W.B. VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

SPECIAL QUASI DUAL NUMBERS AND GROUPOIDS

Special Quasi Dual Numbers and Groupoids

W. B. Vasantha Kandasamy
Florentin Smarandache

ZIP PUBLISHING
Ohio
2012

This book can be ordered from:

Zip Publishing
1313 Chesapeake Ave.
Columbus, Ohio 43212, USA
Toll Free: (614) 485-0721
E-mail: info@zippublishing.com
Website: www.zippublishing.com

Peer reviewers:
Prof. Valeri Kroumov, Okayama Univ. of Science, Japan.
Prof. Gabriel Tica, Bailesti College, Bailesti, Jud. Dolj, Romania.
Prof. Mihàly Bencze, Department of Mathematics,
Áprily Lajos College, Braşov, Romania

Many books can be downloaded from the following
Digital Library of Science:
http://www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm

ISBN-13: 978-1-59973-192-6
EAN: 9781599731926

Printed in the United States of America

CONTENTS

PREFACE

In this book the authors introduce a new notion called special quasi dual number, $x = a + bg$; where a and b are from R or Q or Z or Z_n or $\langle Q \cup I\rangle$ or $\langle R \cup I\rangle$ or $\langle Z \cup I\rangle$ or $\langle Z_n \cup I\rangle$ or $C(Z_n)$ and $g^2 = -g$ is the new element.

Among the reals -1 behaves in this way, for $(-1)^2 = 1 = -(-1)$. Likewise $-I$ behaves in such a way $(-I)^2 = -(-I)$.

These special quasi dual numbers can be generated from matrices with entries from 1 or I using only the natural product $\times_n$. Another rich source of these special quasi dual numbers or quasi special dual numbers is Z_n, n a composite number. For instance 8 in Z_{12} is such that $8^2 = 64 = -8(\text{mod } 12) = 4(\text{mod } 12)$. In chapter two we introduce the notion of special quasi dual numbers. The notion of higher dimensional special quasi dual numbers are introduced in chapter three of this book. We using the dual numbers and special dual like numbers with special quasi dual numbers construct three types of mixed special quasi numbers and discuss their properties.

However the only source of getting higher dimensional special quasi dual numbers and mixed special dual numbers are from the modulo integers Z_n, n a suitable number. We for the first time build non associative algebraic structures using these special quasi dual numbers, dual numbers and special dual like numbers. This forms chapter four of this book.

We give the possible applications of this new concept in chapter five and the final chapter suggests some problems.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

The concept of dual numbers was introduced by W.K. Clifford in 1873. An element $x = a + bg$ is a dual number if a and b are reals and g is a new element such that $g^2 = 0$.

Now if we replace this g by a new element g_1 such that $g_1^2 = g_1$ we call $x = a + bg_1$ to be a special dual like number. Several interesting properties akin to dual numbers are statisfied by special dual like numbers.

In $x = a + bg_1$ a and b reals g_1 the new element such that $g_1^2 = g_1$ for every x the pair (a, b) is uniquely determined. Now this study was very recently made by the authors in their book [24] in the year 2012.

The authors have in this book introduced another new type of dual number called special quasi dual numbers. We call $x = a + bg_2 + cg_3$ to be a special quasi dual number where a, b and c are reals and g_2, a new element such that $g_2^2 = -g_2$ $(=g_3)$. Thus $x = a + bg_2 + c(-g_2)$ is a special quasi dual number. These numbers also behave akin to dual numbers and special dual like numbers.

We in this book study, describe analyse and define properties associated with special quasi dual numbers. So if $x = a + bg_2 + c(-g_2)$ is a special quasi dual number the triple (a, b, c) is uniquely determined for the given x.

Suppose a, b and c are positive reals greater than one.

$$\begin{aligned} x &= a + bg_2 + c(-g_2) \\ x^2 &= a^2 + b^2(-g_2) + c^2(-g_2) + 2abg_2 + 2ac(-g_2) + 2bcg_2 \\ &= a^2 + (2ab + 2bc)g_2 + (b^2 + c^2 + 2ac)(-g_2). \end{aligned}$$

Thus $x, x^2, x^3, x^4, \ldots$ becomes diverging for the positive real values associated with g_2 and $-g_2$; grow larger and larger by raising the power of $x = a + bg_2 + c(-g_2)$. If a, b, c are positive but less than 1 then $x, x^2, x^3, x^4, \ldots$ is such that the coefficient of g_2 and $(-g_2)$ becomes smaller and smaller.

This is the way the powers of $x = a + bg_2 + c(-g_2)$ behave in case of special quasi dual numbers. These can be used in appropriate models.

Chapter Two

QUASI SPECIAL DUAL NUMBERS

The concept of special dual like numbers and mixed dual numbers was recently studied and introduced respectively [22, 24].

Here we introduce the new notion of quasi special dual numbers. A number $x = a_1 + a_2g$ with $a_1, a_2 \in R$ (or Q or C or Z_n or Z) and g a new special element such that $g^2 = -g$ is defined as the quasi special dual numbers. Clearly $(-1)^2 = 1$ (that is $g = -1$ then $-g = 1$ is also a new special element but since this g is in Z or Q or R or C we do not distinguish it separately, it can be taken as a trivial new special element). With this assumption we seek to find quasi special dual numbers.

Let $Z_{12} = \{0, 1, 2, \ldots, 11\}$; $-1 = 11 \pmod{12}$, $-2 = 10 \pmod{12}$, $10 \equiv -2 \pmod{12}$, $3 = -9 \pmod{12}$ or $9 \equiv -3 \pmod{12}$, $8 = -4 \pmod{12}$ or $-8 = 4 \pmod{12}$, $7 = -5 \pmod{12}$, $5 = -7 \pmod{12}$, $6 \equiv 6 \pmod{12}$ as $-6 = 6 \pmod{12}$.

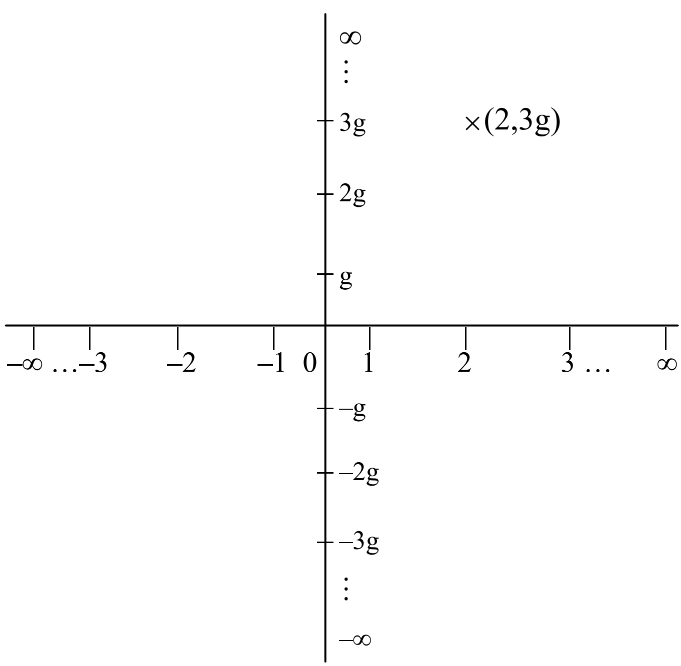
Consider $8 \in Z_{12}$; $8^2 \equiv 64 \pmod{12}$ that is $8^2 \equiv 4 \pmod{12}$ but $4 \equiv -8 \pmod{12}$. Hence $x = a_1 + a_2g$ with $g = 8 \in Z_{12}$ and a, $b \in Q$ is a quasi special dual number.

Consider $x = 5 + 3g$ and $y = 2 - 9g$ two quasi special dual numbers. $x + y = 7 - 6g$ is again a quasi special dual number.

Consider $x \times y = (5 + 3g)\ (2 - 9g)$
$= 10 + 6g - 45g + (-27)g^2$
$= 10 + 16g - 45g - 27 \times -g$
$= 10 - 2g$ is again a quasi special dual number.

Hence we see just like dual numbers quasi special dual numbers also behave.

We can have a plane representation of quasi special dual numbers also.



$x = 2 + 3g$ is represented.

Further if $g^2 = -g$ be a new special element then $g^2.g = -g.g$ that is $g^3 = -(g^2) = -1\ (-g) = g$.

$$g^3.g = g^4 = g.g = g^2 = -g.$$
$$g^4.g = g^5 = -g \times g = -\ (g^2) = g.$$

$$\text{Thus } g = g^3 = g^5 = g^7 = \ldots \text{ and}$$
$$g^2 = g^4 = g^6 = g^8 = \ldots = -g.$$

This is the way powers of g behave.

We see $g = 8 \in Z_{12}$ is such that

$$g^2 \equiv 64 \pmod{12}$$
$$= 4 \pmod{12} = -8 \pmod{12} = -g \pmod{12}.$$

$$g^2.g = g^3 = -g.g = -(g^2) = -(-g) = g$$

and so on. Thus in general if g is a quasi new element which contributes to a quasi special dual element $x = a + bg$, $a, b \in R$ (or Q or Z or Z_n or C) then

$$g = g^3 = g^5 = g^7 = \ldots = \ldots \text{ and}$$
$$g^2 = g^4 = g^6 = g^8 = \ldots = -g.$$

$$g = g^3 = g^5 = g^2 = \ldots = \text{ and}$$
$$g^2 = g^4 = g^6 = g^8 = \ldots = -g.$$

Further we see Z_6 is the first modulo integer which has the quasi special dual number. We see $2 \in Z_6$ is such that $2^2 = 4 = -4 \pmod 6$ and $4^2 = 4$.

We see $S = \{0, 8, 4\} \subseteq Z_{12}$ is a group under addition modulo 12.

+	0	4	8
0	0	4	8
4	4	8	0
8	8	0	4

The table for (S, ×) is as follows:

×	0	4	8
0	0	0	0
4	0	4	8
8	0	8	4

Thus (S, +, ×) is a field isomorphic to Z_3. $\eta: S \to Z_3$

$\eta(4) \mapsto 1$

$\eta(8) \mapsto 2$ and $\eta(0) = 0$ is an isomorphism.

THEOREM 2.1: *Let Z_n be a ring of modulo integers. $g \in Z_n$ be such that*

$$g^2 = -g = g^4 = g^6 = \ldots \text{ and}$$
$$g = g^3 = g^5 = g^7 = g^9 = \ldots \text{ where } g \text{ is a new element of } Z_n.$$

Then Z_n has zero divisors.

Proof: We see $g^2 = -g$ (given for $g \in Z_n$).

Thus $g^2 + g = 0$, $g(g+1) = 0 \pmod n$.

Now $g \neq 0$ and $g + 1 \neq 0$ as $g \neq -1$. Hence Z_n has zero divisors.

Corollary 2.1: Z_p, p a prime has no quasi special element.

Proof follows from the simple fact if $g \in Z_p$ is such that $g^2 = -g$ then Z_p has zero divisors, hence Z_p has no quasi special element.

Example 2.1: Let $Z_{14} = \{0, 1, 2, \ldots, 13\}$ be the ring of modulo integer. Z_{14} has 6 to be a quasi special element, for $6^2 = 36 \pmod{14} = 8 \pmod{14} = -6 \pmod{14}$.

Example 2.2: Let $Z_{15} = \{0, 1, 2, \ldots, 14\}$ be the ring of modulo integers modulo 15. $9 \in Z_{15}$ is a quasi special number, for $9^2 = 81 \pmod{15} = 6 \pmod{15} = -9 \pmod{15} = 6 \pmod{15}$.

Thus $6^2 \equiv 6 \pmod{15}$ is an idempotent and S = {0, 6, 9} is a field.

Example 2.3: Z_{16} = {0, 1, 2, …, 15} the ring of modulo integers has no quasi special number.

Example 2.4: Consider Z_{18} = {0, 1, 2, …, 17}, the ring of modulo integers.

8 is the quasi special new element of Z_{18}.
For $8^2 \equiv 10 \pmod{18}$
$= -8 \pmod{18}$
and $10^2 = 10 \pmod{18}$ and $8 \times 10 \equiv 8 \pmod{18}$.

Example 2.5: Let Z_{20} = {0, 1, 2, …, 19} be the ring of modulo integers 20. 15 is the only quasi special new element of Z_{20}.
$15^2 \equiv 5 \pmod{20}$
$= -15 \pmod{20}$.

Thus in Z_{20}, 15 is a quasi special element and –15 = 5 is an idempotent.

It is observed in all these cases if $t \in Z_n$ is a special quasi element then –t is an idemponent.

Further Z_{16} has no quasi special numbers.
Finally in view of this we have the following theorem.

THEOREM 2.2: *Let Z_{pq}, p and q powers of primes. $pq \geq 6$ ($p \neq q$). Z_{pq} has special quasi elements.*

The proof is simple and exploits only number theoretic techniques.

Example 2.6: Let Z_{30} = {0, 1, 2, …, 29} be the ring of modulo integers. Z_{30} has 4 quasi special elements and 30 = 2.3.5 product of three primes.

Consider $24 \in Z_{30}$, $24^2 = 6 \pmod{30} = -24 \pmod{30}$ and $6^2 \equiv 6 \pmod{30}$.

24 is a quasi special element of Z_{30}.
Consider $9 \in Z_{30}$, $9^2 \equiv 21 \pmod{30} = -9 \pmod{30}$.

Further $21^2 = 21 \pmod{30}$.

So 9 is a quasi special element of Z_{30}.
Now $20 \in Z_{30}$ is such that $20^2 = 10 \pmod{30}$
that is $20^2 = -20 \pmod{30}$ and $10^2 = 10 \pmod{30}$.

Finally $14 \in Z_{30}$ is again another quasi special element of Z_{30}.

We see $14^2 = 16 \pmod{30} = -14 \pmod{30}$ and
$16^2 \equiv 16 \pmod{30}$.
Thus {24, 9, 14 and 20} are quasi special elements.

Let S = {9, 14, 20, 24, 6, 21, 10, 0, 16} be the quasi special elements and the associated idempotents.

Clearly S is not closed under addition modulo 30. We consider × on S.

The table of × on S is as follows.

×	0	6	9	10	14	16	20	21	24
0	0	0	0	0	0	0	0	0	0
6	0	6	24	0	24	6	0	6	24
9	0	24	21	0	6	24	0	9	6
10	0	0	21	10	20	10	20	0	0
14	0	24	6	20	16	24	10	24	6
16	0	6	24	10	24	16	20	6	24
20	0	0	0	20	10	20	10	0	0
21	0	6	9	0	24	6	0	21	24
24	0	24	6	0	6	24	0	24	6

Clearly $(S, \times)$ is a semigroup and will be known as the associated quasi special semigroup of Z_{30}. However $5 \in Z_{30}$ is such that $5^2 = 25 = -5$ and $25^2 = 25$. If we include 5 and 25 we see we have included 15 and the extended semigroup M = {0, 5, 6, 9, 10, 15, 14, 16, 20, 21, 24, 25} $\subseteq Z_{30}$.

Example 2.7: Let Z_{42} = {0, 1, 2, …, 41} be the ring of modulo integers. Consider $35 \in Z_{42}$, $35^2 = 7 \pmod{42}$ that $35^2 = -35 \pmod{42}$ so 35 is a quasi special element with 7 as its associated idempotent.

Consider $14 \in Z_{42}$; clearly $14^2 = 28 \pmod{42}$ that is $14^2 = -14 \pmod{42}$ so 14 is a quasi special element in Z_{42} with 28 as its associated idempotent.

$27 \in Z_{42}$ is a quasi special element as $27^2 \equiv 15 \pmod{42}$. 15 is the associated idempotent element of 27 in Z_{42}. $20 \in Z_{42}$ is also a quasi special element as $20^2 = 22 \pmod{42}$ and $20^2 = -20 \pmod{42}$ with $22 \in Z_{42}$ as its associated idempotent.

Now let P = {0, 35, 7, 14, 28, 27, 15, 20, 22} $\subseteq Z_{42}$, $(P, \times)$ is a semigroup given by the following table.

×	0	7	14	15	20	22	27	28	35
0	0	0	0	0	0	0	0	0	0
7	0	7	14	21	14	28	21	28	35
14	0	14	28	0	28	14	0	14	28
15	0	21	0	15	6	36	27	0	21
20	0	14	28	6	22	20	36	14	28
22	0	28	14	36	20	22	6	28	14
27	0	21	0	27	36	6	14	0	21
28	0	28	14	0	14	28	0	28	14
35	0	35	28	21	28	14	21	14	7

Clearly P is not a semigroup. Consider M = {0, 6, 7, 14, 15, 20, 22, 27, 28, 35, 36, 21} $\subseteq Z_{42}$ is semigroup.

However 21 is an idempotent and 6 and 36 are such that $6^2 = 36 \pmod{42} = -6 \pmod{42}$ and $36^2 = 36 \pmod{42}$ is again a quasi special new element of Z_{42}. However M is not a associated semigroup.

We call M the extended semigroup of the associated special quasi semigroup. From the context one can understand whether the semigroup is an extended one or not. At times we ignore it also.

Example 2.8: Now consider $Z_6 = \{0, 1, 2, 3, 4, 5\}$. $2^2 \equiv 4 \pmod 6$ we have $2^2 = -2 \pmod 6$ as $-2 \equiv 4 \pmod 6$ and $4^2 = 4 \pmod 6$.

Thus 2 is a quasi special element in Z_6. {0, 2, 4} is a semigroup both under '+' as well as × . That is P = {0, 2, 4} $\subseteq Z_6$ is a subring of Z_6.

Example 2.9: Let $S = Z_{10} = \{0, 1, 2, 3, 4, \ldots, 9\}$ be the ring of modulo integers. $4^2 = 6 \pmod{10} = -4 \pmod{10}$ as $-4 = 6 \pmod{10}$ and $6^2 \equiv 6 \pmod{10}$. Take $\{4, 6, 0\} \subseteq Z_{10}$ is only a semigroup under product.

×	0	4	6
0	0	0	0
4	0	6	4
6	0	4	6

Example 2.10: Let $Z_{12} = \{0, 1, 2, \ldots, 12\}$. To find all quasi special elements of Z_{12}. Consider $3 \in Z_{12}$, $3^2 \equiv 9 \pmod{12} = -3 \pmod{12}$; $8^2 = 9 \pmod{12} = -8 \pmod{12}$, $9^2 = 9 \pmod{12}$ and $4^2 = 4 \pmod{12}$. S = {0, 3, 9} $\subseteq Z_{12}$ is such that S is a quasi associated semigroup under product. If we obtain S $\cup$ {6} then T = {0, 3, 6, 9} has the following table.

×	0	6	3	9
0	0	0	0	0
6	0	6	6	6
3	0	6	9	3
9	0	6	3	9

Thus T is a subring and {0, 8, 4} is a field. However W = {0, 3, 4, 6, 9} is not even closed under '+'.

