dense subset.

4.2.5 *Base vectors*

The Hilbert space $\mathbf{H}_t$ is **separable**. That means that a countable row of elements $\{f_n>\}$ exists that **spans** the whole space.

If $<f_n|f_m> = \delta(m,n) = [1$ when $n = m$; 0 otherwise$]$
then $\{|f_n>\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k>\}$of $\mathbf{H}_t$ is a minimal set of ket vectors $|k>$ that together span the Hilbert space $\mathbf{H}_t$.

Any ket vector $|f>$ in $\mathbf{H}_t$ can be written as a linear combination of elements of $\{|k>\}$.

$$|f> = \sum_k (|k>\cdot<k|f>) \quad (1)$$

A bra base $\{<b|\}$of $\mathbf{H}_t^{\dagger}$ is a minimal set of bra vectors $<b|$ that together span the Hilbert space $\mathbf{H}_t^{\dagger}$.

Any bra vector $<f|$ in $\mathbf{H}_t^{\dagger}$ can be written as a linear combination of elements of $\{<b|\}$.

(2)

$$\langle f| = \sum_b (\langle f|b\rangle \cdot \langle b|)$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an othonormal base.

4.2.6 *Operators*

Operators act on a subset of the elements of the Hilbert space.

4.2.6.1 Linear operators

An operator Q is linear when for all vectors $|f\rangle$ and $|g\rangle$ for which Q is defined and for all quaternionic numbers a and b:

$$|Q\cdot a\, f\rangle + |Q\cdot b\, g\rangle = |a\cdot Q\, f\rangle + |b\cdot Q\, g\rangle = |Q\, f\rangle\cdot a + |Q\, g\rangle\cdot b = \quad (1)$$

$$Q\,(|f\rangle\cdot a + |g\rangle\cdot b) = Q\,(|a\, f\rangle + |b\, g\rangle) \quad (2)$$

B is **colinear** when for all vectors $|f\rangle$ for which B is defined and for all quaternionic numbers *a* there exists a quaternionic number *c* such that:

$$|B\cdot a\, f\rangle = |a\cdot B\, f\rangle = |B\, f\rangle\, c\cdot a\cdot c^{-1} \quad (3)$$

If $|f\rangle$ is an eigenvector of operator A with quaternionic eigenvalue *a*, then is $|b\, f\rangle$ an eigenvector of A with quaternionic eigenvalue $b\cdot a\cdot b^{-1}$.

$\langle A|$ = $A^{\dagger}$ is the **adjoint** of the **normal** operator A. $|A$ is the same as A.

$$\langle f\, A|\, g\rangle = \langle fA^{\dagger}|g\rangle^{*} \quad (4)$$

$$A^{\dagger\,\dagger} = A \quad (5)$$

$$(A\cdot B)^{\dagger} = B^{\dagger}\cdot A^{\dagger} \quad (6)$$

|B| is a **self adjoint** operator.

| is a nil operator.

The construct |f><g| acts as a linear operator. |g><f| is its adjoint operator.

$$\sum_n \{|f_n\rangle \cdot a_n \cdot \langle f_n|\}, \qquad (7)$$

where a n is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

A self adjoint operator has real numbers as eigenvalues.

$\{\langle q|f\rangle\}_q$ is a function f(q) of parameter *q*.

$\{\langle g|q\rangle\}_q$ is a function g(q) of parameter *q*.

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.

So, usually |q> is an eigenvector of a normal operator Q with eigenvalues *q*.

{q} is the set of eigenvalues of Q.

$\{q\}_q$ is the ordered field of eigenvalues of *q*.

$\{|q\rangle\}_q$ is the ordered set of eigenvectors of Q.
$\{\langle q|f\rangle\}_q$ is the **Q view** of |f>.

4.2.6.2 Normal operators

The most common definition of continuous operators is:

A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.

Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

If |a> is an eigenvector of normal operator A with eigenvalue *a* then

$$< a|A|a > = < a|a|a > = < a|a > \ a$$

indicates that the eigenvalues are taken from the same number system as the inner products.

A normal operator is a continuous linear operator.

A normal operator in **Ḥ** creates an image of **Ḥ** onto **Ḥ.** It transfers closed subspaces of **Ḥ** into closed subspaces of **Ḥ**.

Normal operators represent continuous quantum logical observables.

The normal operators N have the following property.

$$N: \mathbf{Ḥ} \Rightarrow \mathbf{Ḥ} \qquad (1)$$

N commutes with its **(Hermitian) adjoint** $N^{\dagger}$

$$N \cdot N^{\dagger} = N^{\dagger} \cdot N \qquad (2)$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators: $U^{\dagger} = U^{-1}$, unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^{\dagger} = N$;

- **Anti-Hermitian** or anti-self-adjoint operators: $N^{\dagger} = -N$;
- **Anti-unitary** operators: $I^{\dagger} = -I = I^{-1}$, anti-unitary operators are bounded;
- **positive operators**: $N = MM^{\dagger}$
- **orthogonal projection** operators: $N = N^{\dagger} = N^2$

4.2.6.3 Spectral theorem

For every compact self-adjoint operator *T* on a real, complex or quaternionic Hilbert space $\mathbf{H}_1$, there exists an orthonormal basis of $\mathbf{H}_1$ consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis {en} of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset R$, such that $\lambda_n \to 0$. Due to the fact that $\mathbf{H}_1$ is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of $\mathbf{H}_1$.

If *T* is compact on an infinite dimensional Hilbert space $\mathbf{H}_1$, then *T* is not invertible, hence $\sigma(T)$, the spectrum of *T*, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of *T*, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space $\mathbf{H}_1$.

In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that $\mathbf{H}_1$ is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, in general the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on $\mathbf{H}_1$ has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

4.2.6.4 Eigenspace

The set of eigenvalues {*q*} of the operator Q form the eigenspace of Q

4.2.6.5 Eigenvectors and eigenvalues

For the eigenvector $|q\rangle$ of normal operator Q holds

$$|Q\, q\rangle = |q\, q\rangle = |q\rangle \cdot q \qquad (1)$$

$$<q\ Q^{\dagger}| = <q\ q^{*}| = q^{*}\cdot<q| \tag{2}$$

$$\forall_{|f>\ \in \mathrm{H}} [\{< f|Q\ q >\}_q = \{< f|q > q\}_q = \{< q\ Q^{\dagger}|f >^{*}\}_q = \{q^{*} < q|f >^{*}\}_q] \tag{3}$$

The eigenvalues of 2^n-on normal operator are 2^n-ons

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j Q_i \tag{4}$$

The Q_j are self-adjoint operators.

4.2.6.6 Generalized Trotter formula

For bounded operators $\{A_j\}$ hold:

$$\lim_{n\to\infty}\left(\prod_{j=1}^{p} e^{A_j/n}\right)^n = \exp\left(\sum_{j=1}^{p} A_j\right) = \lim_{n\to\infty}\left(1+\frac{\sum_{j=1}^{p} A_j}{n}\right)^n \quad (1)$$

In general

$$\exp\left(\sum_{j=1}^{p} A_j\right) \neq \prod_{j=1}^{p} e^{A_j} \quad (2)$$

In the realm of quaternionic notion the Trotter formula is confusing.