×	0	3	4	6	8	9
0	0	0	0	0	0	0
3	0	0	0	6	0	3
4	0	0	4	0	8	0
6	0	6	0	6	0	6
8	0	0	8	0	4	0
9	0	3	0	6	0	9

W is only an extended semigroup. Suppose we remove 6 from W. Let V = {0, 3, 4, 8, 9}. Is V a quasi special semigroup?

	0	3	4	8	9
0	0	0	0	0	0
3	0	9	0	0	3
4	0	0	4	8	0
8	0	0	8	4	0
9	0	3	0	0	9

V is infact a quasi special semigroup. However V is not closed under '+'.

Example 2.11: Let Z_{14} = {0, 1, 2, 3, 4, …, 13} be the ring of modulo integers. Clearly $6^2 = 8 \pmod{14}$, $6^2 = -6 \pmod{14}$ and

$8^2 = 8 \pmod{14}$. So 6 is the only quasi special element of Z_{14}. However T = {0, 6, 8} is not a semigroup under '+' only a semigroup under product ×.

Example 2.12: Let Z_{40} be the ring of modulo integers.

Consider $15^2 = 225 \equiv 25 \pmod{40} = -15 \pmod{40}$.

Further $25^2 = 625 = 25 \pmod{40}$. So 15 is a quasi special number.

Take $16 \in Z_{40}$, $16^2 = 16 \pmod{40}$ and $24^2 = 576 \equiv 16 \pmod{40} = -24 \pmod{40}$ so 24 is also a quasi special number.

Does the set W = {15, 25, 16, 20, 0} form a semigroup under product?

×	0	15	16	20	25
0	0	0	0	0	0
15	0	25	0	20	15
16	0	0	16	0	0
20	0	20	0	0	20
25	0	15	0	20	25

W is the special quasi semigroup of Z_{40}.

We can also obtain the algebraic structure enjoyed by these quasi special dual numbers.

Example 2.13: Let $M = \{a + bg \mid a, b \in Z, g = 2 \in Z_6\}$ be the collection of all quasi special dual numbers. M is a ring infact a commutative ring.

Consider $x = -3 + 8g$ and $y = 10 - g$ in M;

$x + y = (-3 + 8g) + (10 - g) = 7 + 7g \in M$.

$x \times y = (-3 + 8g)(10 - g)$

$= -30 + 80g + 3g - 8g^2$ ($\because g^2 = -g$)

$= -30 + 80g + 3g + 8g$

$= -30 + 91g \in M$.

It is easily verified M is a general ring of quasi special dual numbers.

Clearly $Z \subseteq M$. M has subrings which are not ideals.

Example 2.14: Let

$S = \{a + bg \mid a, b \in Q;\ g = 15 \in Z_{40},\ 15^2 = g^2 = 25 = -g\}$
be the general ring of quasi special dual numbers.

$P = \{a + bg \mid a, b \in Z,\ g = 15 \in Z_{40},\ g^2 = -g \in Z_{40}\} \subseteq S$ is only a subring of S and is not an ideal. Infact S has infinitely many subrings which are not ideals.

Take $T = \{ag \mid a \in Q\} \subseteq S$; T is an ideal of S.

Example 2.15: Let

$$S = \{a + bg \mid g = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix};\ a, b \in Z;\ g^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -g\}$$

be the general ring of quasi special dual numbers.

Consider $x = 5 + 2g$ and $y = 7 + 10g$ in S. $x + y = 12 + 12g$ and

$$x \times y = \left(5 + 2\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}\right)\left(7 + 10\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}\right)$$

$$= 35 + 14\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix} + 50\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix} + 20\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix} \times_n \begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}$$

$$= 35 + 64\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix} + 20\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} \quad (g^2 = -g)$$

$$= 35 + 64\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix} - 20\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix} = 35 + 44\begin{bmatrix}-1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}$$

$$= 35 + 44g \in S.$$

S has subrings which are not ideals. S has ideals.

Can S have zero divisors?

Suppose $x = a + bg$ and $y = c + dg$ $(a, b, c, d \in Z \setminus \{0\})$ then

$x \times y = (a + bg)(c + dg) = ac + bcg + dag - dbg$
$= ac + (bc + da - db)g \neq 0$ even if $bc + da - db = 0$.

Thus S is an integral domain and infact S is a Smarandache ring.

Example 2.16: Let S = $\{a + bg \mid a, b \in Z_3, g = 24 \in Z_{40}, g^2 = -g \pmod{40}\}$ = {0, 1, g, 2, 2g, 1 + g, 2+g, 1+2g, 2+2g} be the quasi special dual number general ring table for S \ {0} under × is as follows:

	1	2	g	2g	1+g	2+g	1+2g	2+2g	0
1	1	2	g	2g	1+g	2+g	1+2g	2+2g	0
2	2	1	2g	g	2+2g	1+2g	2+g	1+g	0
g	g	2g	2g	g	0	g	2g	0	0
2g	2g	g	2g	2g	0	2g	g	0	0
1+g	1+g	2+2g	g	0	1+g	2+2g	1+g	2+2g	0
2+g	2+g	1+2g	0	2g	2+2g	1	2	1+g	0
1+2g	1+2g	2+g	g	g	1+g	2	1	2+2g	0
2+2g	2+2g	1+g	2g	0	2+2g	1+g	2+2g	1+g	0
0	0	0	0	0	0	0	0	0	0

Clearly S is only a ring and S has zero divisors.

Example 2.17: Let

$$M = \{a + bg \mid a, b \in Z_6, g = (-1\ -1\ -1\ -1\ -1), g^2 = -g\}$$

be the general ring of quasi special dual numbers. M is a finite order M has zero divisors. Order of M is 36.

Example 2.18: Let

$$P = \{a + bg \mid a, b \in Z_8, g = 2 \in Z_6, g^2 = 4 = -g \pmod{6}\}$$

be a finite general quasi special dual ring.

We have both infinite and finite general quasi special dual rings.

We will illustrate this by examples.

Example 2.19: Let

$W = \{a + bg \mid g = 4 \in Z_{10}, g^2 = 6 = -4 \pmod{10}, a, b \in Z\}$ be an infinite quasi special dual ring which is commutative.

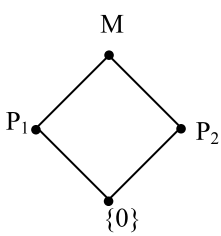
Example 2.20: Let $M = \{a + bg \mid a, b \in Q; g = 5 \in Z_{15}, g^2 = 10 \pmod{15}$ that $g^2 = -5 = -g \pmod{15}\}$ be again an infinite quasi special dual ring.

$T = \{a + bg \mid a, b \in Z, g = 5 \in Z_{15}, g^2 = -g \pmod{15}\} \subseteq M$ is only a subring of M and is not an ideal of M.

Example 2.21: Let $S = \{a + bg \mid a, b \in Z_4, g = 15 \in Z_{40}\}$ be the general quasi special dual ring. Ideals of S are $P_1 = \{0, g, 2g, 3g\} \subseteq S$, $P_2 = \{0, 2g\} \subseteq S$, $P_3 = \{0, 1+g, 2+2g, 3+3g\} \subseteq S$ and $P_4 = \{0, g, 3g, 2g, 2+g, 2, 2+3g, 2+2g\} \subseteq S$ are ideals of S.

Example 2.22: Let $M = \{a + bg \mid a, b \in Z_3, g = 24 \in Z_{40}\}$ be a quasi special dual general ring.

$M = \{0, 1, 2, g, 2g, 1+g, 2+, 1+2g, 2+2g\}$. The ideals of M are $P_1 = \{0, g, 2g\} \subseteq M$ and $P_2 = \{0, 1+g, 2+2g\} \subseteq M$.



The lattice of ideals of M is a distributive lattice with four elements including M and {0}.

We can thus build general quasi special dual number rings of dimension one and study them.

Since '– g' ∈ M for g ∈ M, M a quasi special dual number ring, we see we cannot in general build a semiring ring $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$. This is one of the main limitations while working with quasi special dual numbers. Further if g is such that $g^2 = -g$ then invariably $-g$ happens to be an idempotent so we cannot contain the replacement of –g by h even though gh = hg = 0 and $h^2 = h$, since for a in a semiring –a does not belong to the semiring. Semirings mentioned above we cannot build semiring structures using those standard semirings or even using distributive lattices.

So we to overcome this problem define a new notion called complete quasi special dual number pair.

That is if $g^2 = -g$ then $x = a + bg + c(-g)$ is defined as the complete quasi special dual pair number.

We will first illustrate this situation by some examples. It is observed that we call the dimension as three or pair dimension as two.

Example 2.23: Let M = {a + bg + c (–g) | a, b ∈ Q, g = 15 ∈ Z_{40}, $g^2 = 225 \pmod{40} = 25 = -g \pmod{40}$}. So if x = 8 + 3g + 8 (–g) = 8 + 3g + 8g′ where $g^2 = g' \pmod{40}$, g′ = –g.

(This notational compromise is made to avoid the confusion –8 (–g) = 8g but it is –8g′ so that we will make this notational change) and y = 3 + 4g + 5g′are in M then

$x + y = 11 + 7g + 13g'$

$x \times y = xy = (8 + 3g + 8g') \times (3 + 4g + 5g')$

$= 24 + 9g + 24g' + 32g + 12g^2 + 32g'g + 40g' + 15gg' + 40(g')^2$

$= 24 + 88g + 116g'$

(using the fact $g^2 = 225 \pmod{40}$

$25 = g' = -g$ and $(g')^2 = 625 \pmod{40}$

$= 25 = g' \pmod{40}$ and $gg' = g = g'g \pmod{40}$).

Clearly $xy = 24 + 88g + 116g' \in M$.

Example 2.24: Let P = {a + bg + cg_1 | a, b, c ∈ Q; g = 2, $g_1 = 4 \in Z_6$, $gg_1 = 2 = g_1g$, $2^2 = g^2 = 4 = -2 \pmod 6$ and $g_1^2 = g_1 \pmod 6$} be the complete quasi special dual number pair general ring.

Example 2.25: Let S = {a + bg + cg_1 | a, b, c ∈ Z, g = 4 and $g_1 = 6 \in Z_{10}$, $g_1^2 = 6 \pmod{10}$, $g^2 = 6 = -g \pmod{10}$, $g_1g_2 = g_2g_1 =$

$6 \times 4 = 4 \pmod{10}\}$ be again the general ring of complete quasi special dual number pair.

Example 2.26: Let $S = \{x_1 + x_2g + x_3g_1 \mid x_1, x_2, x_3 \in Q;\ g = 6$ and $g_1 = 8 \in Z_{14}$ are such that $g^2 = 36 \equiv 8 \equiv -6 \pmod{11}$, $8^2 = 8 \pmod{14}$, $8^2 = 8 \pmod{14}$, $g.g_1 = g_1g_2 = 6\}$ be the complete quasi special dual pair general ring.

Example 2.27: Let $S = \{x_1 + x_2g + x_3g_1 \mid x_1, x_2, x_3 \in Z_7$, $g = (-1, -1, -1, -1)$, $g^2 = (1, 1, 1, 1) = -g$ and $g_1 = (1, 1, 1, 1)$, $g_1g = gg_1 = (-1, -1, -1, -1)\}$ be the finite general ring of complete quasi special dual number pair.

Example 2.28: Let $M = \{x_1 + x_2g + x_3g_1 \mid x_i \in Q;\ 1 \leq i \leq 3$, $g=(-1, -1, -1, -1, -1, -1, -1)$ and $g_1=(1, 1, 1, 1, 1, 1, 1)$, $g_1^2 = g_1$, $g^2 = g_1$, $g_1g = gg_1 = g\}$ be the finite general ring of complete quasi special pair.

$$
\begin{aligned}
& a = 5 + 3g + 4g_1 \text{ and } b = 8 + 7g - 8g_1 \in M, \\
& a + b = 13 + 10g - 4g_1 \\
& a \times b = (5 + 3g + 4g_1)\,(8 + 7g - 8g_1) \\
& \quad = 40 + 24g + 32g_1 + 35g + 21g^2 + 28g_1g - 40g_1 - \\
& \qquad 24gg_1 - 32\,g_1^2 \\
& \quad = 40 + 24g + 32g_1 + 35g + 21g_1 + 28g + 40g_1 - 24g - \\
& \qquad 31g_1 \\
& \quad = 40 + 63g + 62g_1 \in M.
\end{aligned}
$$

M is a complete special quasi dual pair.

Example 2.29: Let

$$P = \{x_1 + x_2g + x_3g_1 \mid x_i \in Q;\ 1 \leq i \leq 3,$$

$$g = \begin{pmatrix} -I & -I & -I \\ -I & -I & -I \end{pmatrix} \text{ and } g^2 = \begin{pmatrix} I & I & I \\ I & I & I \end{pmatrix} = g_1$$

$$\text{so that } g^2 = -g = g_1\}$$

be the complete special quasi dual number pair. (I is the indeterminate such that $I^2 = I$).

Now having seen examples of complete quasi special dual number pair we now proceed onto develop algebraic structure enjoyed by them.

(1) $M = \{x_1 + x_2g + x_3g_1 \mid x_i \in Q, 1 \leq i \leq 3, g^2 = g_1 = -g$ and $g_1^2 = g, g_1g = gg_1 = g\}$ is a group under addition, +.
(2) M is a semigroup under product, ×.
(3) (M, +, ×) is a commutative ring.

In case of complete quasi special dual pair numbers we can define semirings / semifields.

We will illustrate this situation by some examples.

Example 2.30: Let $P = \{x_1 + x_2g + x_3g_1 \mid x_i \in Z^+, 1 \leq i \leq 3, g^2 = g_1, g_1^2 = g_1$ and $g_1g = gg_1 = g\} \cup \{0\}$ be a semiring. Infact P is a strict semiring P is infact a semifield of complete quasi special dual pair numbers.

Example 2.31: Let $M = \{x_1 + x_2g + x_3g_1 \mid x_i \in Q^+, 1 \leq i \leq 3, g = 2$ and $g_1 = 4 \in Z_6, g^2 = g_1$ and $g_1^2 = 4, g_1g = gg_1 = g\} \cup \{0\}$ is again a semifield of complete quasi special dual pair numbers.

If $x = 8 + 10g + 3g_1$ and $y = 3 + 7g + 5g_1 \in M$
$x + y = 11 + 17g + 8g_1 \in M$
and $xy = (8 + 10g + 3g_1)(3 + 7g + 5g_1)$
$= 24 + 30g + 9g_1 + 56g + 70g_2 + 21gg_1 + 40g_1 + 50gg_1 + 15g_1^2$
$= 24 + 30g + 9g_1 + 56g + 70g_1 + 21g + 40g_1 + 50g + 15g_1$
$= 24 + 157 + 134g_1 \in M.$

Example 2.32: Let

$$M = \{x_1 + x_2g + x_3g_1 \mid x_i \in R^+, 1 \leq i \leq 3,$$

$$g = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix} \text{ and } g_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix};$$

$$g_1^2 = g_1, g_1g = gg_1 = g, g^2 = g_1\} \cup \{0\}$$

be the complete quasi special dual number pair semifield.

Example 2.33: Let $S = \{x_1 + x_2g + x_3g_1 \mid x_i \in Z^+, 1 \leq i \leq 3, g = 8, g_1 = 16 \in Z_{24}, g^2 = 64 \equiv 16 \pmod{24} \equiv g_1$ and $g_1^2 = 256, g_1g = gg_1 = g, g^2 = g_1\} \cup \{0\}$ be the complete quasi special dual number pair semifield.

Note: If in the above examples we permit $Z^+ \cup \{0\}$ in the place of Z^+ we see the semirings / semifields continue to be semirings / semifield with a small charge; if $x \in S$ the semiring $S = \{x_1 \mid (x_1 \in Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\})$ or x_2g; $x_2 \in Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or x_3g_1 where $x_3 \in Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$))$\}$. But if we take only Z^+ or Q^+ or R^+ every element in S is of the form $x_1 + x_2g + x_3g_1$ ($x_1, x_2, x_3 \in Q^+$ or Z^+ or R^+). That is every element is a complete special quasi dual number pair.

Thus only introduction of complete quasi dual special pair number could lead to semiring / semifield structure in case of quasi special dual numbers we cannot have semiring / semifield structure.

Further at this juncture we can equivalently define a complete quasi special dual pair or quasi special dual number component as follows.

We say a pair (g, g_1) is a complete quasi special dual pair number or a quasi special dual number component if
(i) $g^2 = g_1$ $(= -g)$ (ii) $g_1^2 = g_1$ and $g_1g = gg_1 = g$.

That is g is the quasi special dual number component which contributes to quasi special dual number.

We will illustrate this situation using neutrosophic rings $\langle Z \cup I\rangle$ or $\langle Q \cup I\rangle$ or $\langle R \cup I\rangle$.

Let $g = \underbrace{(-I, -I, \ldots, -I)}_{n-\text{times}}$ ($I^2 = I$ is the indeterminate)

$$g^2 = \underbrace{(I, I, \ldots, I)}_{n-\text{times}} = -g.$$

Let $g_1 = (I, I, \ldots, I)$ then $g_1^2 = g_1$ and $gg_1 = g_1g = g$ with $g^2 = g_1$.

Thus $\{(-I, -I, \ldots, -I), (I, I, \ldots, I)\}$ is the complete quasi special dual pair or quasi special dual component of $x = x_1 + x_2g + x_3g_1$.

Take $g = \begin{bmatrix} -I \\ -I \\ \vdots \\ -I \end{bmatrix}$, $g \times_n g = g^2 = -g = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}$. Let $g_1 = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}$,

then $g^2 = g_1$ and $g_1^2 = g_1$ with $gg_1 = g_1g = g$.

$$\text{Let } g = \begin{bmatrix} -I & -I & \ldots & -I \\ -I & -I & \ldots & -I \\ \vdots & \vdots & & \vdots \\ -I & -I & \ldots & -I \end{bmatrix}_{m \times n}$$

$$(m \neq n) \text{ then } g^2 = g \times_n g = \begin{bmatrix} I & I & \dots & I \\ I & I & \dots & I \\ \vdots & \vdots & & \vdots \\ I & I & \dots & I \end{bmatrix} = -g.$$

Let

$$g_1 = \begin{bmatrix} I & I & \dots & I \\ I & I & \dots & I \\ \vdots & \vdots & & \vdots \\ I & I & \dots & I \end{bmatrix}_{m \times n} \quad (m \neq n),$$

then $g_1 \times_n g_1 = g_1^2 = g_1$ and $g \times_n g_1 = g_1 \times g = g$.

Thus $\{g, g_1\}$ is a complete quasi special dual pair number component.

Finally

$$\text{let } g = \begin{bmatrix} -I & -I & \dots & -I \\ -I & -I & \dots & -I \\ \vdots & \vdots & & \vdots \\ -I & -I & \dots & -I \end{bmatrix}_{n \times n}$$

be a n × n matrix only order the natural product $\times_n$,

$$g^2 = g \times_n g = \begin{bmatrix} I & I & \dots & I \\ I & I & \dots & I \\ \vdots & \vdots & & \vdots \\ I & I & \dots & I \end{bmatrix}_{n \times n} = -g.$$

If $g_1 = \begin{bmatrix} I & I & \dots & I \\ I & I & \dots & I \\ \vdots & \vdots & & \vdots \\ I & I & \dots & I \end{bmatrix} = g_1 \times_n g_1 = g_1^2 = g_1$ and

$g \times_n g_1 = g_1 \times_n g = g$.

Certainly under usual product $g_1 \times g_1 \neq g_1$ and $g \times g \neq -g$.

Also $g \times g_1 \neq g_1 \times g \neq g$.

Thus using these neutrosophic matrices we get complete quasi special dual pair component.

Also if $(-1, -1, \ldots, -1) = g$ then $g \times_n g = g^2 = -g = (1, 1, \ldots, 1)$ and if $g_1 = (1, 1, \ldots, 1)$ then $g^2 = g_1$, $g_1^2 = g_1$, $gg_1 = g_1g = g$. Thus $\{g, g_1\}$ acts as a complete quasi special dual number pair component.

We can use all -1 entries as column matrices so that

$\left(\begin{bmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right)$ is a complete quasi special dual number pair

component.

Likewise $\left(\begin{bmatrix} -1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \dots & -1 \end{bmatrix}_{m \times n} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}_{m \times n} \right)$ $(m \neq n)$

is a complete quasi special dual number pair component.

$$\text{Also } \left(\begin{bmatrix} -1 & -1 & \ldots & -1 \\ -1 & -1 & \ldots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \ldots & -1 \end{bmatrix}_{n\times n} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}_{n\times n} \right)$$

is again a complete quasi special dual pair number component.

All this can be used to build rings, semirings which have elements of the form $x_1 + x_2g + x_3g_1$ with $g^2 = g_1$ $(g_1 = -g)$ and $g_1^2 = g$, $g_1g = gg_1 = g$.

$x_i \in Q$, $(Q^+ \cup \{0\})$ (or Z, $Z^+ \cup \{0\}$ or R or $R^+ \cup \{0\}$).

However all these (1) or (–1) matrices will not and cannot contribute to higher dimensional complete special quasi dual number pair rings (or semifield).

Further the rings of complete special quasi dual number pairs are never fields but they are Smarandache rings.

Certainly using g and g_1 such that $g^2 = g_1 = -g$ and $g_1^2 = g_1$, $g_1g = gg_1 = g$ we can only get complete quasi special dual pair number semiring of dimension three as it is impossible to have the concept of $-g$ in semirings for the structure to be a semiring.

Next we proceed onto describe with examples the concept of vector space and semivector space of quasi special dual numbers and complete quasi special dual number pairs.

Example 2.34: Let $M = \{a + bg \mid g = 2 \in Z_6, g^2 = 4 = -g \in Z_6,$ $a, b \in Q\}$ be the vector space of quasi special dual numbers over the field Q.

Example 2.35: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 4 \in Z_{10} \right.$$

is such that $4^2 = 6 \pmod{10}$ that is $g^2 = -g$, $x_i, y_i \in R$, $1 \leq i \leq 9\}$.

(i) S is a group under +.
(ii) $(S, \times_n)$ is a semigroup with zero divisors.
(iii) $(S, +, \times_n)$ is a ring, commutative and has zero divisors.
(iv) $(S, +, \times_n)$ is a Smarandache ring.
(v) S is a quasi special dual number vector space over R.
(vi) S is a quasi special dual number Smarandache vector space over the S-ring.

$$P = \{a + bg \mid a, b \in R; g = 4 \in Z_{10}, g^2 = -g \in Z_{10}\}.$$

Example 2.36: Let $S = \{(a_1, a_2, \ldots, a_{15}) \mid a_i = x_i + y_i g$ where $x_i, y_i \in Q$; $1 \leq i \leq 15$ and $g = 6 \in Z_{14}$ $6^2 = -6 = 9 \pmod{14}\}$ be the general quasi special dual numbers vector space over the field Q. S has subspaces.

Example 2.37: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \end{bmatrix} \middle| \; a_i = x_i + y_i g;\ 1 \leq i \leq 21,\ x_i, y_i \in Z; \right.$$

$$g = 8 \in Z_{12}, g^2 = -g \in Z_{12}\}$$

be the group under '+' of quasi special dual numbers. S is not a vector space as S is defined only on Z.

If Z is replaced by Q then certainly S is a general vector space of quasi special dual number matrices over the field Q. Infact using the natural product $\times_n$; S will also be a general linear algebra of quasi special dual number matrices over Q.

Example 2.38: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i = x_i + y_i g;\ 1 \leq i \leq 16, \right.$$

$$\left. x_i, y_i \in R;\ g = 10 \in Z_{22},\ g^2 = 100 = 12 = -g \pmod{22} \right\}$$

be a general vector space of quasi special dual numbers over the field Q.

T is only a general non commutative linear algebra of quasi special dual numbers under the usual matrix product, but is a commutative linear algebra of quasi special dual numbers under the natural product $\times_n$. We can construct polynomials with quasi special dual number coefficients.

Let V = $\{\sum a_i x_i \mid a_i = x_i + y_i g$ with $x_i, y_i \in R$, g is such that $g^2 = -g$ is the special new element$\}$ be the polynomial collection with special quasi dual number coefficients. Using this structure we can build vector space / linear algebras of special dual like numbers which will be illustrated by examples.

Example 2.39: Let

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in R; \right.$$

$$\left. g = 2 \in Z_6 \text{ so that } g^2 = -g = 4 \right\};$$

W be a ring called the ring of polynomials in the variable x with coefficients from the quasi special dual numbers. W is also a general vector space of quasi special dual numbers over the field R (or Q). Infact W is a linear algebra of quasi special dual numbers.

Take $p(x) = (5 + 8g) + (3 + g)x^2$ and
$q(x) = (8 + 4g)x + (2+g)x^2 + 4g \in W$.

$p(x) + q(x) = (5 + 12g) + (8+4g)x + (5+2g)x^2 \in W$.

$$\begin{aligned}
p(x) \times q(x) &= (5 + 8g)4g + (3+g)x^2 \times 4g + (5 + 8g) \\
&\quad (8 + 4g)x + (3+g)(8 + 4g)x^3 + (5 + 8g) \\
&\quad (2+g)x^2 + (3+g)(2+g)x^4 \\
&= (20g - 32g) + (12g - 4g)x^2 + (40 + 64g + 20g - \\
&\quad 40g)x + (24 + 8g + 12g - 4g)x^3 + (10 + 16g + \\
&\quad 5g - 8g)x^2 + (6 + 2g + 3g - g)x^4 \\
&= -12g + 8gx^2 + (44g + 40)x + (24 + 16g)x^3 + \\
&\quad (10 + 13g)x^2 + (6 + 4g)x^4 \\
&= -12g + (40 + 44g)x + (10 + 21g)x^2 + (24 + \\
&\quad 16g)x^3 + (6 + 4g)x^4
\end{aligned}$$

is in W.

Properties like ideals, subrings which are not ideals irreducible polynomials, solving for roots of polynomials etc can be carried out as a matter of routine. Infact roots will be from R or Rg or a + bg a, b $\in$ R and g is such that $g^2 = -g$. It is interesting to study these polynomials and finding roots of them.

Next just indicate we can get finite vector spaces of special quasi numbers. We will illustrate these situations by examples.

Example 2.40: Let

$$S = \{a + bg \mid a, b \in Z_{31}, g = 2 \in Z_6, g^2 = -g\}$$

be a general vector space of special quasi dual numbers over the field Z_{31}. S is of finite order and finite dimensional over Z_{31}.

Example 2.41: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i = x_i + y_i g,\ x_i, y_i \in Z_{113}, \right.$$

$$\left. 1 \le i \le 10,\ g = 4 \in Z_{10},\ g^2 = -g \right\}$$

be the general vector space of special quasi dual numbers over the field Z_{113}.

Example 2.42: Let M = $\{(a_1, a_2, \ldots, a_{15})$ where $a_i = x_i + y_i g$; $x_i, y_i \in Z_{47}$, $1 \le i \le 15$, $g = 15 \in Z_{40}$, $g^2 = -g\}$ be a general vector space of special quasi dual numbers over the field Z_{47}.

Example 2.43: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i = x_i + y_i g,\ 1 \le i \le 30, \right.$$

$$\left. x_i, y_i \in Z_{59},\ g = 24 \in Z_{40},\ g^2 = -g = 16 \right\}$$

be the general vector space of special quasi dual numbers over the field Z_{59}.

Example 2.44: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{36} \end{bmatrix} \middle| \; a_i = x_i + y_i g, \right.$$

$$\left. x_i, y_i \in Z_7,\ 1 \le i \le 36;\ g = 5 \in Z_{15},\ g^2 = 10 = -g \right\}$$

be the general vector space of special quasi dual numbers over the field Z_7.

All these vector spaces can also be made into linear algebras of special quasi dual numbers over the respective fields.

Finally we give one example of a non commutative linear algebra of special quasi dual numbers.

Example 2.45: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \text{ where } a_i = x_i + y_i g \text{ with } x_i, y_i \in Z_3, \right.$$

$$\left. g = 2 \in Z_6, g^2 = -g \right\}$$

be the non commutative general linear algebra of special quasi dual numbers under the usual product × of matrices.

Example 2.46: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| a_i = x_i + y_i g;\ x_i, y_i \in Z_5,\ 0 \le i \le \infty; \right.$$

$$\left. g = 5 \in Z_{15}, g^2 = -g = 10 \right\}$$

be a general linear algebra of special quasi dual numbers.

If $p(x) = 3 + 2g + (1+3g)\,x$
and $q(x) = 4 + 3g + (1+g)x^2$ are in M.
$p(x) + q(x) = 2 + (1+3g)x + (1+g)x^2 \in M$.

$$\begin{aligned} p(x) \times q(x) &= [(3 + 2g) + (1+3g)x] \times [(4 + 3g) + (1+g)x^2] \\ &= (3+2g)\,(4+3g) + (1+3g)\,(4 + 3g)x + (3+2g) \\ &\quad (1+g)x^2 + (1+3g)\,(1+g)x^3 \\ &= (2+2g+4g + 4g) + (4 + 2g + 3g + g)x + \\ &\quad (3 + 2g + 3g + 3g)x^2 + (1+3g + g + 2g)x^3 \\ &= 2 + (4+g)x + (3+3g)x^2 + (1+g)x^3 \in M. \end{aligned}$$

Example 2.47: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_i + y_i g;\ x_i, y_i \in Z_5,\ 0 \leq i \leq \infty; \right.$$

$$\left. g = 14 \in Z_{21},\ g^2 = 7 = -14 \right\}$$

be the general linear algebra of special quasi dual numbers over the field Z_5.

Now having see examples of vector spaces / linear algebras we proceed onto give examples of semivector spaces.

Example 2.48: Let
$M = \{x + yg \mid x, y \in Z^+ \cup \{0\},\ g = 14 \in Z_{21},\ g^2 = -g = 7 \in Z_{21}\}$ be a general semivector space of special quasi dual elements over the semifield $Z^+ \cup \{0\}$.

Clearly M is not a semilinear algebra over $Z^+ \cup \{0\}$.

Example 2.49: Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \,\middle|\, a_i = x_i + y_i g \text{ with} \right.$$

$$\left. x_i, y_i \in Q^+ \cup \{0\},\ 1 \leq i \leq 15,\ g = 2 \in Z_6 \right\}$$

be a general semivector space of special quasi dual numbers over the semifield $Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$).

Clearly W is not a general semilinear algebra of special quasi dual numbers over $Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$).

Example 2.50: Let

$$S = \left\{ \begin{pmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{pmatrix} \middle| \; a_i = x_i + y_i g \text{ with } x_i, y_i \in R^+ \cup \{0\}, \right.$$

$$\left. 1 \leq i \leq 10, g = 6 \in Z_{14}, g^2 = 36 = 8 \pmod{14} \right\}$$

be the general semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

Clearly S is not a general semilinear algebra.

Example 2.51: Let

$$P = \left\{ \begin{bmatrix} a_1 & \ldots & a_6 \\ a_7 & \ldots & a_{12} \\ \vdots & & \vdots \\ a_{31} & \ldots & a_{36} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ with} \right.$$

$$\left. x_i, y_i \in Z^+ \cup \{0\}, 1 \leq i \leq 36 \text{ with } g = 2 \in Z_6 \right\}$$

be the general semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$. Clearly P is not a semilinear algebra.

It is pertinent to mention here that we can use instead of semigroups under '+' groups under '+' of special dual numbers and build semilinear algebras.

Example 2.52: Let M = $\{a + bg \mid a, b \in Q, g = 14 \in Z_{21}, g^2 = 19^6 \pmod{21} = 7 = -g\}$ be the semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$. Infact M is also a semilinear algebra of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.53: Let $S = \{(a_1, a_2, a_3) \mid a_i = x_i + y_ig;\ x_i, y_i \in Z;\ 1 \leq i \leq 3,\ g = 2 \in Z_6,\ g^2 = 4 = -g \in Z_6\}$ be the semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$. Infact S is also a semilinear algebra of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.54: Let

$$P = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{19} \end{bmatrix}\right| a_i = x_i + y_ig;\ x_i, y_i \in Z,\ 1 \leq i \leq 19 \text{ with}\right.$$

$$\left. g = 24 \in Z_{40},\ g^2 = 16 = -g \in Z_{40}\right\}$$

be the semilinear algebra of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$ under the natural product $\times_n$ of matrices.

Example 2.55: Let

$$S = \left\{\left.\begin{pmatrix} a_1 & \ldots & a_{10} \\ a_6 & \ldots & a_{20} \end{pmatrix}\right| a_i = x_i + y_ig,\ x_i, y_i \in Z,\ 1 \leq i \leq 20,\right.$$

$$\left. g = 15 \in Z_{40},\ g^2 = 25 = -g \in Z_{40}\right\}$$

be a semilinear algebra of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$ under the natural product $\times_n$ of matrices.

Example 2.56: Let

$$T = \left\{\left.\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right| a_i = x_i + y_ig,\ x_i, y_i \in Z,\ 1 \leq i \leq 4,\right.$$

$$\left. g = 5 \in Z_{15},\ g^2 = 10 \pmod{15} = -g\right\}$$

be the semilinear algebra of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.57: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_i + y_i g \text{ with } x_i, y_i \in Z, g = 2 \in Z_6 \right\}$$

be the semilinear algebra of polynomials of special quasi dual number coefficients over the semifield $Z^+ \cup \{0\}$.

Now we can build both vector spaces and semivector spaces using the notion of complete special quasi dual pair numbers. This we will illustrate by an example or two.

Example 2.58: Let $W = \{a + bg + cg_1 \mid a, b, c \in Q; g = 2$ and $g_1 = 4 \in Z_6, g^2 = g_1 = -g$ and $g_1^2 = 4, gg_1 = g_1g = g\}$ be the vector space of complete special quasi dual number pair over the field Q. Infact W is also a linear algebra.

We see $V = \{a + bg \mid a, b \in Q, g = 2 \in Z_6\}$ and W are identical as vector spaces as $g_1 = -g$.

However we see the difference occurs only when we use semivector space with elements from $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ as $-1 \notin R^+$ or Q^+ or Z^+.

Example 2.59: Let $M = \{a + bg + cg_1 \mid a, b, c \in Q^+ \cup \{0\}; g = 5 \in Z_{15}, g_1 = 10$ so that $g^2 = 10 \pmod{15} = -5 \pmod{15}$ and $g_1^2 = g_1$ with $gg_1 = g_1g = g\}$ be the semivector space of complete special quasi dual numbers over the field $Q^+ \cup \{0\}$.

Example 2.60: Let $T = \{(a_1, a_2, \ldots, a_7)$ with $a_j = x_1 + x_2g_1 + x_3g$ where $1 \leq j \leq 7$, $g = 15$, $g_1 = 25 \in Z_{40}$, $g_1^2 = g_1 = 25 \pmod{40}$, $g_1g = gg_1 = g$ and $g^2 = -g = g_1$; $x_i \in R^+ \cup \{0\}$; $1 \leq i \leq 3\}$ be the semilinear algebra of complete special quasi dual pair numbers over the semifield $Q^+ \cup \{0\}$.

Example 2.61: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \middle| a_i = x_1 + x_2g + x_3g_1 \text{ with } 1 \leq i \leq 15, \right.$$

$$x_k \in Z^+ \cup \{0\}, 1 \leq i \leq 3 \text{ and } g = 4 \text{ and } g_1 = 6 \in Z_{10},$$

$$\left. g^2 = 6 = g_1 = -g \text{ and } g_1^2 = g_1; gg_1 = g_1g = g \right\}$$

be the semilinear algebra of complete quasi special dual pair numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.62: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| a_i = x_1 + x_2g + x_3g_1 \right.$$

$$\text{with } 1 \leq i \leq 30, x_k \in Z^+ \cup \{0\}, 1 \leq k \leq 3 \text{ and } g = 2$$

$$\left. \text{and } g_1 = 4 \in Z_6 \right\}$$

be a semilinear algebra of complete quasi special dual pair numbers over the semifield $Z^+ \cup \{0\}$ under the natural product $\times_n$ of matrices.

Example 2.63: Let

$$S = \left\{ \begin{bmatrix} g_1 & g_2 & g_3 \\ \vdots & \vdots & \vdots \\ g_{13} & g_{14} & g_{15} \end{bmatrix} \middle| g_i = x_1 + x_2g + x_3h \right.$$

$$\text{with } 1 \leq i \leq 15,\ x_k \in Z^+ \cup \{0\},\ 1 \leq k \leq 3 \text{ and}$$

$$g = 15 \text{ and } h = 25 \in Z_{40};\ g^2 = 25 = h \text{ and } h^2 = h \quad gh = hg = g\}$$

be the semilinear algebra of complete special quasi dual pair numbers over the semifield $Z^+ \cup \{0\}$ under the natural product $\times_n$.

Likewise consider $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1 \right.$ where $x_j \in Z^+ \cup \{0\}$, $1 \leq j \leq 3$; $g = 2 \in Z_6$, $4 = g_1$, $g_1^2 = 4$, $g^2 = 4 = -g$; $g_1 g = g g_1 = g\}$; P is a semifield of polynomials with coefficients as complete special quasi dual pair number.