4.2.6.7 Unitary operators

For unitary operators holds:

$$U^{\dagger} = U^{-1} \quad (1)$$

Thus

$$U\cdot U^{\dagger} = U^{\dagger}\cdot U = 1 \quad (2)$$

Suppose $U = I + C$ where U is unitary and C is compact. The equations $U\,U^* = U^*U = I$ and $C = U - I$ show that C is normal. The spectrum of C contains 0, and possibly, a finite set or a sequence tending to 0. Since $U = I + C$, the spectrum of U is obtained by shifting the spectrum of C by 1.

The unitary transform can be expressed as:

$$U = \exp(\tilde{\mathbf{I}}\cdot\Phi/\hbar) \quad (3)$$

$$\hbar = h/(2\cdot\pi) \quad (4)$$

Φ is Hermitian. The constant h refers to the granularity of the eigenspace.

Unitary operators have eigenvalues that are located in the unity sphere of the 2^n-ons field.

The eigenvalues have the form:

$$u = \exp(\underline{\mathbf{i}}\cdot\phi/\hbar) \quad (5)$$

ϕ is real. $\underline{\mathbf{i}}$ is a unit length imaginary number in 2^n-on space. It represents a direction.

u spans a sphere in 2^n-on space. For constant $\underline{\mathbf{i}}$, *u* spans a circle in a complex subspace.

4.2.6.7.1 Polar decomposition

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = ||N||\cdot U = A\cdot U \quad (1)$$

$$N = A\cdot U = U\cdot A \quad (2)$$

$$= A\cdot \exp(\underline{\tilde{\mathbf{I}}}\cdot\Phi)/\hbar)$$

$$= \exp(\Phi_r + \underline{\tilde{\mathbf{I}}}\cdot\Phi)/\hbar)$$

Φ_r is a positive normal operator.

4.2.6.8 Ladder operator

4.2.6.8.1 General formulation

Suppose that two operators X and N have the commutation relation:

$$[N, X] = c \cdot X \tag{1}$$

for some scalar *c*. If |n> is an eigenstate of *N* with eigenvalue equation,

$$|N\ n\rangle = |n\rangle \cdot n \tag{2}$$

then the operator *X* acts on |n> in such a way as to shift the eigenvalue by *c*:

$$\begin{aligned} |N \cdot X\ n\rangle &= |(X \cdot N + [N, X])\ n\rangle \\ &= |(X \cdot N + c \cdot X)\ n\rangle \\ &= |X \cdot N\ n\rangle + |X\ n\rangle \cdot c \\ &= |X\ n\rangle \cdot n + |X\ n\rangle \cdot c \\ &= |X\ n\rangle \cdot (n+c) \end{aligned} \tag{3}$$

In other words, if |n> is an eigenstate of *N* with eigenvalue *n* then |*X* n> is an eigenstate of *N* with eigenvalue *n* + *c*.

The operator *X* is a *raising operator* for *N* if *c* is real and positive, and a *lowering operator* for *N* if *c* is real and negative.

If N is a Hermitian operator then c must be real and the Hermitian adjoint of X obeys the commutation relation:

$$[N, X^{\dagger}] = - c \cdot X^{\dagger} \tag{4}$$

In particular, if *X* is a lowering operator for *N* then $X^{\dagger}$ is a raising operator for *N* and vice-versa.

4.2.7 Unit sphere of Ң

The ket vectors in **Ң** that have their norm equal to one form together the **unit sphere Θ** of **Ң**.

Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.

The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere **Θ**of **Ң**.

4.2.8 Bra-ket in four dimensional space

The Bra-ket formulation can also be used in transformations of the four dimensional curved spaces.

The bra $\langle f$ is then a covariant vector and the ket $g\rangle$ is a contra-variant vector. The inner product acts as a metric.

$$s = \langle f|g\rangle \tag{1}$$

The effect of a linear transformation L is then given by

$$s_L = \langle f|Lg\rangle \tag{2}$$

The effect of a the transpose transformation $L^\dagger$ is then given by

$$\langle fL^\dagger\ |g\rangle = \langle f|Lg\rangle \tag{3}$$

For a unitary transformation U holds:

$$\langle Uf|Ug\rangle = \langle f|g\rangle \tag{4}$$

These definitions work for curved spaces with a Euclidian signature as well as for curved spaces with a Minkowski signature.

$$\langle \nabla f|\nabla g\rangle = \langle f|\nabla^2 \mathrm{g}\rangle = \langle f|\square \mathrm{g}\rangle \tag{5}$$

4.2.9 Closure

The closure of $\mathbf{H}_1$ means that converging rows of vectors converge to a vector of $\mathbf{H}_1$.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.

Thus, the set of eigenvalues of Q is open.

At best the density of the coverage of the set of eigenvalues is comparable with the set of 2^n-ons that have rational numbers as coordinate values.

With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.

The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space $\mathbf{H}$ can offer a solution, but then the direct relation with quantum logic is lost.

4.2.10 Canonical conjugate operator P

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.

Q has eigenvectors $\{|q>\}_q$ and eigenvalues q.

P has eigenvectors $\{|p>\}_p$ and eigenvalues p.

For each eigenvector |q> of Q we define an eigenvector |p> and eigenvalues p of P such that:

$$< q|p > = < p|q >^* = \ exp\ (\hat{\imath} \cdot p \cdot q/\hbar) \tag{1}$$

$\hbar = h/(2\pi)$ is a scaling factor. $< q|p >$ is a quaternion. $\hat{\imath}$ is a unit length imaginary quaternion.

4.2.11 Displacement generators

Variance of the scalar product gives:

$$\boldsymbol{i}\,\hbar\,\delta < q|p > = \ -p < q|p > \delta q \tag{1}$$

$$\boldsymbol{i}\,\hbar\,\delta < p|q > = \ -q < p|q > \delta p \tag{2}$$

In the rigged Hilbert space **ℍ** the variance can be replaced by differentiation.

Partial differentiation of the function <q|p> gives:

$$\boldsymbol{i}\,\hbar\,\partial/\partial q_s < q|p > = \ -p_s < q|p > \tag{3}$$

$$\boldsymbol{i}\,\hbar\frac{\partial}{\partial p_s} < p|q > = \ -q_s < p|q > \tag{4}$$

4.3 Quaternionic L^2 space

The space of quaternionic measurable functions is a separable quaternionic Hilbert space. For example quaternionic probability amplitude distributions are measurable.[12]

[12] http://en.wikipedia.org/wiki/Lp_space#Lp_spaces

This space is spanned by an orthonormal basis of quaternionic measurable functions. The shared affine versions of the parameter space of these functions is called **Palestra**[13]. When the Palestra is non-curved, then this base has a canonical conjugate, which is the quaternionic Fourier transform of the original base.

As soon as curvature of the Palestra arises, this relation is disturbed.

With other words: “In advance the Palestra has a virgin state.”

[13] The name Palestra is suggested by Henning Dekant’s wive Sarah. It is a name from Greek antiquity. It is a public place for training or exercise in wrestling or athletics

5 Gelfand triple

The separable Hilbert space only supports countable orthonormal bases and countable eigenspaces. The rigged Hilbert space $\mathbb{H}$ that belongs to a separable Hilbert space $\mathbf{H}$ is a Gelfand triple. It supports non-countable orthonormal bases and continuum eigenspaces.