If $Z^+ \cup \{0\}$ is replaced by $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ still we continue to get semifield of polynomials with coefficients as complete special quasi dual pair numbers.

Example 2.64: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1 \text{ with} \right.$$

$$\left. x_1, x_2, x_3 \in Q^+ \cup \{0\},\ g = 15 \text{ and } g_1 = 25 \in Z_{40} \right\}$$

be the semilinear algebra of polynomials with complete quasi special dual pair numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.65: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1 \text{ where } x_k \in R^+ \cup \{0\}, \right.$$

$$1 \leq k \leq 3,\ g = 24,\ g_1 = 16 \in Z_{40} \text{ with } g g_1 = g_1 g = g,$$

$$\left. g_1^2 = g_1 = 16,\ g^2 = 24^2 = g_1 = -g \right\}$$

be the semilinear algebra of polynomials with complete quasi special dual pair numbers over the semifield $Q^+ \cup \{0\}$ (or $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$).

Example 2.66: Let

$$P = \left\{ \sum_{i=0}^{5} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1 \text{ where } x_j \in R^+ \cup \{0\}, \right.$$

$$1 \leq j \leq 3,\ g = 14,\ g_1 = 7 \in Z_{21} \text{ with } g^2 = 14 = -g,\ g_1^2 = g_1,$$

$$\left. g_1 g = g g_1 = g = 14 \right\}$$

be only a semivector space of complete quasi special dual number pairs. P is clearly not a semilinear algebras as

$p(x) = (8 + 3g + 6g_1)x^4 + (2 + 2g + g_1)$ and
$q(x) = (3 + g_1 + g)x^3 + (2 + g + 2g_1)x \in P$.

$$\begin{aligned} \text{But } p(x) \times q(x) = {} & (8 + 3g + 6g_1) \times (3 + g_1 + g)\, x^7 + \\ & (2 + 2g + g_1)\,(3 + g_1 + g)x^3 + \\ & (8 + 3g + 6g_1)\,(2 + g + 2g_1)x^5 + \\ & (2 + 2g + g_1)\,(2 + g + 2g_1)x \notin P. \end{aligned}$$

Hence P is only a semivector space of complete special quasi dual pair of numbers.

All properties associated with semivector spaces, semilinear algebras, linear algebras and vector space can be easily derived in case of complete special quasi dual pair without any difficulty. Interested reader can work with them, however several problems in this direction are suggested in the last chapter of this book.

Chapter Three

HIGHER DIMENSIONAL QUASI SPECIAL DUAL NUMBERS

In this chapter we for the first time introduce the notion of t-dimensional quasi special dual numbers t ≥ 3. However it is pertinent to keep on record that apart from these modulo integers the other source are from the neutrosophic numbers.

We will first illustrate by examples or –1 and 1 in matrix form.

Let x = (–I, –I, –I, –I), $x^2 = (I^2, I^2, I^2, I^2)$ = (I, I, I, I) = –x.

Thus a + bx, a, b ∈ R or C or Q or Z or Z_n is a quasi special dual number.

Likewise x = $\begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}$;

x under natural product $\times_n$ is given by

$$x \times_n x = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ so } x \times_n x = x^2 = -x.$$

This is yet another way of building quasi special dual numbers by a + bx with a, b $\in$ R or C or Z or Z_n.

$$\text{Let } y = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \text{ be such that}$$

$$y \times_n y = y^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -y \text{ and}$$

a + by, a, b $\in$ R (or Q or Z_n or Z or C) is a quasi special dual number.

$$\text{Let } x = \begin{pmatrix} -I & -I & -I & -I & -1 \\ -I & -I & -I & -I & -1 \end{pmatrix} \text{ be such that}$$

$$x^2 = \begin{pmatrix} I & I & I & I & 1 \\ I & I & I & I & 1 \end{pmatrix} = -x \text{ under the natural product } \times_n.$$

Thus using these matrices we cannot get any desired number of quasi special elements.

Example 3.1: Let $x = a + bg_1 + cg_2$ with $g_1 = 3$ and $g_2 = 8$, g_1, $g_2 \in Z_{12}$. We see x is a quasi special dual number.

$$\begin{aligned} x &= (a + bg_1 + cg_2) \text{ and } y = c + dg_1 + eg_2 \\ xy &= (a + bg_1 + cg_2)(c + dg_1 + eg_2) \end{aligned}$$

$= ac + bcg_1 + c^2g_2 + dag_1 + db\,g_1^2 + dcg_1g_2 +$
$eag_2 + beg_1g_2 + ce\,g_2^2$

$= ac + bcg_1 + dag_1 – dbg_1 + eag_2 – ceg_2 + c^2g_2$
$= ac + (bc + da – db)\,g_1 + (ea – ce + c^2)g_2$

is again a three dimensional quasi special dual number.

Example 3.2: Let M = {a + bg_1 + cg_2 where a, b, c ∈ Z, g_1 = (–I, 0, 0, 0); g_2 = (0, 0, 0, –I); g_1^2 = (I, 0, 0, 0) = $–g_1$ and g_2^2 = (0, 0, 0, I) = $–g_2$ and $g_1g_2 = g_2g_1$ = (0, 0, 0, 0)} be a three dimensional quasi special dual number.

Example 3.3: Let

$$P = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid a_j \in Z,$$

$$1 \le j \le 5;$$

$$g_1 = \begin{bmatrix} -I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 \\ -I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ g_3 = \begin{bmatrix} 0 \\ 0 \\ -I \\ 0 \\ 0 \\ 0 \end{bmatrix},\ g_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -I \\ 0 \\ 0 \end{bmatrix},$$

$$g_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -I \\ 0 \end{bmatrix} \text{ and } g_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -I \end{bmatrix}$$

with $g_j^2 = -g_j$; $1 \le j \le 6$ and $g_i \times_n g_j = g_j \times_n g_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ if $i \ne j$;

$1 \le i, j \le 6\}$ be the collection of all seven dimensional quasi special dual numbers.

Example 3.4: Let

$$W = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 + a_9g_8 \mid$$

$$a_i \in Q, 1 \le i \le 9;$$

$$g_1 = \begin{bmatrix} -I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & -I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 \\ -I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$g_4 = \begin{bmatrix} 0 & 0 \\ 0 & -I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -I & 0 \\ 0 & 0 \end{bmatrix}, g_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix},$$

$$g_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -I & 0 \end{bmatrix} \text{ and } g_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} \text{with } g_k^2 = -g_k;$$

$$1 \le k \le 8 \text{ and } g_i \times_n g_j = g_j \times_n g_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } i \ne j;\ 1 \le i, j \le 8\}$$

be the nine dimensional quasi special dual numbers.

Only this method allows one to construct any desired dimensional quasi special dual numbers.

Now we can have several such numbers.

Justlike neutrosophic numbers helped in constructing special dual like numbers neutrosophic numbers help in constructing quasi special dual numbers of higher dimension.

We will illustrate this situation by some examples.

Example 3.5: Let

$$W = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 +$$

$$a_6g_5 + a_7g_6 \mid g_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_2 = \begin{bmatrix} -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ g_3 = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix},\ g_4 = \begin{bmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \text{ and } g_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \end{bmatrix}.$$

$$g_i \times_n g_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \ne j,\ g_i^2 = g_{i-1},\ i = 2, 3, 4, 5, 6,$$

that is

$$g_4^2 = \begin{bmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} = g_3; \text{ with}$$

$g_{i+1} = -g_i$, i = 1, 2, 3, 4, 5. $g_j \in Z^+ \cup \{0\}$; $1 \leq j \leq 7\}$
be the complete quasi special neutrosophic dual number pair.

Example 3.6: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \text{ where } a_i = x_1 + x_2g + x_3g_1 + x_4h + \right.$$

x_5h_1 where $x_j \in Q^+ \cup \{0\}$, g = 15, g_1 = 25, h = 24 and

$h_1 = 16 \in Z_{40}$, $1 \leq j \leq 5$, $1 \leq i \leq 6\}$

be the complete quasi special dual number pair.

Clearly M is a semigroup under + also M is a semigroup under $\times_n$. Infact (M, +, $\times_n$) is a commutative strict semiring.

Example 3.7: Let M = $\{a_1 + a_2g + a_3g_1 + a_4h + a_5h_1 + a_6k + a_7k_1 \mid a_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 7$, g = (–I, –I, 0, 0, 0, 0), g_1 = (I, I, 0, 0, 0, 0), h = (0, 0, –I, –I, 0, 0) h_1 = (0, 0, I, I, 0, 0), k = (0, 0, 0, 0, –I, –I) and k_1 = (0, 0, 0, 0, I, I)} be the semigroup of complete quasi special dual number pair under product.

Example 3.8: Let S = $\{a_1 + a_2g_1 + a_3g + a_4h + a_5h_1 + a_6k + a_7k_1 \mid a_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 7$, g = (–I, 0, 0), g_1 = (I, 0, 0), h = (0, –I, 0), h_1 = (0, I, 0), k = (0, 0, –I) and k_1 = (0, 0, I)} be the semigroup under product.

We see $g + g_1 = g + g_1 = 0$.

However we do not add g + h or $g_1 + h_1$ or $g + h_1$ and so on $g^2 = g_1 = -g$, $h^2 = +h_1 = -h$ and $k^2 = k_1 = -k$.

However we cannot add $h_1 + g = (–I, I, 0)$ where $(h_1 + g)^2 = (I, I, 0) \neq h_1 + g$ or $= –(h_1 + g)$ or $= (0, 0, 0)$.

Thus we do not perform addition of g with h or h_1 or k or k_1, however $g^2 = –g_1$.

Example 3.9: Let $S = \{a_1 + a_2g + a_3g_1 + a_4h + a_5h_1 \mid a_i \in Z, 1 \leq i \leq 5, g = 15, g_1 = 25 = –g \pmod{40}, h = 24$ and $h_1 = 6 = –h \pmod{40}\}$. We see $P = \{g, g_1, h, h_1, 0\}$ is a semigroup under product. However P is not closed under '+'. However $P \cup \{1\}$ is a monoid under $\times$.

We call $P \cup \{1\}$ as the semigroup associated with quasi special dual numbers. Using $P \cup \{1\}$ we can construct semigroup ring and semigroup semiring which will form the collection of complete quasi special dual number pairs rings or semirings respectively.

We will illustrate this situation by some examples.

Example 3.10: Let $S = \{1, 0, 3, 4, 8, 9\} \subseteq Z_{12}$ is the associated semigroup of special quasi dual number component.

Let Q be the field of rationals QS be the semigroup ring of S over Q.

Suppose
$S = \{1 = g_1 \;\; g_2 = 3, g_3 = 4, \;\; g_4 = 8$ and $g_5 = 9, 0\} \subseteq Z_{12}$.

Then $QS = \{x_1 + x_2g_2 + x_3g_3 + x_4g_4 + x_5g_5 \mid x_i \in Q; g_j \in S$ and $g_1 = 1$ so $x_1g_1 = x_1, 1 \leq i \leq 1; 2 \leq j \leq 5\}$. QS is the general ring of complete quasi special dual number pairs.

QS has zero divisors, units and idempotents.

Thus as we get using complex number $C = \{a + bi \mid i^2 = –1\}$ quasi special dual numbers $a + bg, g^2 = –g$ and $a, b \in C$.

$S = \{a + bg \mid g^2 = -g$ with a, b are complex numbers$\}$ and quasi special dual complex modulo integers.

$P = \{a + bg \mid g^2 = -g$ with $a, b \in Z_n$, Z_n the modulo integers$\}$.

We see in case of complex numbers or neutrosophic numbers we cannot extend it higher dimension.

But in case of quasi special dual numbers we can extend the notion to any desired dimension. That is if $\{g_1, g_2, \ldots, g_t\}$ are t-distinct quasi special dual numbers such that $g_i^2 = -g_i$ and $g_ig_j = g_i$ or g_j or 0 if $i \neq j$, $1 \leq i \leq t$.

So $Q(g_1, g_2, \ldots, g_t) = \{x_1 + x_2g_1 + \ldots + x_{t+1} g_t\}$ is the t+1 dimensional quasi special dual numbers.

$Q(g_1, g_2, \ldots, g_t)$ is a ring and not a field.

Let us consider $(Q^+ \cup \{0\}) (g_1, \ldots, g_t)$ we see we cannot give any structure except $(Q^+ \cup \{0\}) (g_1, \ldots, g_t)$ is just a semigroup under '+'.

However if we denote the collection $(-g_1, \ldots, -g_t)$ as say $(h_1, h_2, \ldots, h_t)$ then with such modification we can build.

$V = (Q^+ \cup \{0\}) (g_1, \ldots, g_t, h_1, h_2, \ldots, h_t)$

$= \{x_1 + x_2g_1 + \ldots + x_{t+1}g_t + y_1h_1 + \ldots +y_th_t \mid x_i\ y_j \in Q^+\cup \{0\}$ with $g_i^2 = h_i$; $1 \leq i \leq t$; $g_ig_j = h_j$ or h_i or g_i or g_j or 0, $1 \leq i, j \leq t\}$. Clearly V is a semigroup under $\times$ infact V is a semiring.

In case of rings R, the addition of $h_1, \ldots, h_t$ is not essential as for every $a \in R$, $-a \in R$ so we can say even if we write $Q(g_1, g_2, \ldots, g_t; h_1, h_2, \ldots, h_t)$ yet both $Q(g_1, \ldots, g_t, h_1, \ldots, h_t)$ is isomorphic with $Q(g_1, \ldots, g_t)$ as rings.

Now we can have $Z_n(g_1, \ldots, g_t)$ is isomorphic with $Z_n(g_1, \ldots, g_t, h_1, \ldots, h_t)$ as rings.

Thus the study of rings and semirings in case of special quasi dual numbers can be taken as a matter of routine.

We only indicate by some simple examples how vector spaces, semivector spaces and Smarandache semivector spaces can be constructed using the notion of complete special quasi dual pairs of numbers.

Example 3.11: Let $M = \{(a_1, a_2, \ldots, a_6) \mid a_i = x_i + y_ig + z_ig_1 + m_ih + n_ih_1$ where $x_i, y_i, z_i, m_i, n_i \in Q$; $1 \leq i \leq 6$; $g = 15$, $g_1 = 25$, $h = 24$ and $h_1 = 16$ in $Z_{40}\}$ be the complete vector space of quasi special dual numbers pairs over the field Q.

Take $M_1 = Q(g, g_1, h, h_1) = \{x_1 + x_2g + x_3g_1 + x_4h + x_5h_1 \mid x_i \in Q; 1 \leq i \leq 5\}$.

Clearly suppose we take $S = \{0, 1, 15, 25, 16, 24\} \subseteq Z_{40}$ we see $(S, \times)$ is a semigroup given by the following table.

$\times$	0	1	15	16	24	25
0	0	0	0	0	0	0
1	0	1	15	16	24	25
15	0	15	25	0	0	15
16	0	16	0	16	24	0
24	0	24	0	24	16	0
25	0	25	15	0	0	25

Consider the semigroup ring QS of the semigroup S over the ring Q.

Clearly $QS \cong M_1$, so infact we can say QS the semigroup ring is a vector space of complete special quasi dual pairs over the field Q.

It is clear QS is a linear algebra.

Also we can say M_1 is isomorphic with QS as well as Q(g, h) as rings or linear algebras where $g = -g = g_1$ and $h_2 = -h = h_1$. Thus without loss of generality we can work with

$N = \{(a_1, a_2, \ldots, a_6) \mid a_i = x_1 + x_2g + x_3h, 1 \leq i \leq 6\}$ as M is isomorphic with N as linear algebras however they are not isomorphic as vector spaces.

Example 3.12: Let

$$P = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \text{ where } a_i = x_1 + x_2g + x_3g_1\right.$$

$$\left. \text{where } x_1, x_2, x_3 \in R, g = 8, g_1 = 4 \in Z_{12}, 1 \leq i \leq 16\right\}$$

be a vector space over R of complete special quasi dual number pair over the field R.

We see P is a commutative linear algebra over the field R under natural product $\times_n$ and a non commutative linear algebra over the field $\times$.

$$\text{We see if } S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i = x_1 + x_2g\right.$$

$$\left. \text{where } x_1, x_2 \in R, g = 8 \text{ with } g^2 = 4 \in Z_{12}, 1 \leq i \leq 16\right\}$$

is again a commutative linear algebra over R under $\times_n$.

We see S and P are isomorphic as linear algebras but are not isomorphic as vector spaces.

Example 3.13: Let

$$S = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \end{pmatrix} \middle| \; a_i = x_1 + x_2g + x_3g_1 + x_4h + x_5h_1; \right.$$

$1 \leq i \leq 12$, $x_j \in Q$; $1 \leq j \leq 5$, $g = 6$, $g_1 = 15$, $h = 14$, $h_1 = 7 \in Z_{21}\}$

be a vector space of special quasi dual pairs over the field Q.

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \end{pmatrix} \middle| \; a_i = x_1 + x_2g + x_3h \text{ where} \right.$$

$$g = 6 \text{ and } h = 14 \in Z_{21}, x_1, x_2, x_3 \in Q, 1 \leq i \leq 12\}$$

is a linear algebra of quasi dual pairs over the field Q.

Example 3.14: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i = x_1 + x_2g + x_3g_1 + x_4h + x_5h_1 + x_6k + x_7k_1 + \right.$$

$x_8p + x_9p_1$ where $1 \leq i \leq 4$, $x_j \in Q^+ \cup \{0\}$; $1 \leq j \leq 9$, $g = (-I, 0, 0, 0)$, $g_1 = (I, 0, 0, 0)$, $h = (0, -I, 0, 0)$, $h_1 = (0, I, 0, 0)$, $k = (0, 0, -I, 0)$, $k_1 = (0, 0, I, 0)$ and $p = (0, 0, 0, -I)$ and $p_1 = (0, 0, 0, I)\}$ be a vector space / linear algebra of complete quasi special dual number pairs over the semifield $Q^+ \cup \{0\}$.

Example 3.15: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_6 & a_{11} \\ a_2 & a_7 & a_{12} \\ a_3 & a_8 & a_{13} \\ a_4 & a_9 & a_{14} \\ a_5 & a_{10} & a_{15} \end{bmatrix} \middle| \; a_i = x_0 + x_1 g + x_2 h \text{ where} \right.$$

$$g = \begin{pmatrix} -I & -I & -I \\ 0 & 0 & 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 0 & 0 & 0 \\ -I & -I & -I \end{pmatrix},$$

$$\left. x_0, x_1, x_2 \in Z^+ \cup \{0\}, 1 \leq i \leq 15 \right\}$$

be the semivector space of quasi special dual numbers over $Z^+ \cup \{0\}$.

S is not a linear algebra.

Example 3.16: Let

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{pmatrix} \middle| \; a_i = x_0 + x_1 g + x_2 h \text{ where} \right.$$

$$\left. g_1 = 15 \text{ and } h = 24 \in Z_{40}, x_0, x_1, x_2 \in Q^+ \cup \{0\}; 1 \leq i \leq 30 \right\}$$

be a semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

Clearly P is not a semilinear algebra.

Thus we have semivector spaces which are not semilinear algebras, however if these semivector spaces of complete quasi special dual number pairs then certainly these semivector spaces will be semilinear algebras over $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or $Q^+ \cup$

{0}. Now we can also have the simple notion of polynomial rings of quasi special dual pair numbers and polynomial semirings of complete special dual pair numbers.

We will just illustrate this situation.

Example 3.17: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = y_1 + y_2 g + y_3 h \text{ where} \right.$$

$$g = \begin{pmatrix} -I & -I & 0 \\ -I & -I & 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 0 & 0 & -I \\ 0 & 0 & -I \end{pmatrix} \text{ and } y_i \in Q;$$

$$\left. g^2 = -g \text{ and } h^2 = -h,\ 1 \le i \le 3 \right\}$$

be the polynomial ring of quasi special dual numbers.

Example 3.18: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 k \text{ where} \right.$$

$$\left. x_1, x_2, x_3 \in R;\ g = 15 \text{ and } k = 24 \in Z_{40} \right\}$$

be the quasi special dual number ring of polynomials.

All concepts of reducibility / irreducibility and roots; etc can be done as a matter of routine. However roots of polynomials can also be special quasi dual number.

Further Q or R can also be replaced by C and still the conclusions hold good.

Suppose we now use Z_n instead of C or Z or Q or R; we give a few examples of them.

Example 3.19: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2g + x_3k \text{ where } g = 14 \right.$$

$$\left. \text{and } k = 6 \in Z_{21},\ x_j \in Z_{240},\ 1 \le j \le 3 \right\}$$

be the special quasi dual numbers polynomial ring.

Example 3.20: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2g + x_3k \text{ where } g = 6, \right.$$

$$\left. k = 14 \in Z_{21},\ x_j \in Q^+ \cup \{0\},\ 1 \le j \le 3 \right\}.$$

M is only a semigroup under '+' and M is not closed under product for

$p(x) = 9gx^3$, $q(x) = 2g + 3kx$ in M

$$\begin{aligned} p(x) \times q(x) &= 9gx^3\,(2g + 3kx) \\ &= 18g^2x^3 + 27gkx^4 \\ &= -18gx^3 + 0 \notin M. \end{aligned}$$

Example 3.21: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2g + x_3k + x_4h \text{ where} \right.$$

$$g = \begin{pmatrix} -I & 0 & 0 & 0 \\ -I & 0 & -I & 0 \end{pmatrix},\ k = \begin{pmatrix} 0 & -I & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix} \text{ and}$$

$$\left. h = \begin{pmatrix} 0 & 0 & -I & -I \\ 0 & 0 & 0 & -I \end{pmatrix};\ x_j \in Z^+ \cup \{0\};\ 1 \le j \le 4 \right\}$$

be the semigroup under '+'.

Clearly P is not a semigroup under ×.

Take $p(x) = 3 + 2gx + 4hx^2$ and
$q(x) = 4g + 5hx^3 + 2gx^5$ in P.

Consider

$$\begin{aligned} p(x) \times q(x) &= (3 + 2gx + 4hx^2) \times (4g + 5hx^3 + 2gx^5) \\ &= 12g + 8g^2x + 16ghx^2 + 15hx^3 + 10ghx^4 + \\ &\quad 20h^2x^5 + 6gx^5 + 4g^2x^6 + 8ghx^7 \\ &= 12g + 8(-g)x + 0 + 15hx^3 + 20(-h)x^5 + 6gx^5 + 4(-g)x^6 \end{aligned}$$

Clearly $p(x) \times q(x) \notin P$. Inview of this we have the following result. Only if we take the collection of all complete special quasi dual number pairs then only we get a semigroup under × and hence a semiring.

We will just illustrate this situation by some examples.

Example 3.22: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = x_1 + x_2 g + x_3 g_1 + x_4 h + x_5 h_1 \right.$$

where $x_j \in Q^+ \cup \{0\}$

$$1 \le j \le 5,\ g = \begin{pmatrix} -I & 0 & 0 & 0 \\ -I & 0 & -I & 0 \end{pmatrix}\ g_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ I & 0 & I & 0 \end{pmatrix},$$

$$\left. h = \begin{pmatrix} 0 & 0 & -I & -I \\ 0 & 0 & 0 & -I \end{pmatrix},\ h_1 = \begin{pmatrix} 0 & 0 & I & I \\ 0 & 0 & 0 & I \end{pmatrix} \right\}.$$

M is a semigroup under product and infact a semiring. However M is not a semifield as M has zero divisors, $p(x) = 3gx^3$ and $q(x) = 4hx^7 \in M$ then

$$p(x).q(x) = 3gx^3 \times 4hx^7 = 12ghx^{10} = 0 \text{ as}$$

$$gh = \begin{pmatrix} -I & 0 & 0 & 0 \\ -I & 0 & -I & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & -I & -I \\ 0 & 0 & 0 & -I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the claim.

Example 3.23: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = x_1 + x_2 g + x_3 g_1 + x_4 h + x_5 h_1;\ x_j \in Z^+ \cup \{0\}, \right.$$

$1 \leq j \leq 5$ $g = 6$, $g_1 = 15$, $h = 14$ and $h_1 = 7 \in Z_{21}$. $gg_1 = g$,

$hh_1 = h$, $g \times h = 0 \pmod{21}$, $g_1 \times h = 0 \pmod{21}$,

$g \times h_1 = 0 \pmod{21}$ and $g_1 \times h_1 = 0 \pmod{21}\}$.

P is a semiring of complete special quasi dual number pairs and P is not a semifield.

Thus unless we take complete quasi special dual number pairs as coefficients of the polynomials we would not be in a position to get semirings we only can get semigroup under '+'.

Next we proceed onto study the semigroup counter part of special quasi dual numbers in $C(Z_n)$. First we study some examples. At the outset the authors think $a + bi_F \in C(Z_n)$ $a \neq 0$ $b \neq 0$ cannot be such that

$$(a + bi_F)^2 = -(a + bi_F) = (n-1)(a+bi_F).$$

Thus at this juncture the authors suggest the following problem.

Problem: Let $C(Z_n) = \{a + bi_F \mid a, b \in Z_n,\ i_F^2 = n-1\}$.

Does $C(Z_n)$ contain $x = a + bi_F$; $a \neq 0$, $b \neq 0$ such that $(a + bi_F)^2 = -(a + bi_F) \pmod n = (n-1)(a + bi_F)$.

We at this stage do not discuss about complex modulo integer dual numbers.

Consider $C(Z_5) = \{a + bi_F \mid a, b \in Z_5,\ i_F^2 = 4\}$

$$\begin{aligned}
&\text{Take } (2 + i_F) \in C(Z_5)\\
&(2 + i_F)^2 = 4 + i_F^2 + 4i_F\\
&\qquad = 4 + 4 + 4i_F \ (i_F^2 = 4)\\
&\qquad = 3 + 4i_F\\
&\qquad = -(2 + i_F) \pmod 5.
\end{aligned}$$

Consider $(3 + 4i_F)^2 = 9 + 16 i_F^2 + 24i_F$
$$\begin{aligned}
&= 9 + 16 \times 4 + 24i_F\\
&= (73 + 24i_F) \pmod 5\\
&= 3 + 4i_F.
\end{aligned}$$

Consider $(3 + 4i_F)(2 + i_F)$
$$\begin{aligned}
&= 6 + 8i_F + 3i_F + 4 i_F^2\\
&= 6 + 11i_F + 4 \times 4\\
&= (22 + 11i_F) \pmod 5\\
&= (2 + i_F).
\end{aligned}$$

Thus $2 + i_F$ contributes a quasi special dual number.

Consider $C(Z_{10}) = \{a + bi_F \mid a, b \in Z_{10},\ i_F^2 = 9\}$. $7 + 6i_F$ is a component of a dual number

$(7 + 6i_F)^2 = 3 + 4i_F = -(7 + 6i_F)$ and

$(2 + 4i_F)$ is a component of the dual number; we have $(2 + 4i_F)^2 = 8 + 6i_F = -(2 + 4i_F)$.

Let S = $\{7 + 6i_F, 3 + 4i_F, 2 + 4i_F, 8 + 6i_F, 0\}$. Clearly (S, +) is not a semigroup. We find out whether (S, $\times$) is a semigroup. Consider the following table of S under $\times$.

$\times$	0	$2+4i_F$	$3+4i_F$	$7+6i_F$	$8+6i_F$
0	0	0	0	0	0
$2+4i_F$	0	$8+6i_F$	0	0	$2+4i_F$
$3+4i_F$	0	0	$3+4i_F$	$7+6i_F$	0
$7+6i_F$	0	0	$7+6i_F$	$3+4i_F$	0
$8+6i_F$	0	$2+4i_F$	0	0	$8+6i_F$

(S, $\times$) is a semigroup we can add 1 with S so that $\{S \cup \{1\}, \times\}$ is a monoid.

Example 3.24: $C(Z_4)$ has no special quasi dual number component.

Likewise $C(Z_6)$ has no complex special quasi dual number component.

Thus the study of existence of special quasi dual number component in case of $C(Z_n)$ happens to be an interesting problem.

Example 3.25: Consider $C(Z_{17}) = \{a + bi_F \mid i_F^2 = 16, a, b \in Z_{17}\}$ be the ring of modulo integers.

Consider
$S = \{0, 1, 8 + 2i_F, 9 + 15i_F, 9 + 2i_F, 8 + 15i_F\} \subseteq C(Z_{17})$.

We see clearly S is not closed under the operation '+'.

Now we find the table of S under '$\times$' which is as follows:

$\times$	0	1	$8+2i_F$	$9+15i_F$	$9+2i_F$	$8+15i_F$
0	0	0	0	0	0	0
1	0	1	$8+2i_F$	$9+15i_F$	$9+2i_F$	$8+15i_F$
$8+2i_F$	0	$8+2i_F$	$9+15i_F$	$8+2i_F$	0	0
$9+15i_F$	0	$9+15i_F$	$8+2i_F$	$9+15i_F$	0	0
$9+2i_F$	0	$9+2i_F$	0	0	$9+2i_F$	$8+15i_F$
$8+15i_F$	0	$8+15i_F$	0	0	$8+15i_F$	$9+2i_F$

S is the special quasi dual number component semigroup of $C(Z_{17})$. However we have not found all such semigroups of $C(Z_{17})$.

Now using components of quasi special dual complex modulo integer numbers we can construct quasi special dual complex modulo integer numbers as well as complete quasi special dual complex modulo integer numbers pairs.

We will only illustrate these situations by some examples.

Example 3.26: Let $S = \{a + bg \mid a, b \in Q, g = 2 + i_F \in C(Z_5)\ g^2 = -g\}$ be the collection of quasi special complex modulo integer dual numbers.

S is a commutative ring with units and zero divisors. Infact S is a Smarandache ring.

Example 3.27: Let $M = \{(a_1, a_2, a_3) \mid a_i = x + yb$ where $b = 7 + 6i_F \in C(Z_{10})$; $1 \leq i \leq 3$, $x, y \in Q$ with $b^2 = -b\}$ be the ring of quasi special dual numbers of complex modulo integers.

M is a S-ring with units, idempotents and zero divisors.

Example 3.28: Let

$$S = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i = x_i + y_i g \text{ where}\right.$$

$$g = 8 + 2i_F \in C(Z_{17}) = \{a + bi_F \mid a, b \in Z_{17},\ i_F^2 = 16\}$$

$$\left.\text{with } g^2 = -g,\ x_i, y_i \in Q;\ 1 \leq i \leq 9\right\}$$

be the non commutative ring of quasi special dual numbers of complex modulo integers. S is also a Smarandache ring with unit.

Example 3.29: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \, a_i = x_i + y_i g \text{ with } g = 8 + 15i_F \in C(Z_{17}), \right.$$

$$\left. g^2 = -g,\ x_i, y_i \in Z;\ 1 \le i \le 15 \right\}$$

be a commutative special dual complex modulo integer ring; under the natural product $\times_n$ on S. S is also a S-ring with zero divisors and idempotents.

It is pertinent to mention here that it is not easy to construct semiring of special dual complex modulo integers; only those structure are rings as every ring is a semiring and not vice versa. To over come this as before we have only complete quasi special dual pair number semirings only. We do not define this as it is a matter of routine. However we give examples of them.

Example 3.30: Let $S = \{a + bg + cg_1 \mid a, b, c \in Z^+ \cup \{0\}, g = 2 + i_F$ and $g_1 = 3 + 4i_F \in C(Z_5)$; $g^2 = g_1$, $g_1^2 = g_1$ with $gg_1 = g = g_1g\}$ be the complete quasi special dual pair number semiring. Clearly S is a strict semiring.

Example 3.31: Let $M = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i = x_i + y_ig + z_ig_1, x_i, y_i, z_i \in Q^+ \cup \{0\}, 1 \le i \le 6, g = 7 + 6i_F, g_1 = 3+4i_F \in C(Z_{10})$ with $g^2 = g_1$, $g_1^2 = g_1$, $gg_1 = g_1g = g_1\}$ be the semiring of complete special quasi dual number pairs M has zero divisors and units but M is not a semifield.

Example 3.32: Let

$$T = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{29} & a_{30} & a_{31} & a_{32} \end{bmatrix} \right| a_i = x_i + y_i g + z_i g_1, \right.$$

$$x_i, y_i, z_i \in Q^+ \cup \{0\}, 1 \leq i \leq 32,$$

$$\left. g = 8+2i_F, g_1 = 9 + 15i_F \in C(Z_{17}); g^2 = g_1, gg_1 = g_1 g = g \right\}$$

be the semiring of complete dual special quasi number pairs under natural product $\times_n$. T has zero divisors and units. However T is not a semifield.

Example 3.33: Let

$$S = \left\{ \left. \begin{pmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{pmatrix} \right| a_i = x_i + y_i g + z_i g_1, \right.$$

$$x_i, y_i, z_i \in Z^+ \cup \{0\}, 1 \leq i \leq 14, g = 8 + 15i_F,$$

$$\left. g_1 = 9 + 2i_F \in C(Z_{17}), g^2 = g_1, g_1 g = gg_1 = g, g_1^2 = g_1 \right\}$$

be the semiring of complete quasi special dual pair number.

Now having seen examples of quasi special dual number rings and complete quasi special dual number pair semiring we proceed onto give examples of vector space of quasi special dual number pair and semivector space of quasi special dual number pair.

Example 3.34: Let $S = \{(a_1, a_2, a_3) \mid a_i = x + yg$ where $x, y \in Q$, $1 \leq i \leq 3$, $g = 8 + 2i_F \in C(Z_{17})\}$ be a vector space of quasi special dual number over the field Q.

S has subspaces and infact S can be realized as a linear algebra of quasi special dual numbers.

Example 3.35: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right| a_i = x_i + y_i g;\ x_i, y_i \in R,\ 1 \leq i \leq 12 \right\}$$

be a vector space of special quasi dual numbers over R $(M, \times_n)$ becomes a general linear algebra of special quasi dual numbers.

Example 3.36: Let

$$S = \left\{ \left. \sum_{i=0}^{\infty} a_i x^i \right| a_i = x_i + y_i g,\ x_i, y_i \in Q;\ g = 2 + 4i_F \in C(Z_{10}) \right\}$$

be the general semilinear algebra of special quasi dual numbers over Q.

Example 3.37: Let

$$W = \left\{ \left. \sum_{i=0}^{\infty} a_i x^i \right| a_i = x_i + y_i g,\ x_i, y_i \in Z_{11};\ g = 2 + i_F \in C(Z_5) \right\}$$

be the general linear algebra of special quasi dual numbers over the field Z_{11}.

Example 3.38: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in Z_3;\ 1 \leq i \leq 10, \right.$$

$$\left. g = 7 + 6i_F \in C(Z_{10}) \right\}$$

be a general vector space of special quasi dual numbers over the field Z_3. Clearly under $\times_n$; P is a linear algebra; P is a finite dimensional as well as finite order linear algebra / vector space over Z_3.

Example 3.39: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i = x_i + y_i g,\ 1 \leq i \leq 4,\ i, j \in Z_3, \right.$$

$$\left. g = 8 + 2i_F \in C(Z_{17}) \right\}$$

be the non commutative linear algebra of special quasi dual numbers over the field Z_2.

Let $x = \begin{bmatrix} 1+g & 2 \\ 2g & g+2 \end{bmatrix}$ and $y = \begin{bmatrix} 2g & 1+g \\ 1+g & 1 \end{bmatrix}$ be in P.

$$x + y = \begin{bmatrix} 1 & g \\ 1 & g \end{bmatrix} \text{ and } x \times y = \begin{bmatrix} 2+2g & g \\ g+2 & 2g+2 \end{bmatrix}.$$

Now $y \times x = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$. Clearly $x \times y \neq y \times x$.

Suppose we take the natural product $\times_n$ on P we see

$$x \times_n y = \begin{bmatrix} 1+g & 2 \\ 2g & g+2 \end{bmatrix} \begin{bmatrix} 2g & 1+g \\ 1+g & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2g(1+g) & 2(1+g) \\ 2g(1+g) & (g+2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2(1+g) \\ 0 & (g+2) \end{bmatrix}.$$

We see $x \times y \neq x \times_n y$ and $(P, \times_n)$ is a commutative linear algebra.

Now having seen examples of special quasi dual vector spaces / linear algebras we now proceed on to describe semivector spaces / semilinear algebras of quasi special dual number pairs.

Example 3.40: Let $S = \{(a_1, a_2, \ldots, a_{10}) \mid a_i = x_i + y_i g + z_i g_1, z_i, x_i, y_i \in Q^+ \cup \{0\}, 1 \leq i \leq 10, g_1 = 9 + 15i_F$ and $g = 8 + 2i_F \in C(Z_{17})$ with $g^2 = g_1, g_1^2 = g_1, g_1 g = gg_1 = g\}$ be a semivector space (as well as semilinear algebra) of special quasi dual number pair over the semifield $Z^+ \cup \{0\}$.

Example 3.41: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \, a_i = x_i + y_i g + z_i g_1 \text{ where} \right.$$

$x_i, y_i, z_i \in R^+ \cup \{0\}, 1 \leq i \leq 30$ $g = 2 + 4i_F$ and

$g_1 = 10 + 6i_F \in C(Z_{10})\}$

be the complete special quasi dual pair number general semivector space over the semifield $Z^+ \cup \{0\}$.

Infact P is also a general linear algebra of complete special quasi dual pair of numbers.