A rigged Hilbert space is a pair (H, Φ) with $\mathbf{H}$ a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous. Its name is not correct, because it is not a Hilbert space.

Identifying $\mathbf{H}$ with its dual space $\mathbf{H}^*$, the adjoint to i is the map

$$i^*: \mathrm{H} = \mathrm{H}^* \to \Phi^* \tag{1}$$

The duality pairing between Φ and Φ^* has to be compatible with the inner product on $\mathbf{H}$, in the sense that:

$$\langle u, v \rangle_{\Phi \times \Phi^*} = (u, v)_{\mathrm{H}} \tag{2}$$

whenever $u \in \Phi \subset \mathrm{H}$ and $v \in \mathrm{H} = \mathrm{H}^* \subset \Phi^*$.

The specific triple $(\Phi \subset \mathrm{H} \subset \Phi^*)$ is often named after the mathematician Israel Gelfand).

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion i with its adjoint i^*

$$i^* i: \Phi \subset \mathrm{H} = \mathrm{H}^* \to \Phi^* \tag{3}$$

6 Fourier transform

The Fourier transformation is a linear operator. This transform transfers functions to another parameter space. As a consequence the Fourier transform has no eigenvalues, but the Fourier transform knows functions that are invariant under Fourier transformation.

The Fourier transform cannot cope with functions that have curved parameter spaces. However, it is possible to reduce the parameter space to a domain in which the Fourier transform keeps acceptable accuracy. Another possibility is that the target function is flattened, such that its parameter space becomes flat.

The Fourier transform transfer a orthonormal set of base functions into a new a orthonormal set such that each member of the new set can be written as a linear combination of members of the old set such that none of the coefficients is zero. In fact all coefficients have the same norm.

The Fourier transform converts the nabla operator into an operator that does not differentiate but multiplies the converted function with a factor. That operator will be called a momentum operator.

The Fourier transform has an inverse. It turns the momentum operator into the nabla operator.

The Fourier transform converts convolution of two functions into the multiplication of the two functions and vice versa.

In order to simplify the discussion we restrict it to the case that the parameter spaces of the functions are not curved.

6.1 Fourier transform properties

6.1.1 Linearity

The Fourier transform is a linear operator

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{1}$$

$$\mathcal{F}(a\, g(q) + b\, h(q)) = a\, \tilde{g}(p) + b\, \tilde{h}(p) \tag{2}$$

1.1.1 Differentiation

Fourier transformation converts differentiation into multiplication with the canonical conjugated coordinate.

$$\mathrm{g}(q) = \nabla f(q) \tag{1}$$

$$\tilde{\mathrm{g}}(p) = \mathrm{p}\tilde{f}(p) \tag{2}$$

$$\mathrm{g}(q) = \nabla f(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right) \tag{3}$$

$$\tilde{\mathrm{g}}(k) = \mathrm{k}\tilde{f}(k) = \mathrm{k}_0 \tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k) \rangle \pm \mathrm{k}_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k} \tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k) \right) \tag{4}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm\nabla_0 \boldsymbol{f}(q) + \mathbf{\nabla} f_0(q) \pm \left(\pm\mathbf{\nabla} \times \boldsymbol{f}(q)\right) \tag{5}$$

$$\tilde{\mathbf{g}}(k) = \pm \mathrm{k}_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm\mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) \tag{6}$$

By using

$$\mathbf{\nabla} \times \mathbf{\nabla} f_0(q) = \mathbf{0} \tag{7}$$

and

$$\langle \mathbf{\nabla}, \mathbf{\nabla} \times \boldsymbol{f}(q) \rangle = 0 \tag{8}$$

It can be seen that for the static part ($\nabla_0 f(q) = 0$) holds:

$$\mathbf{g}(q) = \mathbf{\nabla} f_0(q) \pm \left(\pm\mathbf{\nabla} \times \boldsymbol{f}(q)\right) \tag{9}$$

$$\tilde{\mathbf{g}}(k) = \mathbf{k}\tilde{f}_0(k) \pm \left(\pm\mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) \tag{10}$$

1.1.2 Parseval's theorem

Parseval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left|\tilde{f}(p)\right|^2 \cdot dV_p \tag{2}$$

1.1.3 Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{1}$$

6.2 Helmholtz decomposition

The Helmholtz decomposition splits the **static** vector field $\boldsymbol{F}$ in a (transversal) divergence free part $\boldsymbol{F_t}$ and a (one dimensional longitudinal) rotation free part $\boldsymbol{F_l}$.

$$\boldsymbol{F} = \boldsymbol{F_t} + \boldsymbol{F_l} = \boldsymbol{\nabla} \times \boldsymbol{f} - \boldsymbol{\nabla} f_0 \tag{1}$$

Here f_0 is a scalar field and $\boldsymbol{f}$ is a vector field. In quaternionic terms f_0 and $\boldsymbol{f}$ are the real and the imaginary part of a quaternionic field f. $\boldsymbol{F}$ is an imaginary quaternionic distribution.

The significance of the terms “longitudinal” and “transversal” can be understood by computing the local three-dimensional Fourier transform of the vector field $\boldsymbol{F}$, which we call $\widetilde{\boldsymbol{F}}$. Next decompose this field, at each point $\boldsymbol{k}$, into two components, one of which points longitudinally, i.e. parallel to $\boldsymbol{k}$, the other of which points in the transverse direction, i.e. perpendicular to $\boldsymbol{k}$.

$$\widetilde{\boldsymbol{F}}(\boldsymbol{k}) = \widetilde{\boldsymbol{F}}_{\boldsymbol{l}}(\boldsymbol{k}) + \widetilde{\boldsymbol{F}}_{\boldsymbol{t}}(\boldsymbol{k}) \tag{2}$$

$$\langle \boldsymbol{k}, \widetilde{\boldsymbol{F}}_{\boldsymbol{t}}(\boldsymbol{k}) \rangle = 0 \tag{3}$$

$$\boldsymbol{k} \times \widetilde{\boldsymbol{F}}_{\boldsymbol{l}}(\boldsymbol{k}) = \boldsymbol{0} \tag{4}$$

The Fourier transform converts gradient into multiplication and vice versa. Due to these properties the inverse Fourier transform gives:

$$\boldsymbol{F} = \boldsymbol{F_l} + \boldsymbol{F_t} \tag{5}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{F_t} \rangle = 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F_l} = \boldsymbol{0} \tag{7}$$

So, this split indeed conforms to the Helmholtz decomposition.

This interpretation relies on idealized circumstance in which the decomposition runs along straight lines. This idealized condition is not provided in a curved parameter space. In curved parameter space the decomposition and the interpretation via Fourier transformation only work locally and with reduced accuracy.

6.2.1 Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform. It depends on the selection of a radial line $\boldsymbol{k}(q)$ in p space that under ideal conditions runs along a straight line.

$$\mathcal{F}_{\mathbf{k}}\big(g(q)\big) = \mathcal{F}(g(q), \boldsymbol{k}(q)) \tag{1}$$

Or

$$\mathcal{F}_{\|}\big(g(q)\big) \stackrel{\text{def}}{=} \mathcal{F}\left(g_{\|}(q)\right) \tag{2}$$

It relates to the full quaternionic Fourier transform $\mathbb{F}$

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}(\tilde{g}(p)) = g(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

6.3 Fourier integral

For the bra-ket inner product holds:

$$< q|\check{P}\, f > = \hbar \cdot \nabla_q < q|f > = \hbar \cdot \nabla_q f^*(q) = \mathrm{g}(q) \tag{1}$$

$$= \int_{\boldsymbol{p}} < q|p > \cdot < p|g >$$

The static imaginary part is

$$< q|\check{\boldsymbol{P}}\, f > = \hbar \cdot \boldsymbol{\nabla}_q < q|f > = \hbar \cdot \boldsymbol{\nabla}_q \boldsymbol{f}^*(q) = \mathbf{g}(q) \tag{2}$$

$$= Im\left(\int_{\boldsymbol{p}} < q|p > \cdot < p|\boldsymbol{g} >\right) = \int_{\boldsymbol{p}} Im(< q|p > \cdot < p|\boldsymbol{g} >)$$

$$= \int_{\boldsymbol{p}} Im(< q|p > \cdot < p|\boldsymbol{g_l} >) + \int_{\boldsymbol{p}} Im(< q|p > \cdot < p|\boldsymbol{g_t} >)$$

$$= \int_{\boldsymbol{p}} Im(< q|p >\cdot \tilde{\boldsymbol{g}}_{\boldsymbol{l}}(p)) + \int_{\boldsymbol{p}} Im(< q|p >\cdot \tilde{\boldsymbol{g}}_{\boldsymbol{t}}(p))$$

The left part is the longitudinal inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.

The right part is the transverse inverse Fourier transform of field $\tilde{\boldsymbol{g}}(p)$.

For the Fourier transform of $\mathbf{g}(q)$ holds the split:

$$\tilde{\boldsymbol{g}}(p) = \int_{\boldsymbol{q}} Im(< p|q >\cdot \boldsymbol{g}_{\boldsymbol{l}}(q)) + \int_{\boldsymbol{p}} Im(< p|q >\cdot \boldsymbol{g}_{\boldsymbol{t}}(q)) \tag{3}$$

$$= \int_{\boldsymbol{q}} Im(< p|q >\cdot \boldsymbol{g}(q))$$

The longitudinal direction is a one dimensional (radial) space. The corresponding transverse direction is tangent to a sphere in 3D. Its direction depends on the field $\mathbf{g}(q)$ or alternatively on the combination of field f and the selected (ideal) coordinate system $\breve{Q}$.

For a weakly curved coordinate system $\breve{\Omega}$ the formulas hold with a restricted accuracy and within a restricted region.

6.3.1 Alternative formulation

The reference S. Thangavelu[14] provides an alternative specification of the multidimensional Fourier transform .

6.4 Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.

See http://en.wikipedia.org/wiki/Hermite_polynomials.

There exist two types of Hermite polynomials: (1, 2)

[14] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(½z^2)\ \frac{d^n}{dz^n}\ \exp(-½z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2) \frac{d^n}{dx^n}\ \exp(-z^2) = \exp(½z^2)\ \left(z - \frac{d}{dz}\right)\ \exp(-½z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\ H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function ϕ(z) defined by

$$\varphi(x)\ =\ exp(-\pi\ z^2) \tag{4}$$

is an eigenfunction of F. It means that its Fourier transform has the same form.

As $\mathcal{F}^4 =$ I any λ in its spectrum $\sigma\ (\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma\ (\mathcal{F})\ =\ \{1; -1;\ i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-½\ z^2 +\ 2\ z\ c -\ c^2)\ =\ \sum_{n=0}^{\infty} exp(-½\ z^2)\ H_n(z)\ c^n/n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z)\, exp(-½\, z^2 \;+\; 2\, z\, c - c^2)\; dz \tag{7}$$

$$= \; exp(-½\, p_z^2 \;-\; 2\, \boldsymbol{k}\, p_z\, c \;+\; c^2)$$

$$= \; \sum_{n=0}^{\infty} exp(-½\, p_z^2)\; H_n(p_z)\; (-\boldsymbol{k}\, c)^n / n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\; H_n(z)\; c^n / n!\; dz \tag{8}$$

Equating like powers of c in the transformed versions of the left- and right-hand sides gives

$$\tfrac{1}{\sqrt{2\pi}} \textstyle\int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\; H_n(z)\; c^n / n!\; dz \tag{9}$$

$$= \; (-\boldsymbol{k})^n \cdot \exp(-½\, p_z^2)\; H_n(p_z)\; \frac{c^n}{n!}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \;\stackrel{\text{def}}{\equiv}\; < z|\psi_n > = \mathrm{c}_n \exp(-½\, z^2)\; H_n(z) \tag{10}$$

$$|\mathcal{F}\, \psi_n > = \; |\psi_n > \; (-\boldsymbol{k})^n \tag{11}$$

with suitably chosen c_n so as to make

$$\|\psi_n\|^2 \;=\; 1 \tag{12}$$

$$\mathrm{c}_n = \frac{1}{\sqrt{2^n n!\, \sqrt{\pi}}} \tag{13}$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions ψ_n; $n \in \mathbb{N}$ form an orthonormal basis for $L^2(\mathbb{R})$"

Consider the operator

$$H = -\tfrac{1}{2}\frac{d^2}{dz^2} + \tfrac{1}{2} z^2 \tag{14}$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) = (\tfrac{1}{2} + n)\, \psi_n(z) \tag{15}$$

Thus, ψ_n is an eigenfunction of *H*.

Let $f = \psi_{4k+j}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\left(-2\pi \boldsymbol{k}\, x\, (y+n)\right) \tag{16}$$

$$= (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n)\, exp(2\pi \boldsymbol{k}\, n\, y)$$

The vectors $|\psi_n\rangle$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$. The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n\rangle$ that span the complex Hilbert space $\mathbf{H}_{\boldsymbol{k}}$.

For higher *n* the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.

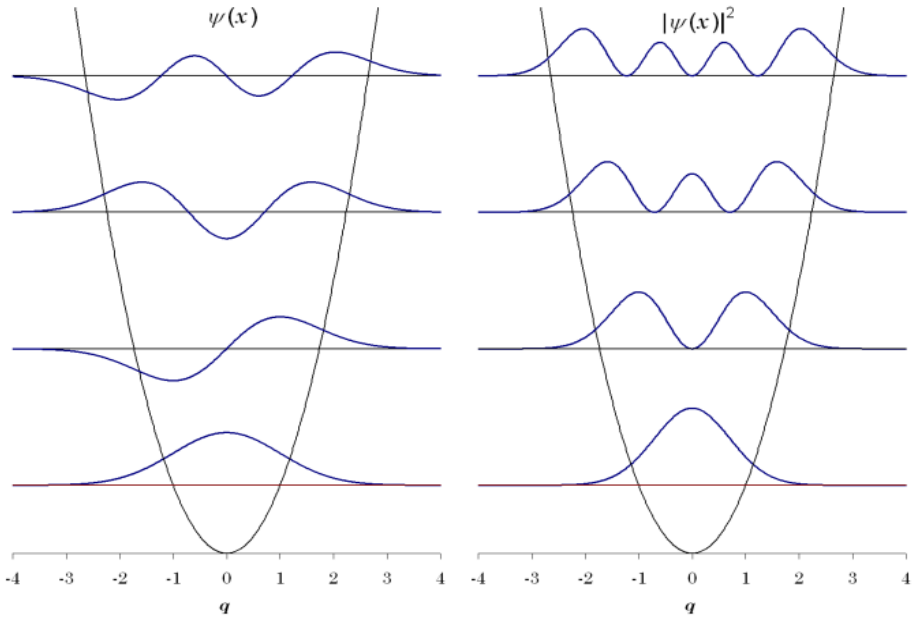

Figure 4

A **coherent state**[15] is a specific kind of state[16] of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state[17].