Example 3.42: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i = x_i + y_i g + z_i g_1 \text{ with} \right.$$

$$x_i, y_i, z_i \in Z^+ \cup \{0\},\ 1 \leq i \leq 9,\ g = 7 + 6i_F,\ g_1 = 3 + 4i_F \in C(Z_{10})$$

$$\left. \text{where } g^2 = g_1,\ g_1^2 = g_1,\ g_1 g = g g_1 = g \right\}$$

be the complete non commutative linear algebra of special quasi dual pair of numbers over the semifield $Z^+ \cup \{0\}$.

Example 3.43: Let

$$S = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{pmatrix} \middle| \; a_i = x_i + y_i g + z_i g_1 \text{ with} \right.$$

$$g = 2 + i_F,\ g_1 = 3 + 4i_F,\ x_i, y_i, z_i \in Z^+ \cup \{0\},\ 1 \leq i \leq 30,$$

$$\left. g, g_1 \in C(Z_5),\ g^2 = g_1,\ g_1^2 = g_1 \text{ and } g_1 g = g g_1 = g \right\}$$

be the complete general semilinear algebra of special quasi dual like pair of numbers over the semifield $Z^+ \cup \{0\}$.

Now interested reader can study the properties like subspaces, linear (semilinear operator) operator, transformation, direct sum, pseudo direct sum and linear functionals both in case of general vector spaces of special quasi dual numbers and general complete semivector space of special dual like number pairs respectively.

Next we proceed onto give examples of t-dimensional semivector spaces / vector spaces of special quasi dual complex modulo numbers.

Example 3.44: Let S = $\{a_1 + a_2g + a_3g_1 + a_4h + a_5h_1 \mid a_i \in Q^+ \cup \{0\}, 1 \leq i \leq 5, g = 2 + 4i_F, g_1 = 8 + 6i_F, h = 7 + 6i_F$ and $h_1 = 3 + 4i_F \in C(Z_{10})$ with $gg_1 = g_1g = g, g^2 = g_1, g_1^2 = g_1, h_1^2 = h_1, h_2 = h_1, hh_1 = h_1h = h$ and $gh_1 = h, gh = 0 \ g, h_1 = h_1gk = 0 \ gk_1 = 0 \ g_1k = 0, g_1k_1 = 0\}$ be the general quasi dual numbers.

Example 3.45: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \ a_i = x_i + y_ig + z_ik, \right.$$

$$x_i, y_i, z_i \in Q, 1 \leq i \leq 12, g = 7 + 6i_F, \text{ and}$$

$$\left. h = 2 + 4i_F \in C(Z_{10}); gh = 0, g^2 = -g \text{ and } h^2 = -h \right\}$$

be the 3-dimensional general ring of quasi special dual numbers under the natural product $\times_n$ of matrices. Clearly S is a commutative ring with zero divisors, units and idempotents.

Example 3.46: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \ a_i = x_i + y_ig + z_ih \text{ with } x_i, y_i, z_i \in Z, \right.$$

$1 \leq i \leq 6$, $g = 8 + 2i_F$, and $h = 8 + 15i_F \in C(Z_{17})$ with $gh = 0\}$

be the general ring of special quasi dual numbers of complex modulo integers of dimension three under the natural product $\times_n$.

Clearly P has ideals, subrings zero divisors and idempotents.

Example 3.47: Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i = x_i + y_ig + z_ih\right.$ where $x_i, y_i, z_i \in Z_{12}$, $1 \leq i \leq 6$, $g = 2 + 4i_F$, and $h = 8 + 6i_F \in C(Z_{10})\}$ be the general quasi special dual number Smarandache ring of dimension three.

Clearly M is of finite order and is a commutative ring with $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ as the unit element.

Example 3.48: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \cdots & a_{100} \end{bmatrix} \middle| \; a_i = x_i + y_ig + z_ih \text{ where}\right.$$

$g = 8 + 2i_F$ and $h = 8 + 15i_F \in C(Z_{17})$; $x_i \in Z_5$, $1 \leq i \leq 5\}$

be the finite general special quasi dual number ring of modulo integers of dimension three.

Now having seen examples of complex modulo integer quasi special dual numbers. We now proceed onto describe mixed quasi special dual numbers, mixed quasi special dual like numbers and finally strongly mixed dual number and illustrate them with examples.

We have already defined mixed dual numbers $x = a + bg + cg_1$ where a, b, c are reals and g and g_1 are new elements such that $g^2 = 0$, $g_1^2 = g_1$ with $gg_1 = g_1g = 0$ or g_1 or g.

DEFINITION 3.1: *Let $x = a + bg + cg_1$ where a, b and c are reals and g and g_1 are new elements such that g is a dual number component that is $g^2 = 0$ and $g_1^2 = -g_1$ is a special quasi dual number component. We define x as a mixed special quasi dual number.*

We will first illustrate this situation and see where from we can generate such numbers.

Example 3.49: Consider $x = a + bg + cg_1$ where $g = 6$ and $g_1 = 8$ in Z_{12} we see $g^2 = 0 \pmod{12}$ and $g_1^2 = 4 = -8 \pmod{12}$; with a, b, c $\in$ R (or Q or Z) is a mixed special quasi dual number. Clearly $gg_1 = 0$. Let $x = 7 + 3g + 2g_1$ and $y = -3 - 4g + 8g_1$ to find x + y and x × y.

$x + y = 4 - g + 10g_1$ is again mixed special quasi dual number.

$$\begin{aligned} x \times y &= (7 + 3g + 2g_1) \times (-3 - 4g + 8g_1) \\ &= -21 - 9g - 6g_1 - 28g - 12g_2 - 8gg_1 + 56g_1 + \\ &\qquad\qquad 24gg_1 + 16\, g_1^2 \\ &= -21 - 9g - 6g_1 - 28g - 0 - 0 + 56g_1 + 0 + (-16g_1) \\ &= -21 - 34g + 34g_1 \end{aligned}$$
is again a mixed special quasi dual number.

Consider $x = 8 + 3g + 7g_1$ and $y = -8 + g + g_1$ two mixed special quasi dual numbers.

$x + y = 4g + 8g_1$ is a mixed special quasi dual number with $a = 0$.

Consider $x = 3 - 5g + 2g_1$ and $y = 15 + 5g + 8g_1$ two mixed quasi special dual numbers $x + y = 18 + 10g_1$; x + y is not a mixed special quasi dual number infact only a special quasi dual number.

Let $p = 8 + 5g - 18g_1$ and $q = 7 + 2g + 18g_1$ be two mixed special quasi dual numbers.

$p + q = 15 + 7g$, that is $p + q$ is only a dual number. Finally let $m = 3 - 3g + 4g_1$ and $n = 8 + 2g - 4g_1$ two mixed special quasi dual numbers.

$m + n = 11$; that is $m + n$ is just a real number.

Now we have seen the definition and description of mixed special quasi dual numbers.

We proceed on to give some examples of them.

Example 3.50: Let $S = \{a + bg + cg_1 \mid a, b, c \in Q, g = 6$ and $g_1 = 3$ in Z_{12}. Clearly $g^2 = 0$, $g_1^2 = 9 = -g_1 \in Z_{12}$, $gg_1 = g_1g = 6 \pmod{12} = g \pmod{12}\}$ be the mixed special quasi dual numbers collection. Clearly S is a group under addition and semigroup under multiplication. Infact S is a ring defined as the general ring of mixed special quasi dual numbers. S is commutative ring with units has zero divisors and units.

Example 3.51: Let $P = \{a + bg + cg_1 \mid a, b, c \in Z, g = 20$ and $g_1 = 15 \in Z_{40}$, $g^2 = 0 \pmod{40}$, $g_1^2 = -g_1 \pmod{40}$, $gg_1 = g_1g = 0 \pmod{40}\}$ be the general ring of mixed quasi special dual numbers. P is a commutative ring with unit and with zero divisors. However only -1 is the invertible for $(-1)^2 = 1$; thus -1 is a self inversed element of P.

Example 3.52: Let $S = \{a + bg + cg_1 \mid a, b, c \in Z_{17}$; $g = 3$ and $g_1 = 6 \in Z_{12}$; $g^2 = +9 = -g \pmod{12}$, $g_1^2 = 6^2 = 0 \pmod{12}$, $6 \times 3 = 6 \pmod{12}\}$ be the general ring of mixed special quasi dual numbers. Clearly S is of finite cardinality and S is a characteristic 17.

Example 3.53: Let M = $\{a + bg + cg_1 \mid g = 6 \in Z_{12}$ and $g_1 = 8 \in Z_{12}$, $g^2 = 6 \pmod{12}$, $g_1^2 = -g \pmod{12}$, $g_1g = gg_1 = 0 \pmod{12}$, a, b, c $\in Z_{10}\}$ be the general ring of mixed special quasi dual numbers of finite order. M is of characteristic 10 and M has units zero divisors and idempotents.

Example 3.54: Let M = $\{a + bg + cg_1 \mid a, b, c \in R; g = 20, g^2 \equiv 0 \pmod{40}$ $g_1 = 24$, $g_1^2 = -g_1 \pmod{40}$ $gg_1 = 0 \pmod{40}\}$ be the general ring of mixed special quasi dual numbers. M is of infinite order. M has zero divisors and units.

Now let $x = a + bg + cg_1$ where a, b, c $\in R^+ \cup \{0\}$, where g and g_1 are now elements such that $g^2 = 0$ and $g_1^2 = -g_1$ with $gg_1 = g_1g$ = (g or 0 or g_1). We make the following observations.

(i) If we take the collection of all mixed special quasi dual numbers with the coefficient from $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ we see that collection is only a semigroup under '+' however the collection is not closed under product.

For let $x = 3 + 2g + 5g_1$ and $y = 2 + 5g + 4g_1$ be two elements of S = $\{a + bg + cg_1 \mid a, b, c \in Z^+ \cup \{0\}, g^2 = 0, g_1^2 = -g_1, g_1g = gg_1 = 0, g = 20, g_1 = 24 \in Z_{40}\}$. $x \times y = (3 + 2g + 5g_1) \times (2 + 5g + 4g_1)$

$$= 6 + 4g + 10g_1 + 15g + 10g_2 + 25gg_1 + 12g_1 + 20gg_1 + 20\ g_1^2$$

$$= 6 + 4g + 10g_1 + 15g + 0 + 0 + 12g_1 + 0 + 20 \times -g_1$$
$$= 6 + 19g + 22g_1 - 20g_1 \notin S \text{ as if } n \in M, -n \notin M$$
$$(n \in Z^+ \cup \{0\}).$$

Thus the set M is not closed under product. How to overcome this difficulty?

Before we over come this problem it is important to make the following observation.

Suppose $x = a + bg + cg_1$ is a mixed quasi special dual number then we see it is essential x is of dimension three, so a mixed special quasi dual number has its dimension to be three.

Now consider $P = \{a + bg + cg_1 + dg_2 \mid a, b, c, d \in Q^+ \cup \{0\}$ (or $Z^+ \cup \{0\}$ or $R^+ \cup \{0\}$) with $g^2 = 0$, $g_1^2 = g_2$; $g_1g_2 = g_2g_1 = g_1$ and $gg_1 = g_1g = g$ (or g_1 or g_2) $g_2g = gg_2$ g (or g_1 or g_2)$\}$. We call P be the collection of complete mixed quasi special dual number. Clearly a complete quasi special dual number has least dimension four if entries (coefficients) are taken from $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ otherwise the term complete is not essential and the dimension is only three.

We now can give algebraic structure to P. (P, $\times$) is a semigroup and (P, +) is also a semigroup. Thus (P, +, $\times$) is a semiring need not be a semifield.

We will first illustrate this situation by some simple examples.

Example 3.55: Let $S = \{a + bg + ch + dh_1 \mid a, b, c, d \in Z^+ \cup \{0\}$, $g = 20$, $h = 15$ and $h_1 = 25 \in Z_{40}$ with $g^2 = 0 \pmod{40}$, $h^2 = 25 = h_1 \pmod{40}$ and $h_1^2 = h_1 \pmod{40}$, $gh = hg = 0 \pmod{40}$, $gh = h_1g = 0 \pmod{40}$, $hh_1 = h = h_1h \pmod{40}\}$ be the general semiring of mixed special quasi dual like numbers.

We see how operations on S are performed. Let $x = 3 + 2g + 5h + 8h_1$ and $y = 2 + 5g + 10h + h_1$ be in S. To find x + y and x $\times$ y.

$$x + y = 5 + 7g + 15h + 9h_1 \in S$$

$$\begin{aligned} x \times y &= (3 + 2g + 5h + 8h_1) \times (2 + 5g + 10h + h_1) \\ &= 6 + 4g + 10h + 16h_1 + 15h + 10g^2 + 25hg + \\ &\quad 40gh1 + 30h + 20gh + 50h^2 + 80hh_1 + 3h_1 + \\ &\quad 2gh_1 + 5hh_1 + 8\ h_1^2 \end{aligned}$$

$$= 6 + 4g + 10h + 16h_1 + 15h + 0 + 0 + 0 + 30h + 0 + 50h_1 + 80h + 3h_1 + 0 + 5h + 8h_1$$
$$= 6 + 4g + 140h + 77h_1 \in S.$$

Thus (S, +, ×) is a semigroup S is not a semifield for S has zero divisors.

Example 3.56: Let $S = \{a + bg + ch + dh_1 \mid a, b, c, d \in Z^+ \cup \{0\}, g = 6, h = 8 \in Z_{12}, g^2 = 0 \pmod{12}, 8^2 = h^2 = h_1 \pmod{12}; gh = hg = 0 \pmod{12}, gh_1 = h_1g \equiv 0 \pmod{12}$ and $hh_1 = h = h_1h_1 \pmod{12}\}$ be the complete general dual like numbers. S is not a semifield. Dimension of S is four.

Example 3.57: Let $S = \{a + bg + ch + dh_1 \mid a, b, c, d \in R^+ \cup \{0\}, g = 56, h = 3, h_1 = 9 \in Z_{12}, g^2 = 0 \pmod{12}, h_2 = h_1, h_1^2 = h_1; gh = hg = g, gh_1 = h_1g = g\}$ be the general semiring of complete special quasi dual numbers of dimension four.

Consider $x = a + bg + ch + dh_1 + ek + fk_1$ where g = 6, h = 3, $h_1 = 9$, k = 8 and $k_1 = 4 \in Z_{12}$. The table for P = {0, 3, 9, 8, 4} = $\{0, g, h, h_1, k, k_1\} \subseteq Z_{12}$ is as follows:

×	0	3	9	8	4	6
0	0	0	0	0	0	0
3	0	9	3	0	0	6
9	0	3	9	0	0	6
8	0	0	0	4	8	0
4	0	0	0	8	4	0
6	0	6	6	0	0	0

Now $x = 3+ 2g + h + 5h_1 + 3k + 2k_1$ and $y = 2 + 7g + 2h + h_1 + k + 5k_1$ be two mixed complete quasi special dual numbers of dimension six.

Clearly $x + y = 5 + 12g + 3h + 6h_1 + 4k + 7k_1$.

$$\begin{aligned} x \times y &= (3+2g+h+5h_1+3k+2k_1) \times (2+7g+2h+h_1+k+5k_1) \\ &= 6 + 4g + 2h + 10h_1 + 6k + 4k_1 + 21g + 14g^2 + \\ &\quad 7gh + 35h_1g + 21kg + 14k_1g + 6h + 4gh + 2h^2 + \\ &\quad 10hh_1 + 6kh + 4k_1h + 3k + 2kg + hk + 5h_1k + \\ &\quad 3k^2 + 2k_1k + 15k_1 + 10gk_1 + 5k_1h + 25h_1k_1 + \\ &\quad 15kk_1 + 10\ k_1^2 + 3h_1 + 2h_1g + h_1h + 5\ h_1^2 + \\ &\quad 3kh_1 + 2k_1h_1 \\ \\ &= 6 + 4g + 2h + 10h_1 + 6k + 4k_1 + 21g + 0 + \\ &\quad 2h + 10h_1 + 6k + 4k_1 + 21g + 0 + 7g + 35g + \\ &\quad 0 + 0 + 6h + 4g + 2h_1 + 10h + 6k + 0 + 3k + 0 + \\ &\quad 0 + 0 + 3k_1 + 2k + 15k_1 + 2h_1 + k_1 + 5h_1 + 0 + 0 \\ \\ &= 6 + 71g + 18h + 12h_1 + 26k + 29k_1 \end{aligned}$$

is again a five dimensional complete mixed quasi special dual number.

We will present one or two examples of mixed quasi special dual numbers of higher order.

Example 3.58: Let S = $\{a + bg + ch + dh_1 + ek + fk_1 \mid a, b, c, d, e, f \in Q^+ \cup \{0\}, g = 20, h_1 = 25, h = 15, k = 24$ and $k_1 = 16 \in Z_{40}\}$ be the 6-dimensional complete mixed dual quasi special number general semiring.

The product table for P = $\{0, 20, 15, 16, 24, 25\} \subseteq Z_{40}$ is as follows:

×	0	15	16	24	25	20
0	0	0	0	0	0	0
15	0	25	0	0	15	20
16	0	0	16	24	0	0
24	0	0	24	16	0	0
25	0	15	0	0	25	20
20	0	20	0	0	0	0

Using this table interested reader can find the product of any two elements in S.

Now we proceed onto give one or two examples of higher dimensional rings.

Example 3.59: Let
$M = \{a + bg + ch + dk \mid a, b, c, d, k \in Z, g = 6, h = 8, k=3 \in Z_{12}\}$ be the general ring of special mixed quasi dual numbers of dimension / order four.

One of the natural question would be can we have higher than four dimensional special quasi mixed dual numbers.

The answer is 'yes'.

We illustrate this situation by some examples.

Example 3.60: Let $S = \{a + bg + ch + dk + em + fn \mid a, b, c, d, e, f \in Z, g = (6, 6, 0), n = (0, 0, 6), h = (8, 0, 8), k = (3, 3, 3), m = (0, 8, 0); 3, 8, 6 \in Z_{12}$ with $g^2 = (0, 0, 0), n^2 = (0, 0, 0)$ $gn = ng = (0, 0, 0), \; m^2 = (0, 4, 0) = –m; \; h^2 = (4, 0, 4) = –h, hm = mh = 0, k^2 = (9, 9, 9) = –k$ and so on$\}$ be the 6-dimensional general ring of mixed special quasi dual numbers.

Example 3.61: Let $S = \{a + bg + ch + dm + en + fs + pr + qt + vw \mid a, b, c, d, e, f, p, q, v \in Q; g = (20, 20, 20, 20, 20), h = (0, 20, 0, 20, 20), m = (15, 15, 0, 0, 0), n = (0, 0, 15, 15, 15), s = (0, 15, 0, 15, 0), r = (0, 16, 0, 16, 0), t = (16, 0, 16, 0, 16)$ and with $20, 24, 15, 16, 25 \in Z_{40}\}$ be the 9 dimensional mixed dual quasi number general ring.