[15] http://en.wikipedia.org/wiki/Coherent_state
[16] States
[17] Canonical conjugate: Heisenberg's uncertainty

6.5 Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

6.6 Complex Fourier transform invariance properties

Each even function $f(q) \Leftrightarrow \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) + \tilde{f}(q). \tag{1}$$

$$\tilde{h}(q) = \sqrt{2\pi}\, h(q) \tag{2}$$

Each odd function $f(q) \Leftrightarrow \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) - \tilde{f}(q). \tag{3}$$

A function $f(q)$ is invariant under Fourier transformation *if and only if* the function f satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha\, f(q), \text{ for some scalar } \alpha \in C. \quad (4)$$

The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1, ***k***, -1 or **–*k***. ***k*** is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is, $\Delta_p = \Delta_q$ in the Heisenberg's uncertainty relation.

For proves see: http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf.

7 Quaternionic probability amplitude distributions

Continuous quaternionic distributions contain a scalar field in their real part and an associated vector field in their imaginary part. In a quaternionic probability amplitude distribution (QPAD), the scalar field can be interpreted as a scalar potential and will then correspond to a distribution of the density of property carriers. The associated vector field can be interpreted as a vector potential and will then correspond to a distribution of the current density of these carriers. The squared modulus of the value of the QPAD can be interpreted as the probability density of the presence of the carrier of the charge at the location that is specified by the parameter. The charge can be any property of the carrier or it stands for the ensemble of the properties of the carrier. The QPAD inherits the sign flavor of the quaternionic distribution that defines the curvature of its parameter space.

7.1 C-type QPAD

If a QPAD is a type C quaternionic distribution, then an A-type distance function defines the curvature of the parameter space of the QPAD. The carriers can be interpreted as the function values of this distance function. In this case the carriers are tiny patches of the parameter space of the QPAD. Their charge is formed by the discrete symmetry set (sign flavor) of the QPAD. This type of QPAD is suitable for application in quantum fluid dynamics.

7.2 D-type QPAD

If a QPAD is a type D quaternionic distribution, then an B-type distance function defines the curvature of the parameter space of the QPAD. The carriers can be interpreted as elements of a medium like a gas or a fluid. This type of QPAD is suitable for application in conventional fluid dynamics.

7.3 Differential equation

For QPAD's the equation for the differential can be interpreted as a differential continuity equation. Another name for continuity equation is balance equation. The differential continuity equation is paired by an integral continuity equation. The differential equation runs:

$$g(q) = g_0(q) + \boldsymbol{g}(q) = \nabla f(q)$$

$$= \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle$$

$$\pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right)$$

7.4 Continuity equation

Let us approach the balance equation from the integral variety of the balance equation.

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge[18] is given by the continuity equation:

Total change within *V* = flow into *V* + production inside *V* (1)

In formula this means:

$$\frac{d}{d\tau} \int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}} \rho_0 \frac{\boldsymbol{v}}{c} \, dS + \int_V s_0 \, dV \tag{2}$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \tag{3}$$

The conversion from formula (2) to formula (3) uses the Gauss theorem[19]. Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V.* In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

[18] Also see Noether's laws: http://en.wikipedia.org/wiki/Noether%27s_theorem

[19] http://en.wikipedia.org/wiki/Divergence_theorem

The combination of $\rho_0(\tau, \boldsymbol{q})$ and $\boldsymbol{\rho}(\tau, \boldsymbol{q})$ is a quaternionic skew field $\rho(\tau, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

$\rho(\tau, \boldsymbol{q})\rho^*(\tau, \boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(\tau, \boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(\tau, \boldsymbol{q})$ is the current density distribution.

This results in the law of charge conservation:

$$s_0(\tau, \boldsymbol{q}) = \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \big(\rho_0(\tau, \boldsymbol{q})\boldsymbol{v}(\tau, \boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})\big)\rangle \tag{6}$$

$$= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(\tau, \boldsymbol{q}) + \boldsymbol{A}(\tau, \boldsymbol{q})\rangle$$

$$= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{v}(\tau, \boldsymbol{q}), \boldsymbol{\nabla}\rho_0(\tau, \boldsymbol{q})\rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(\tau, \boldsymbol{q})\rangle\, \rho_0(\tau, \boldsymbol{q})$$

$$\mp \langle \boldsymbol{\nabla}, \boldsymbol{A}(\tau, \boldsymbol{q})\rangle$$

The blue colored ± indicates quaternionic sign selection through conjugation of the field $\rho(\tau, \boldsymbol{q})$. The field $\boldsymbol{a}(\tau, \boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})\rangle = 0 \tag{7}$$

$\boldsymbol{A}(\tau, \boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(\tau, \boldsymbol{q})$.

Equation (6) represents a balance equation for charge density. What this charge actually is, will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

Up to this point the investigation only treats the real part of the full equation. The full continuity equation runs:

$$s(\tau,\boldsymbol{q}) = \nabla\rho(\tau,\boldsymbol{q}) = s_0(\tau,\boldsymbol{q}) + \boldsymbol{s}(\tau,\boldsymbol{q}) \tag{8}$$

$$= \nabla_0\rho_0(\tau,\boldsymbol{q}) \mp \langle\boldsymbol{\nabla},\boldsymbol{\rho}(\tau,\boldsymbol{q})\rangle \pm \nabla_0\boldsymbol{\rho}(\tau,\boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q}) \pm \left(\pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(\tau,\boldsymbol{q})\right)$$

$$= \nabla_0\rho_0(\tau,\boldsymbol{q}) \mp \langle\boldsymbol{v}(\tau,\boldsymbol{q}),\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla},\boldsymbol{v}(\tau,\boldsymbol{q})\rangle\,\rho_0(\tau,\boldsymbol{q})$$

$$\pm\nabla_0\boldsymbol{v}(\tau,\boldsymbol{q}) + \nabla_0\rho_0(\tau,\boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})$$

$$\pm\left(\pm(\rho_0(\tau,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(\tau,\boldsymbol{q}) - \boldsymbol{v}(\tau,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\right)$$

$$s_0(\tau,\boldsymbol{q}) = 2\nabla_0\rho_0(\tau,\boldsymbol{q}) \mp \langle\boldsymbol{v}(q),\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla},\boldsymbol{v}(\tau,\boldsymbol{q})\rangle\,\rho_0(\tau,\boldsymbol{q}) \tag{9}$$

$$\boldsymbol{s}(\tau,\boldsymbol{q}) = \pm\nabla_0\boldsymbol{v}(\tau,\boldsymbol{q}) \pm \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q}) \tag{10}$$

$$\pm\left(\pm\left(\rho_0(\tau,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(\tau,\boldsymbol{q}) - \boldsymbol{v}(\tau,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\right)\right)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

In its simplest form the full continuity equation runs:

$$s(\boldsymbol{q},\tau) = \nabla\rho(\boldsymbol{q},\tau)$$

Thus the full continuity equation specifies a quaternionic distribution s as a flat differential $\nabla\rho$.

When we go back to the integral balance equation, then holds for the imaginary parts:

$$\frac{d}{d\tau}\int_V \boldsymbol{\rho}\, dV = -\oint_S \hat{\boldsymbol{n}}\rho_0\, dS - \oint_S \hat{\boldsymbol{n}} \times \boldsymbol{\rho}\, dS + \int_V \boldsymbol{s}\, dV \tag{4}$$

$$\int_V \nabla_0\, \boldsymbol{\rho}\, dV = -\int_V \boldsymbol{\nabla}\rho_0\, dV - \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho}\, dV + \int_V \boldsymbol{s}\, dV \tag{5}$$

For the full integral equation holds:

$$\frac{d}{d\tau}\int_V \rho\, dV + \oint_S \hat{\boldsymbol{n}}\rho\, dS = \int_V s\, dV \tag{6}$$

$$\int_V \nabla\, \rho\, dV = \int_V s\, dV \tag{7}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V.* In the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{8}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . t stands for progression (not coordinate time).

7.5 Fluid dynamics

The quaternionic continuity equation is the foundation of quaternionic fluid dynamics. Depending on the nature of the streaming medium, this branch of physics exists in two forms.

- In conventional fluid dynamics the streaming charge carriers are elements of a gas or a liquid.
- In quantum fluid dynamics the streaming charge carriers are tiny patches of the parameter space of the QPAD. They correspond to the target values of an A-type quaternionic distance function $\wp(x)$. This function has a flat parameter space that is spanned by the rational quaternions.

It means that in quantum fluid dynamics the coupling of QPAD's can affect the local curvature.

7.5.1 Coupling equation

In its simplest form the continuity equation runs:

$$\nabla\psi = \varphi$$

The continuity equation couples the local distribution ψ to a source φ.

The coupling strength can be made explicit. This results in the coupling equation.

$$\nabla\psi = m\,\phi$$

Here m is the coupling factor and ϕ is the adapted source.

8 Conservation laws

The following holds for all QPAD's!!!

Only the interpretation tells whether the QPAD concerns a quantum state function, a photon, a gluon or the field of a single charge, a field of a set of charges or a field corresponding to the density distribution of eventually moving charge carriers.

8.1 Differential potential equations

Let $\phi(q)$ define a quaternionic potential. The potential corresponds to a charge density distribution $\phi_0(q)$ and a current density distribution $\boldsymbol{\phi}(q)$.

Note: This means that the following holds for any QPAD!

$$\phi(q) = \rho_0(q) + \boldsymbol{\rho}(q) = \rho_0(q) + \rho_0(q)\boldsymbol{v}(q) \tag{1}$$

The gradient and curl of $\phi(q)$ are related. In configuration space holds:

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \pm \nabla_0\boldsymbol{\phi}(q) \pm \boldsymbol{\nabla}\phi_0(q) \pm \left(\pm\boldsymbol{\nabla} \times \boldsymbol{\phi}(q)\right) \tag{2}$$

$$\mathfrak{E}(q) \stackrel{\text{def}}{=} -\boldsymbol{\nabla}\phi_0(q) \tag{3}$$

$$\mathfrak{B}(q) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \phi(q) \tag{4}$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \mathfrak{F}_0(q) + \mathfrak{F}(q) \tag{5}$$

$$\mathfrak{F}_0(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \tag{6}$$

$$\mathfrak{F}(q) = \mp\mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \tag{7}$$

Note: When the velocity $\boldsymbol{v}$ in $\boldsymbol{\phi}$ changes, then an extra term $\nabla_0\boldsymbol{\phi}(q)$ is added to equation (7).

8.1.1 Maxwell

In Maxwell equations, the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ is defined as:

$$\boldsymbol{E}(\boldsymbol{r}, t) \equiv -\boldsymbol{\nabla}\phi_0(\boldsymbol{r}, t) - \frac{\partial\boldsymbol{\phi}(\boldsymbol{r}, t)}{\partial t} = \mathfrak{E}(\boldsymbol{r}, t) - \frac{\partial\boldsymbol{\phi}(\boldsymbol{r}, t)}{\partial t} \tag{1}$$

This is a remarkable decision, because $\dot{\boldsymbol{\phi}}$ can have components along $\mathfrak{E}$ and components along $\mathfrak{B}$, while $\mathfrak{E}$ and $\mathfrak{B}$ are mutually perpendicular.

Further:

$$\langle \boldsymbol{\nabla}, \boldsymbol{E}(\boldsymbol{r},t)\rangle = -\boldsymbol{\nabla}^2\phi_0(\boldsymbol{r},t) - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(\boldsymbol{r},t)\rangle}{\partial t} \tag{2}$$

$$= \frac{\rho_0(\boldsymbol{r},t)}{\varepsilon_0} - \frac{\partial\langle \boldsymbol{\nabla}, \boldsymbol{\phi}(\boldsymbol{r},t)\rangle}{\partial t}$$

In Maxwell equations, ***B***(***r***) is defined as:

$$\boldsymbol{B}(\boldsymbol{r},t) \equiv \boldsymbol{\nabla} \times \boldsymbol{\phi}(\boldsymbol{r},t) = \mathfrak{B}(\boldsymbol{r},t) \tag{3}$$

Further:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \tag{4}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{B}(\boldsymbol{r},t)\rangle = 0 \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_0(\boldsymbol{\rho} + \varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t}) \tag{6}$$

8.2 Flux vector

The longitudinal direction **k** of field $\mathfrak{E}(q)$ and the direction **i** of field $\mathfrak{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\mathbf{k} \times \mathbf{i}$**.** With other words what happens with $\mathfrak{E}(q) \times \mathfrak{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \stackrel{\text{def}}{=} \mathfrak{E}(q) \times \mathfrak{B}(q) \tag{1}$$

8.3 Conservation of energy

$$\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = \langle \mathfrak{B}(q), \boldsymbol{\nabla} \times \mathfrak{E}(q) \rangle - \langle \mathfrak{E}(q), \boldsymbol{\nabla} \times \mathfrak{B}(q) \rangle \tag{1}$$

$$= -\langle \mathfrak{B}(q), \nabla_0 \mathfrak{B}(q) \rangle - \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle - \langle \mathfrak{E}(q), \nabla_0 \mathfrak{B}(q) \rangle$$

$$= -½\nabla_0(\langle \mathfrak{B}(q), \mathfrak{B}(q) \rangle + \langle \mathfrak{E}(q), \mathfrak{E}(q) \rangle) - \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle$$

The **field energy density** is defined as:

$$u_{field}(q) = ½(\langle \mathfrak{B}(q), \mathfrak{B}(q) \rangle + \langle \mathfrak{E}(q), \mathfrak{E}(q) \rangle) = u_{\mathfrak{B}}(q) + u_{\mathfrak{E}}(q) \tag{2}$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.