It is pertinent to mention here that in S if we replace Q by $Q^+ \cup \{0\}$ clearly S is not closed under $\times$.

It is left as an exercise to the reader to construct semiring using row vectors which contribute to mixed special dual quasi semirings.

Example 3.62: Let

$$P = \{a + bg + cd + eh + fq + sr + mn + ut + vw + xy \mid a, b, c, e, f, s, m, u, v, x \in R$$

$$\text{with } g = \begin{bmatrix}3\\3\\3\\0\\0\\0\\3\\0\end{bmatrix}, h = \begin{bmatrix}0\\0\\0\\3\\3\\3\\0\\3\end{bmatrix}, d = \begin{bmatrix}3\\3\\3\\3\\3\\3\\3\\3\end{bmatrix}, q = \begin{bmatrix}6\\6\\0\\0\\6\\6\\0\\0\end{bmatrix}, r = \begin{bmatrix}0\\0\\6\\6\\0\\0\\6\\6\end{bmatrix}, m = \begin{bmatrix}8\\8\\8\\8\\8\\8\\8\\8\end{bmatrix},$$

$$t = \begin{bmatrix}8\\8\\8\\8\\0\\0\\0\\0\end{bmatrix}, w = \begin{bmatrix}0\\0\\0\\0\\8\\8\\8\\8\end{bmatrix}, y = \begin{bmatrix}6\\6\\6\\6\\6\\6\\6\\6\end{bmatrix} \text{ where } 3, 6, 8 \in Z_{12}\}$$

be the general 10 dimensional general commutative ring of mixed dual quasi special numbers.

Clearly P is also a Smarandache ring. We use the natural product $\times_n$ on P.

$$\text{Further } g \times_n d = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 0 \\ 0 \\ 0 \\ 9 \\ 0 \end{bmatrix} = - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = -g \text{ and so on.}$$

Example 3.63: Let

$$P = \{a + bx + cy + dm + en + gq + hp + kr + fs \mid a, b, c, d, e, f, g, h, k \in Q;$$

$$x = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 3 & 3 & 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}, m = \begin{bmatrix} 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix},$$

$$n = \begin{bmatrix} 6 & 6 & 0 & 0 \\ 0 & 0 & 6 & 6 \\ 6 & 6 & 0 & 0 \end{bmatrix}, q = \begin{bmatrix} 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 6 \end{bmatrix}, p = \begin{bmatrix} 8 & 8 & 0 & 8 \\ 0 & 0 & 8 & 0 \\ 8 & 8 & 0 & 0 \end{bmatrix},$$

$$r = \begin{bmatrix} 0 & 0 & 8 & 0 \\ 8 & 8 & 0 & 8 \\ 0 & 0 & 8 & 8 \end{bmatrix}, s = \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} \text{ where } 3, 6, 8 \in Z_{12}\}$$

be the nine dimensional general commutative ring of mixed special quasi dual numbers under the natural product $\times_n$.

$$p \times_n s = \begin{bmatrix} 4 & 4 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 4 & 4 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} 8 & 8 & 0 & 8 \\ 0 & 0 & 8 & 0 \\ 8 & 8 & 0 & 0 \end{bmatrix} = -p.$$

$$p \times_n r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and so on.}$$

Example 3.64: Let

$S = \{x_1 + x_2a + x_3b + x_4c + x_5d + x_6e + x_7f + x_8g \mid x_i \in R;$

$$1 \leq i \leq 8,\ a = \begin{bmatrix} 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \end{bmatrix},\ b = \begin{bmatrix} 20 & 0 & 0 & 20 \\ 0 & 20 & 20 & 0 \\ 20 & 0 & 20 & 0 \\ 0 & 20 & 0 & 20 \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & 20 & 20 & 0 \\ 20 & 0 & 0 & 20 \\ 0 & 20 & 0 & 20 \\ 20 & 0 & 20 & 0 \end{bmatrix},\ d = \begin{bmatrix} 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \end{bmatrix},$$

$$e = \begin{bmatrix} 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \end{bmatrix},\ f = \begin{bmatrix} 15 & 0 & 15 & 0 \\ 0 & 15 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 15 & 0 & 15 & 0 \end{bmatrix},$$

$$g = \begin{bmatrix} 0 & 15 & 0 & 15 \\ 15 & 0 & 15 & 0 \\ 15 & 15 & 15 & 15 \\ 0 & 15 & 0 & 15 \end{bmatrix},\ 20, 15, 16 \in Z_{40}\}$$

be a general ring of mixed quasi special dual numbers of dimension eight.

Now we can get any desired dimensional mixed special quasi dual number rings.

Under the assumption if g and h are two distinct components of a mixed special quasi dual number than we just write g + h as g + h and $\underbrace{g+g+\ldots+g}_{n-\text{times}} = ng$ and so on.

Let P = {a + bg + cd + ef + ph | a, b, c, e, p ∈ Q, $g^2 = 0$, $d^2 = -d$, $f^2 = -f$ and $h^2 = 0$, gh = 0 df = 0, gd = d, gf = f} be the collection of five dimensional mixed quasi dual numbers. Then P is an abelian group under addition and (P, ×) is a commutative semigroup.

Infact (P, +, ×) is a ring which is commutative, P is a Smarandache ring. So using such P we can construct mixed quasi dual number vector spaces.

We will illustrate this situation by some examples.

Example 3.65: Let M = {$(a_1, a_2, \ldots, a_9)$ | $a_i = x_1 + x_2g + x_3k + x_4k_1 + x_5h + x_6h_1$ where $x_j \in Q$, $1 \leq i \leq 9$; $1 \leq j \leq 6$ g = 6, k = 3, $k_1 = 9$, h = 8 and $h_1 = 4 \in Z_{12}$} be a general mixed special quasi dual vector space of M over the field Q. Clearly M is also a general mixed special quasi dual linear algebra over the field Q.

Example 3.66: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i = x_1 + x_2g + x_3p + x_4p_1 + x_5h + x_6h_1 + \right.$$

$x_7m_1 + x_8m + x_9t + x_{10}q$, $x_j \in Q$; $1 \le i \le 12$ and $1 \le j \le 10$, where $g = (6, 6, 6, 6)$, $t = (6, 6, 0, 0)$, $q = (0, 0, 6, 6)$, $p = (3, 3, 0, 0)$, $p_1 = (0, 0, 3, 3)$, $h = (8, 8, 8, 8)$, $h_1 = (8, 8, 0, 0)$, $m_1 = (0, 0, 8, 8)$, $m = (3, 3, 3, 3)$, $6, 3, 8 \in Z_{12}\}$ be the general group under '+' of mixed special quasi dual numbers of dimension 11 over the field Q.

Infact P is a general linear algebra of mixed special quasi dual numbers over the field under the natural product $\times_n$.

Example 3.67: Let

$$S = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{array} \right] \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + \ldots + x_{10}g_9 \right.$$

with $1 \le i \le 30$, $x_j \in Z_{19}$, $1 \le j \le 10$ where $g_1 = (20, 20, 20, 20, 20, 20)$, $g_2 = (0, 0, 0, 0, 20, 20)$, $g_3 = (20, 20, 20, 20, 0, 0)$, $g_4 = (15, 15, 15, 15, 15, 15)$, $g_5 = (15, 15, 15, 15, 0, 0)$, $g_6 = (0, 0, 0, 0, 15, 15)$, $g_7 = (25, 25, 25, 25, 25, 25)$, $g_8 = (25, 25, 25, 25, 0, 0)$, $g_9 = (0, 0, 0, 0, 25, 25)$ with $20, 15, 25 \in Z_{40}\}$ be the general vector space of special quasi dual number of dimension ten over the field Z_{19}. Clearly S is of finite order. Under usual product $\times_n$; S is a general linear algebra of mixed special quasi dual numbers over the field Z_{19}.

Example 3.68: Let

$$M = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array} \right] \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5; \right.$$

$1 \le i \le 9$, $x_j \in Z_5$, $1 \le j \le 6$, $g_1 = 3$, $g_2 = 9$, $g_3 = 8$, $g_4 = 4$ and $g_5 = 6 \in Z_{12}\}$ be the general vector space of mixed special quasi dual numbers of dimension six over the field Z_5.

Now we proceed onto give examples of semivector space of mixed special quasi dual numbers over a semifield.

Example 3.69: Let P = $\{(a_1, a_2, \ldots, a_7) \mid a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_3 + x_6g_5, 1 \leq i \leq 7, x_j \in R^+ \cup \{0\}; 1 \leq j \leq 5, g_1 = 6, g_2 = 8, g_4 = 12, g_3 = 3$ and $g_5 = 9 \in Z_{12}\}$ be a general semivector space of mixed semivector space of mixed special quasi dual number over the semifield $R^+ \cup \{0\}$.

Example 3.70: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + \ldots + x_{10}g_9, 1 \leq i \leq 12; \right.$$

$x_j \in Z^+ \cup \{0\}; 1 \leq j \leq 10, g_1 = (20, 20, 20), g_2 = (20, 0, 0), g_3 = (0, 20, 20), g_4 = (15, 15, 15), g_5 = (0, 15, 15), g_6 = (15, 0, 0), g_7 = (25, 25, 25), g_8 = (25, 0, 0)$ and $g_9 = (0, 25, 25)$ with $20, 15, 25 \in Z_{40}\}$ be a general semivector space of mixed special quasi number over the semifield $Z^+ \cup \{0\}$.

Example 3.71: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + \right.$$

$x_5g_4 + x_6g_5; 1 \leq i \leq 16, g_1 = 6, g_2 = 4, g_3 = 8, g_4 = 3$ and $g_5 = 9 \in Z_{12}; x_j \in Q^+ \cup \{0\}, 1 \leq j \leq 6\}$ be a general semivector space of mixed special quasi dual numbers over the semifield $Z^+ \cup \{0\}$. $(T, \times_n)$ is a semilinear algebra over the semifield $Z^+ \cup \{0\}$.

Example 3.72: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2g_1 + x_3g_2 + g_4g_3 + g_5g_4 + g_6g_5,\ g_1 = 20, \right.$$

$g_2 = 15$, $g_3 = 25$, $g_4 = 16$ and $g_5 = 24 \in Z_{40}$; $x_j \in R^+ \cup \{0\}$, $1 \leq j \leq 6\}$ be a general semivector space of mixed special quasi dual numbers over the semifield $R^+ \cup \{0\}$.

Example 3.73: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_6 & a_7 & a_8 \end{bmatrix} \,\middle|\, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5; \right.$$

$x_j \in Q^+ \cup \{0\}$; $1 \leq j \leq 6$; $1 \leq i \leq 9$, $g_1 = 6$, $g_2 = 4$, $g_3 = 8$, $g_4 = 3$ and $g_5 = 9 \in Z_{12}\}$ be general semivector space of mixed special quasi dual numbers over the semifield $Q^+ \cup \{0\}$.

Clearly under the usual product '$\times$'; P is a non commutative semilinear algebra and under the natural product $\times_n$, P is a semilinear algebra over the semifield $Z^+ \cup \{0\}$.

All properties associated with vector spaces and semivector spaces can be without any difficulty derived in the case of general vector space / semivector space of mixed special quasi dual numbers. This task is left as an exercise to the reader.

Now we proceed onto describe the new notion mixed special quasi dual like numbers.

DEFINITION 3.2: *Let $x = a + bg + cg_1$ where a, b, c $\in$ R or Q or Z or C and g and g_1 are new elements such that $g^2 = g$ and $g_1^2 = -g_1$ with $gg_1 = g_1g = g$ or g_1. We define x to be a mixed special quasi dual like number.*

We will illustrate this situation by some examples.

Example 3.74: Let $x = a + bg + cg_1$ where $a, b, c \in R$, $g = 9$ and $g_1 = 8 \in Z_{12}$. Clearly x is a mixed quasi special dual like number. Further a mixed special quasi like number is of dimension three, that is the least dimension possible is three.

Let $x = 5 + 3g + 8g_1$ and $y = -8 - 5g + 2g_1$ be any two mixed special quasi dual like numbers.

$$x + y = -3 - 2g + 10g_1 \text{ and}$$
$$\begin{aligned} x \times y &= (5 + 3g + 8g_1) \times (-8 -5g + 2g_1) \\ &= -40 - 24g - 64g_1 - 25g - 15g^2 - 40gg_1 + \\ &\quad 10g_1 + 6gg_1 + 16\ g_1^2 . \end{aligned}$$

Using $g^2 = 9^2 = 9 \pmod{12}$, $g_1^2 = 64 = -g_1 \pmod{12}$

$gg_1 = gg_1 = 8 \times 9 = 0 \pmod{12}$

$$\begin{aligned} x \times y &= -40 - 24g - 64g_1 - 25g - 15g - 0 + 10g_1 + \\ &\quad 0 + 16\,(-g_1) \\ &= -40 - 64g - 70g_1 \end{aligned}$$

is again a mixed special quasi dual like number.

Let $p = 8 + 5g + 3g_1$ and $q = 3 - 5g + g_1$ be any two mixed special quasi dual like numbers.

$p + q = 11 + 4g_1$. Clearly p + q is only a special quasi dual number and is not a mixed special quasi dual like number.

Consider $a = 4 + 8g - 3g_1$ and $b = -3 + g + 3g_1$ be any two mixed special quasi dual like numbers.

Clearly $a + b = 1 + 9g$ and a + b is only a special dual like number and not a mixed special dual like number.

Finally let $m = 3 - g + 5g_1$ and $n = 8 + g - 5g_1$ be mixed special quasi dual like number. $m + n = 11$ is only a real number and is not a mixed special quasi dual like number. Thus sum of two mixed special quasi dual like numbers can be a real number or a special quasi dual number or a special dual like number. We accept $a + bg + cg_1$ with $a = 0$ to be also a mixed special quasi dual like number.

Example 3.75: Let $M = \{a + bg + cg_1 \mid a, b, c \in Q, g = 15$ and $g_1 = 16 \in Z_{40}$ where $g^2 = -g \pmod{40}$, $g_1^2 = g_1 \pmod{40}$, $g \times g_1 = 0 \pmod{40}\}$ be the collection of all mixed special quasi dual like numbers. $(M, +)$ is a group. $(M, \times)$ is a commutative semigroup.

Example 3.76: Let $S = \{a + bg + cg_1 \mid a, b, c \in Z_{20}$; $g = 3$ and $g_1 = 4$, $g^2 = -g$ and $g_1^2 = g_1 \in Z_{12}\}$ be the semigroup under $\times$ and group under addition $+$.

Clearly $(S, +, \times)$ is a ring of finite order, commutative; has units and zero divisors.

Example 3.77: Let $M = \{a + bg + cg_1 \mid g = 9$ and $g_1 = 8 \in Z_{12}$, $a, b, c \in Z$, $g^2 = g$ and $g_1g = 0$, $g_1^2 = -g_1\}$ be the ring of mixed special quasi dual like numbers.

Example 3.78: Let $P = \{(a_1, a_2, \ldots, a_{25}) \mid a_i = x_1 + x_2g + x_3g_1$ where $1 \leq i \leq 25$, $x_j \in Q$, $1 \leq j \leq 3$, $g = 15$ and $g_1 = 16 \in Z_{40}$, $g^2 = -g$, $g_1^2 = g_1$, $gg_1 = g_1g = 0\}$ be the mixed special quasi dual like number ring of infinite order. This ring has zero divisors, ideals and subrings which are not ideals.

Example 3.79: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{61} & a_{62} & a_{63} & a_{64} \end{bmatrix} \right| a_i = x_1 + x_2g + x_3g_1, 1 \leq i \leq 64,$$

$x_1, x_2, x_3 \in Z_5$ and $g = 7$ and $g_1 = 6 \in Z_{21}$ we see $g^2 = g \pmod{21}$; $g_1^2 = 15 = -g_1 \pmod{21}$ and $gg_1 = g_1g = 0 \pmod{21}\}$ be the finite ring of mixed special quasi dual like number under the natural product $\times_n$.

Example 3.80: Let

$$W = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{pmatrix} \middle|\; a_i = x_1 + x_2g + x_3g_1,\ 1 \leq i \leq 14, \right.$$

$x_1, x_2, x_3 \in Z$ and $g = 6$ and $g = 5$, $g^2 = g \pmod{15}$, $g_1^2 = 10 = -5 \pmod{15}$ and $gg_1 = g_1g = 0 \pmod{15}\}$ be the finite ring of mixed special quasi dual like numbers under the natural product $\times_n$ of infinite order.

Now we just indicate which of the modulo integer rings that pave way to the construction of mixed special quasi dual like numbers.

Consider $Z_6 = \{0, 1, 2, 3, 4, 5\}$ the ring of integers modulo 6.

Take $g = 3$ and $g_1 = 2$ we see $g^2 = g \pmod 6$ and $g_1^2 = 4 = -g_1 \pmod 6$ with $gg_1 = g_1g = 0 \pmod 6$.

Thus $x = a + bg + cg_1$ is a mixed special quasi dual like number.

Consider $g = 5$ and $g_1 = 4$ in Z_{10} we see $g^2 = g \pmod{10}$ and $g_1^2 = 4^2 = 6 \pmod{10} = -g_1 \pmod{10}$.

Further $gg_1 = g_1g = 0 \pmod{10}$.

So $x = a + bg + cg_1$ is a mixed special quasi dual like number.

Consider $Z_{12} = \{0, 1, 2, \ldots, 11\}$ the ring of modulo integers 12.

$g_1 = 3$ such that $g_1^2 = -g_1 \pmod{12}$, $g = 4$ in Z_{12} is such that $g^2 = g \pmod{12}$ and $g_1g = gg_1 = 0 \pmod{12}$.
So $x = a + cg + bg_1$ is a mixed quasi special dual like number.

$g = 8$ and $g_1 = 9$ in Z_{12} are such that $g^2 = -g \pmod{12}$ and $g_1^2 = g \pmod{12}$, $gg_1 = g_1g = 0 \pmod{12}$.

We have two sets of mixed quasi special dual like number components in Z_{12}. So $S = \{0, 3, 4, 8\} \subseteq Z_{12}$ is the semigroup under multiplication modulo 12 called the associated component semigroup of mixed special dual like numbers.

Consider $Z_{14} = \{0, 1, 2, \ldots, 13\}$, ring of modulo integers 14. We see $g = 7$ and $g_1 = 6$ in Z_{14} are such that $g^2 = g \pmod{14}$ and $g_1^2 = 7^2$; $w = 8 = -g_1$; $gg_1 = g_1g = 0 \pmod{14}$.
Thus $x = a + bg + cg_1$ is mixed special quasi dual like number.