The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q) \langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle \tag{3}$$

This means that $\langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle$ can be interpreted as a source term.

8.3.1 Interpretation in physics

Despite the fact that the above equations hold for any QPAD, we give here the physical interpretations when $\mathfrak{E}$ is the electric field and $\mathfrak{B}$ is the magnetic field.

$\phi_0(q)\mathfrak{E}(q)$ represents **force** per unit volume.

$\phi_0(q)\langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$**.**

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \tag{4}$$

$$\nabla_0 u_{mechanical} = \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q) \langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle \tag{5}$$

$$\nabla_0 \Big(u_{field}(q) + u_{mechanical}(q) \Big) = -\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle \tag{6}$$

Total change within *V* = flow into *V* + production inside *V* (7)

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \tag{8}$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \tag{9}$$

$$\frac{d}{dt} \int_V u \, dV = \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{S} \rangle dS + \int_V s_0 \, dV \tag{10}$$

Here the source s_0 is zero.

8.3.2 How to interprete $U_{mechanical}$

$U_{mechanical}$ is the energy of the private field (state function) of the involved particle(s).

8.4 Conservation of linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{g}_{field}(q) = \nabla_0 \, \mathfrak{E}(q) \times \mathfrak{B}(q) + \mathfrak{E}(q) \times \nabla_0 \mathfrak{B}(q) \tag{1}$$

$$= \big(\boldsymbol{\nabla} \times \mathfrak{B}(q) - \boldsymbol{\rho}(q) \big) \times \, \mathfrak{B}(q) - \mathfrak{E}(q) \times \boldsymbol{\nabla} \times \mathfrak{E}(q) \tag{2}$$

$$\boldsymbol{G}(\mathfrak{E}) = \mathfrak{E} \times (\boldsymbol{\nabla} \times \, \mathfrak{E}) = \langle \boldsymbol{\nabla}\mathfrak{E} \, , \mathfrak{E} \rangle - \langle \mathfrak{E}, \mathfrak{E} \rangle = \tfrac{1}{2} \boldsymbol{\nabla} \langle \mathfrak{E} \, , \mathfrak{E} \rangle - \langle \mathfrak{E}, \mathfrak{E} \rangle \tag{3}$$

$$= -\boldsymbol{\nabla}(\mathfrak{E}\mathfrak{E}) + \tfrac{1}{2}\boldsymbol{\nabla}\langle \mathfrak{E}\,, \mathfrak{E}\rangle + \langle \boldsymbol{\nabla}\,, \mathfrak{E}\rangle \mathfrak{E}$$

$$= -\boldsymbol{\nabla}(\mathfrak{E}\mathfrak{E} + \tfrac{1}{2}\mathbf{1_3}\langle \mathfrak{E}\,, \mathfrak{E}\rangle) + \langle \boldsymbol{\nabla}\,, \mathfrak{E}\rangle \mathfrak{E}$$

$$\boldsymbol{G}(\mathfrak{B}) = \mathfrak{B} \times (\boldsymbol{\nabla} \times \mathfrak{B}) = -\boldsymbol{\nabla}(\mathfrak{B}\mathfrak{B} + \tfrac{1}{2}\mathbf{1_3}\langle \mathfrak{B}\,, \mathfrak{B}\rangle) + \langle \boldsymbol{\nabla}\,, \mathfrak{B}\rangle \mathfrak{B} \tag{4}$$

$$\boldsymbol{H}(\mathfrak{B}) = -\boldsymbol{\nabla}(\mathfrak{B}\mathfrak{B} + \tfrac{1}{2}\mathbf{1_3}\langle \mathfrak{B}\,, \mathfrak{B}\rangle) \tag{5}$$

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{G}(\mathfrak{B}) + \boldsymbol{G}(\mathfrak{E}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) \tag{6}$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{B}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) + \langle \boldsymbol{\nabla}\,, \mathfrak{B}\rangle \mathfrak{B} + \langle \boldsymbol{\nabla}\,, \mathfrak{E}\rangle \mathfrak{E}$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{B}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) - \rho_0(q)\, \mathfrak{E}(q)$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{E}) - \boldsymbol{f}(q) = \boldsymbol{\mathcal{T}}(q) - \boldsymbol{f}(q)$$

$\mathcal{T}(\mathrm{q})$ is the linear momentum flux tensor.

The linear momentum of the field contained in volume *V* surrounded by surface *S* is:

$$\boldsymbol{P}_{field} = \int_V \boldsymbol{g}_{field}\, dV = \int_V \rho_0 \boldsymbol{\phi}\, dV + \int_V \langle \nabla \boldsymbol{\phi}, \mathfrak{E}\rangle\, dV + \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{E}\boldsymbol{A}\rangle dS \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \mathfrak{B}(q) + \rho_0(q)\, \mathfrak{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q)\boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume *V* is:

$$\boldsymbol{F} = \int_V \boldsymbol{f}\, dV = \int_V (\boldsymbol{\rho} \times \mathfrak{B} + \rho_0\, \mathfrak{E})\, dV \tag{10}$$

Brought together this gives:

$$\nabla_0 \left(\boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q)\right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q)\rangle \tag{11}$$

This is the continuity equation for linear momentum.

The component $\mathcal{T}_{\mathrm{ij}}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

Total change within *V* = flow into *V* + production inside *V* (12)

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \tag{13}$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g}\, dV \tag{14}$$

$$\frac{d}{dt}\int_V \boldsymbol{g}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}}\rangle dS + \int_V \boldsymbol{s}_{\boldsymbol{g}}\, dV \tag{15}$$

Here the source $\boldsymbol{s_g} = 0$.

8.5 Conservation of angular momentum

8.5.1 *Field angular momentum*

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}(q) \tag{1}$$

$$\boldsymbol{h}_{field}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{field}(q) \tag{2}$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{mechanical}(q) \tag{3}$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{\mathcal{T}}(q) \tag{4}$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left(\boldsymbol{h}_{field}(\boldsymbol{q}_c) + \boldsymbol{h}_{mechanical}(\boldsymbol{q}_c) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) \rangle \tag{5}$$

Total change within V = flow into V + production inside V

$$\boldsymbol{J} = \boldsymbol{J}_{field} + \boldsymbol{J}_{mechanical} = \int_V \boldsymbol{h}\, dV \tag{6}$$

$$\frac{d}{dt} \int_V \boldsymbol{h}\, dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{\mathcal{K}} \rangle dS + \int_V \boldsymbol{s_h}\, dV \tag{7}$$

Here the source $\boldsymbol{s_h} = 0$.

For a localized charge density contained within a volume *V* holds for the mechanical torsion:

$$\tau(\boldsymbol{q}_c) = \int_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \boldsymbol{f}(q')dV \tag{8}$$

$$= \int_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \left(\rho_0(q')\mathfrak{E}(q') + \boldsymbol{j}(q') \times \mathfrak{B}(q')\right)dV$$

$$= Q(\boldsymbol{q} - \boldsymbol{q}_c) \times (\mathfrak{E}(q) + \boldsymbol{v}(q) \times \mathfrak{B}(q))$$

$$\boldsymbol{J}_{field}(\boldsymbol{q}_c) = \boldsymbol{J}_{field}(\boldsymbol{0}) + \boldsymbol{q}_c \times \boldsymbol{P}(q) \tag{9}$$

Using

$$\langle \boldsymbol{\nabla a}, \boldsymbol{b} \rangle = \boldsymbol{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{10}$$

$$\langle \boldsymbol{b}, \boldsymbol{\nabla a} \rangle = \boldsymbol{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{11}$$

holds

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \mathfrak{S}(q')dV = \int_V \boldsymbol{q}' \times \mathfrak{E}(q') \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q')\, dV \tag{12}$$

$$= \int_V (\boldsymbol{q}' \times \langle (\boldsymbol{\nabla}\boldsymbol{\phi}), \mathfrak{E} \rangle - \langle \boldsymbol{q}' \times \mathfrak{E}, (\boldsymbol{\nabla}\boldsymbol{\phi}) \rangle)\, dV$$

$$= \int_V \boldsymbol{q}' \times \langle (\boldsymbol{\nabla}\boldsymbol{\phi}), \mathfrak{E} \rangle dV$$

$$+ \int_V \mathfrak{E} \times \boldsymbol{\phi}\, dV - \int_V \langle \boldsymbol{\nabla}, \mathfrak{E}\boldsymbol{q}' \times \boldsymbol{\phi} \rangle dV + \int_V (\boldsymbol{q}' \times \boldsymbol{\phi}) \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle dV$$

8.5.2 Spin

Define the non-local spin term, *which does not depend on* $\boldsymbol{q}'$ as:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathfrak{E}(q) \times \boldsymbol{\phi}(q) dV \tag{13}$$

Notice

$$\boldsymbol{\phi}(q) \times \boldsymbol{\nabla}\phi_0(q) = \phi_0 \boldsymbol{\nabla} \times \boldsymbol{\phi}(q) + \boldsymbol{\nabla} \times \left(\phi_0(q)\boldsymbol{\phi}(q)\right)$$

And

$$\boldsymbol{L}_{field}(\mathbf{0}) = \int_V \boldsymbol{q}' \times \langle (\boldsymbol{\nabla}\boldsymbol{\phi}), \mathfrak{E} \rangle dV + \int_V \boldsymbol{q}' \times \rho_0 \boldsymbol{\phi} dV \tag{14}$$

Using Gauss:

$$\int_V \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle dV = \oint_S \langle \hat{\boldsymbol{n}}, \boldsymbol{a} \rangle dS \tag{15}$$

And

$$\rho_0 = \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \tag{16}$$

Leads to:

$$\boldsymbol{J}_{field}(\mathbf{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\mathbf{0}) + \oint_S \langle \hat{\boldsymbol{n}}, \mathfrak{E}\boldsymbol{q}' \times \boldsymbol{\phi} \rangle dS \tag{17}$$

8.5.3 Spin discussion

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathfrak{E}(q) \times \boldsymbol{\phi}(q) dV \tag{1}$$

In free space the charge density ρ_0 vanishes and the scalar potential ϕ_0 shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with q_0. Thus:

$$\mathfrak{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0 \boldsymbol{\phi} \approx -\nabla_0 \boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_V (\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q) dV \tag{3}$$

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^+{}_{field} \\ \boldsymbol{\Sigma}^-{}_{field} \end{bmatrix} \tag{4}$$

If $\frac{\boldsymbol{\phi}(q)}{|\boldsymbol{\phi}(q)|}$ can be interpreted as tantrix (q_0)) and $\frac{\nabla_0 \boldsymbol{\phi}(q)}{|\nabla_0 \boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal $\boldsymbol{N}(q_0)$, then $\frac{(\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)}{|(\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)|}$ can be interpreted as the binormal $\mathfrak{B}(q_0)$.

From these quantities the curvature and the torsion[20] can be derived.

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{5}$$

9 Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.

With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic is congruent to an orthocomplemented weakly modular lattice. Another name for that lattice is orthomodular lattice.

9.1 Lattices

A subset of the axioms of the logic characterizes it as a half ordered set. A larger subset defines it as a lattice.

A lattice is a set of elements $a, b, c, \ldots$that is closed for the connections ∩ and ∪. These connections obey:

- The set is partially ordered. With each pair of elements a, b belongs an element c, such that $a \subset c$ and $b \subset c$.
- The set is a ∩ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cap b$.
- The set is a ∪ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cup b$.
- The set is a lattice if it is both a ∩ half lattice and a ∪ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{1}$$

[20] Path characteristics

$$(a \cap b) \cap c = a \cap (b \cap c) \quad (2)$$

$$a \cap (a \cup b) = a \quad (3)$$

$$a \cup b = b \cup a \quad (4)$$

$$(a \cup b) \cup c = a \cup (b \cup c) \quad (5)$$

$$a \cup (a \cap b) = a \quad (6)$$

The lattice has a partial order inclusion ⊂:

$$a \subset b \Leftrightarrow a \subset b = a \quad (7)$$

A complementary lattice contains two elements n and e with each element a an complementary element a' such that:

$$a \cap a' = n \quad (8)$$

$$a \cap n = n \quad (9)$$

$$a \cap e = a \quad (10)$$

$$a \cup a' = e \quad (11)$$

$$a \cup e = e \tag{12}$$

$$a \cup n = a \tag{13}$$

An orthocomplemented lattice contains two elements n and e and with each element a an element $a"$ such that:

$$a \cup a" = e \tag{14}$$

$$a \cap a" = n \tag{15}$$

$$(a")" = a \tag{16}$$

$$a \subset b \Leftrightarrow b" \subset a" \tag{17}$$

e is the unity element; n is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{18}$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{19}$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \tag{20}$$

A weak modular lattice supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c \tag{21}$$
$$= a \cup (b \cap c) \cup (d \cap c)$$

where d obeys:

$$(a \cup b) \cap d = d \tag{22}$$

$$a \cap d = n \tag{23}$$

$$b \cap d = n \tag{24}$$

$$[(a \subset g) \, and \, (b \subset g) \Leftrightarrow d \subset g \tag{25}$$

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \tag{26}$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \tag{27}$$

$$\Rightarrow (x = a \; or \; x = a \cap p)\}$$

p is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space **Ң** have the structure of an orthomodular lattice. In this respect these sets are congruent.

In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

9.2 Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five logical connectives[21], and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector |f> is inside state s*".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned be a single vector.

[21] http://en.wikipedia.org/wiki/Logical_connective

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

9.3 Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. This paper distinguishes between measurements and observations.

- With an observation the state is considered as a linear combination of eigenvectors of the observable. An observation returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring equipment. The measurement can be seen as a combination of a transformation and an observation.

Depending on the characteristics of the measuring equipment a measurement and a clean observation can give the same result.

With this interpretation of the concept of observation it is possible to let states observe other states. A state might do a transformation before doing an observation but in general it fails the equipment to arrange that transformation. In nature observations are far more common than measurements.