We now consider $Z_{15} = \{0, 1, 2, \ldots, 14\}$, ring of modulo integers.

$g_1 = 5$, $g_1^2 = -g_1$, $g_2 = 6$, $g_2^2 = 6 = g_2$, $g_3 = 9$, $g_3^2 = -g_3 = 6$, $g_4 = 10$, $g_4^2 = g_4$ are new elements which contribute to mixed special quasi dual like numbers.

Consider $S = \{0, 5, 6, 9, 10\} \subseteq Z_{15}$, clearly S is not closed under addition modulo 15.

The table for S is as follows:

×	0	5	6	9	10
0	0	0	0	0	0
5	0	10	0	0	5
6	0	0	6	9	0
9	0	0	9	6	0
10	0	5	0	0	10

Thus $x = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$ is a five dimensional mixed quasi special dual like number. Here $x_i \in Q$ or R or Z or Z_n; $1 \leq i \leq 5$.

Let $Z_{18} = \{0, 1, 2, \ldots, 17\}$ be the ring of integers modulo 18. Consider $g_1 = 8$, $g_1^2 = -g_1 = 10$, $g^2 = 10$ and $g_2^2 = 10$. Thus Z_{18} does not contribute to mixed quasi special dual like number. It gives only a quasi special dual number.

Consider $Z_{20} = \{0, 1, 2, \ldots, 19\}$, the ring of integers modulo 20. $g_1 = 4$, $g_1^2 = g_1$, $g_2 = 5^2 = 5 \pmod{20}$; $g_3 = 15$, $g_3^2 = -5 \pmod{20}$, $g_4 = 16$, $g_4^2 = 16$.

We see Z_{20} has a mixed special quasi dual like number component.

Take $x = x_1 + x_2g_1 + x_3g_3 + x_4g_4$; $g_1 = 4$, $g_3 = 15$ and $g_4 = 16$, x is a mixed special dual like number of dimension four.

One can work with any suitable Z_n and find the mixed special dual like numbers.

Also we see if we take $g_1 = 2$ is such that $g_1^2 = 4 = -g_1$ and $g_2 = 3$; $g_2^2 = 9 = g_2 \pmod{6}$, clearly $3.2 = g_1g_2 = 0 \pmod{6}$.

Take $h_1 = (3, 3, 3, 3, 3)$, $h_2 = (2, 2, 2, 2, 2)$, $h_3 = (3, 0, 3, 0, 3)$, $h_4 = (2, 0, 2, 0, 2)$, $h_5 = (0, 3, 0, 3, 0)$ and $h_6 = (0, 2, 0, 2, 0)$ are components of mixed special dual like numbers.

$x = x_1 + x_2h_1 + x_3h_2 + x_4h_3 + x_5h_4 + x_6h_5 + x_7h_6$; $x_i \in Q$; $1 \leq i \leq 7$, is a mixed special dual quasi like number of dimension seven. Thus we can get any desired dimensional mixed special dual like numbers.

Using these we can build all other algebraic structures as in case of usual dual numbers, special dual like numbers and special quasi dual numbers.

This task of studying algebraic structures such mixed special dual like numbers is left as an exercise to the reader.

Now we proceed onto define yet another mixed dual numbers as follows.

Suppose $x = x_1 + x_2g_1 + x_3g_2 + x_4g_3$ where $x_j \in R$; $1 \leq j \leq 4$. g_1 is such that $g_1^2 = 0$, $g_2^2 = g_2$ and $g_3^2 = -g$ with $g_ig_j = g_jg_i = 0$ or g_1 or g_2 or g_3, $1 \leq i, j \leq 3$.

Let us consider Z_{12}, $g_1 = 9$ with $g_1^2 = g_1 \pmod{12}$, $g_2 = 8$, $g_2^2 = -g_2 \pmod{12}$, $g_3 = 6$ and $g_3^2 = 0 \pmod{12}$.

Consider $x = x_1 + x_2g_1 + x_3g_2 + x_4g_3$; $x_j \in R$; $1 \leq j \leq 4$; we define x to be a strongly mixed special quasi dual like numbers.

We will illustrate them by examples.

Example 3.81: Let $x = x_1 + x_2g_1 + x_3g_2 + x_4g_3$ where $g_1 = 6$, $g_2 = 3$ and $g_3 = 4$ in Z_{12}, we see $g_1^2 = 0 \pmod{12}$, $g_2^2 = 9 = -g_2$ and $g_3^2 = g_3 \pmod{12}$ be the strongly mixed special dual quasi like number. The only generating algebraic structure of these strongly mixed special dual quasi like number components are Z_n, $(1 < n < \infty)$. Z_6 has no such component.

Z_p, p a prime has no such component.

Z_{12} is the first smallest n such that Z_{12} has mixed special quasi dual like component.

Consider Z_{20}, $g_1 = 4$, $g_2 = 5$ and $g_3 = 10$ in Z_{20} are such that $g_1^2 = 16 = -4 \pmod{20}$, $g_2^2 = g^2 \pmod{20}$ and $g_3^2 = 0 \pmod{20}$, $g_2g_3 = g_3 \pmod{20}$, $g_1g_3 = 0 \pmod{20}$ and $g_2g_1 = 0 \pmod{20}$.

Thus $x = a + bg_1 + cg_2 + dg_3$ (a, b, c, d $\in$ Z or Q or R) is a strong mixed special quasi dual like number. Z_{21} has no strong mixed special quasi dual like number component. Z_{22} has no strong mixed special quasi dual like number component.

Consider $Z_{24} = \{0, 1, 2, \ldots, 23\}$ be the ring of integers modulo 24. $g_1 = 8$, $g_1^2 = -16 = -g_1 \pmod{24}$, $g_2 = 9$, $g_2^2 = 9 \pmod{24}$, $g_3 = 12$, $g_3^2 = 0 \pmod{24}$, $g_4 = 15$, $g_4^2 = -g_4 \pmod{24}$, $g_5 = 16$ and $g_5^2 = g_5 \pmod{24}$.

$x = a + bg_1 + cg_2 + dg_3$ is a strong mixed special quasi dual like number.

$x = a + bg_2 + cg_4 + dg_3$ is a strong mixed special quasi dual like number. Thus Z_{24} has a component semigroup of strong mixed special quasi dual like numbers.

Consider $Z_{40} = \{0, 1, 2, \ldots, 39\}$ the ring of modulo integers.

$g_1 = 15$, $g_1^2 = -g_1$, $g_2 = 16$, $g_2^2 = g_2$, $g_3 = 20$, $g_3^2 = 0$, $g_4 = 24$, $g_4^2 = -g_4$, $g_5 = 25$ and $g_5^2 = g_5$.

Using $S = \{0, g_1, g_2, g_3, g_4, g_5\} \subseteq Z_{40}$ we can build strongly mixed special quasi dual like numbers. The table of S under $\times$ is as follows:

$\times$	0	15	16	20	24	25
0	0	0	0	0	0	0
15	0	25	0	20	0	15
16	0	0	16	0	24	0
20	0	20	0	0	0	20
24	0	0	24	0	16	0
25	0	15	0	20	0	25

Thus using Z_n (n a composite number) we can get a component semigroup of strongly mixed special quasi dual like numbers.

It is observed if $n = 2^m p$ where $m \geq 2$, p an odd prime we are sure to get a component semigroup. Working with lattices or neutrosophic number I alone cannot yield such elements. Also $(x_1, \ldots, x_n)$ with x_i in R or Q or C or Z do not contribute for the study of mixed special quasi dual like numbers.

But to get higher dimension of strong mixed special quasi dual like numbers we can use matrices with entries from the component semigroup of strong mixed special quasi dual like number associated with Z_n.

We will illustrate this situation by an example or two.

Example 3.82: Let Z_{12} be the ring of modulo integers. Take $g_1 = 4$, $g_1^2 = g_1$, $g_2 = 3$, $g_2^2 = 9 = -g^2$, $g_3 = 6$, $g_3^2 = 0$, $g_4 = 9$ and $g_4^2 = g_4$ in Z_{12}. $x = a + bg_1 + cg_2 + dg_3$ is a strong mixed quasi special dual like number of dimension four.

Take $h_1 = (4, 4, 4, 4, 4)$, $h_2 = (4, 4, 0, 4, 4)$, $h_3 = (0, 0, 4, 0, 0)$,
$h_4 = (3, 3, 3, 3, 3)$, $h_5 = (3, 3, 0, 3, 3)$, $h_6 = (0, 0, 3, 0, 0)$,
$h_7 = (6, 6, 6, 6, 6)$, $h_8 = (0, 0, 6, 0, 0)$ and $h_9 = (6, 6, 0, 6, 6)$.

Now $x = x_1 + x_2h_1 + \ldots + x_{10}h_9$ is a strong mixed special quasi dual like number of dimension ten.

Using these elements 4, 3 and 6 we can have column vectors say (like)

$$p_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ 4 \end{bmatrix}, p_5 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix},$$

$$p_6 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ p_7 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, p_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}, p_9 = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}, p_{10} = \begin{bmatrix} 6 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$p_{11} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 6 \\ 0 \\ 0 \end{bmatrix} \text{ and } p_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 6 \end{bmatrix}, p_i \times_n p_j = p_i \text{ or } p_j \text{ or } 0, 1 \leq i, j \leq 12.$$

Thus $x = x_1 + x_2 g_1 + \ldots + x_{13} g_{12}$ is a 13-dimensional strong special mixed quasi dual number where $x_k \in R$ or Q or Z or Z_t, $0 \leq t \leq \infty$.

Now having seen how column matrix is used to get strong mixed special quasi dual like number component we now proceed onto give some more ways of generating strong mixed special quasi dual like number component.

$$\text{Let } v_1 = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 \end{bmatrix},$$

$$v_{16} = \begin{bmatrix} 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix},$$

$$v_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, v_5 = \begin{bmatrix} 3 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 3 \end{bmatrix},$$

$$v_6 = \begin{bmatrix} 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 \end{bmatrix}, v_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$v_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, v_9 = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 \end{bmatrix},$$

$$v_{10} = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 & 6 \end{bmatrix}, v_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$v_{12} = \begin{bmatrix} 6 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 6 \end{bmatrix}, v_{13} = \begin{bmatrix} 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 & 0 \end{bmatrix},$$

$$v_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, v_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$v_{17} = \begin{bmatrix} 6 & 0 & 6 & 0 & 6 \\ 6 & 0 & 6 & 0 & 6 \\ 6 & 0 & 6 & 0 & 6 \end{bmatrix}, v_{18} = \begin{bmatrix} 0 & 6 & 0 & 6 & 0 \\ 0 & 6 & 0 & 6 & 0 \\ 0 & 6 & 0 & 6 & 0 \end{bmatrix},$$

$$v_{19} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}, v_{20} = \begin{bmatrix} 4 & 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 & 4 \end{bmatrix},$$

$$v_{21} = \begin{bmatrix} 0 & 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 & 0 \end{bmatrix}, v_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$v_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, v_{24} = \begin{bmatrix} 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \end{bmatrix},$$

$$v_{25} = \begin{bmatrix} 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 4 \end{bmatrix}, v_{26} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$v_{27} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}$$

be the components of a strong mixed special quasi dual like number with $v_i \times_n v_j = (0)$ or v_k; $(1 \leq i, j, k \leq 27)$.

$x = x_1 + x_2v_1 + x_3v_2 + \ldots + x_{28} v_{27}$ is a 28 dimensional strong mixed quasi special dual like number.

Thus using any appropriate Z_n we can build any desired dimemsion. We can also use the notion of square matrices with entries from the mixed special strong semigroup component of numbers to construct special strong mixed quasi dual number of any desired dimension.

Now we just indicate using these strong mixed special quasi dual number of any dimension build algebraic structures (both finite as well as infinite) like rings, vector spaces, linear algebras, S-rings, S-vector spaces and S-linear algebra. Interested reader can work on these structures and find nice applications and study their substructures. Further this work is considered as a matter of routine and hence is left as an exercise to the reader.

If on the other hand Z or Q or R or C or Z_n is replaced by $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ then we get other special algebraic structures like semiring, semivector spaces and semilinear algebras. Also the Smarandache analogoue of them can be worked out. This task is left as exercise to the reader.

Finally we describe modulo finite complex integer strong mixed special dual like numbers using

$C(Z_n) = \{a + bi_F \mid a, b \in Z_n, i_F^2 = 11\}$, $g_1 = 6 + 6i_F \in C(Z_{12})$ is such that $g_1^2 = 0$, $g_2 = 8$, $g_2^2 = -g_2$ and $g_3 = 9$, $g_3^2 = g_3$ in $C(Z_{12})$.

Thus $x = x_1 + x_2g_1 + x_3g_2 + x_4g_3$ is a strong mixed special dual like quasi number $g_1 \times g_3 = g_1 \pmod{12}$, $g_1 \times g_2 = 0 \pmod{12}$, $g_2 \times g_3 = 0 \pmod{12}$.

Consider $C(Z_{10}) = \{a + bi_F \mid a, b \in Z_{10}, i_F^2 = 9\}$.

Take $g_1 = (2 + 4i_F)$, $g_1^2 = 4 + 54 + 16i_F = 8 + 6i_F = -g_1$, $g^2 = 5 + 5i_F$, $g_2^2 = (5 + 5i_F)^2 = 25 + 25 \times 9 + 25 \times 2i_F = 0$. $g_3 = 3 + 4i_F$, $g_3^2 = (3 + 4i_F)^2 = 9 + 144 + 24i_F = 3 + 4i_F = g_3$.

Thus $x = x_1 + x_2g_1 + x_3g_2 + x_4g_3$ is a strong mixed special quasi dual number.

$$g_1g_2 = 0,\ g_3g_2 = g_2,\ g_1 \times g_3 = 0.$$

Thus $C(Z_{10})$ has complex modulo integers which leads to a strong mixed special quasi dual like number.

It is pertinent to mention the only source of getting strong mixed special quasi dual like numbers are from $C(Z_n)$ for an appropriate n. However using those new elements from $C(Z_n)$ we can construct row matrices or column matrices, or m × m matrices and $m \times n$ $(m \neq n)$ matrices and use them as new elements to construct strong mixed special quasi dual like numbers of complex modulo integers.

Likewise only Z_n is the only source of getting strong mixed special quasi dual like numbers.

Using these strong mixed special quasi dual like numbers we can construct algebraic structures like ring, semiring, S-ring, vector spaces, linear algebras, S-semirings, S-vector spaces, S-linear algebras, semivector spaces, S-semivector spaces, semilinear algebras and S-semilinear algebras. All these work is a matter of routine and hence is left as an exercise to the reader.

We need to construct a strong mixed special quasi dual like number three types of new elements g, g_1 and g_2 such that $g^2 = 0$, $g_1^2 = g_1$, $g_2^2 = -g_2$ together with the multiplicative compatability like $gg_1 = g_1g = 0$ or g or g_1 or g_2, $g_1g_2 = g_2g_1 = 0$ or g_1 or g_2 or g and $gg_2 = g_2g = 0$ or g_1 or g or g_2. We need also compatability of product among them or in short $\{0, g, g_1, g_2\}$ should form a semigroup under product. Interested reader can study analyse and find example describe / define / develop the related properties.

It is left as an open problem, do we have any source other than Z_n or $C(Z_n)$ or abstractly defined semigroups with three distinct elements g, g_1, satisfying the conditions.

$$g^2 = 0,\ g_1^2 = g_1,\ g_2^2 = -g_2,$$
$$g_ig_j = g_jg_i = 0 \text{ or } g_i \text{ or } g_j,\ g_i, g_j \in \{g, g_1, g_2\}.$$

With these we proceed on to construct non associative structures using dual numbers, special dual like numbers and special quasi dual numbers.

Chapter Four

GROUPOID OF DUAL NUMBERS

It is important using dual numbers we are not in a position to build non associative algebraic structures like loops or rings. The main reason for this is for all the three types of dual numbers we cannot find inverse. We build in this chapter groupoids of dual numbers. Further we see special quasi dual number is one for which it square is the negative of its value. We see only the complex number i is such that it square is negative how ever not the negative of its value. We see $i^2 = -1$. For a new element g to contribute to a quasi special dual number we need $g^2 = -g$; this is not possible in reals.

However the only source of such elements are the modulo integers Z_n. $3 \in Z_{12}$ is such that $3^2 = -3 \pmod{12}$, $8 \in Z_{12}$ is such that $8^2 = -8 = 4 \pmod{12}$ and so on

We first construct groupoids using dual numbers, then with special dual like numbers and then with special quasi dual numbers. Finally with mixed dual numbers.

Let $R(g) = \{a + bg \mid a, b \in R \text{ and } g = 3 \in Z_9\}$.

Define on R(g) an operation *.
If x, y $\in$ R(g) define

x * y = 5x + 2y; (5, 2) is a fixed pair used for every pair of elements in R(g) under the operation *.

Let x = 12 + g and y = 7 + 3g be in R(g).
x * y = 5 (12 + g) + 2(7 + 3g)
= 60 + 5g + 14 + 6g
= 74 + 11g $\in$ R(g).

Thus (R(g), *) is a groupoid of infinite order.
On R(g) define * as x * y
= –3y + 2x for x, y $\in$ R(g) then (R(g), *) is a groupoid.

Consider x = 1 + g, y = 3 – 2g and z = 3g are in R(g).

$$
\begin{aligned}
(x * y) * z &= [(1 + g) * (3 - 2g)] * 3g \\
&= [-3 (3 - 2g) + 2(1+g)] * 3g \\
&= (-9 + 6g + 2 + 2g) * 3g \\
&= (-7 + 8g) * 3g \\
&= -3(3g) + 2(-7+8g) \\
&= -9g - 14 + 16g \\
&= -14 + 7g. \qquad \text{I}
\end{aligned}
$$

$$
\begin{aligned}
x * (y * z) &= (1 + g) * [(3 - 2g) * 3g] \\
&= (1 + g) * [-3 \times 3g + 2 (3 - 2g)] \\
&= (1 + g) * (-9g + 6 - 4g) \\
&= (1 + g) * (6 - 13g) \\
&= -3 (6 - 13g) + 2 (1+g) \\
&= -18 + 39g + 2 + 2g \\
&= -16 + 41g. \qquad \text{II}
\end{aligned}
$$

Clearly I and II are not equal that is (x * y) * z $\neq$ x * (y * z) in general in R(g).

Consider x, y $\in$ R(g) define x * y = 3x + 0y.
Take x = –2 + g and y = 7 + 5g in R(g)

$$x * y = 3(-2+g) + 0\,(7 + 5g)$$
$$= -6 + 3g. \qquad \text{(I)}$$

$$y * x = 3y + 0x$$
$$= 3\,(7 + 5g) + 0\,(-2 + g)$$
$$= 21 + 15g \qquad \text{(II)}$$

Clearly $x * y \neq y * x$ in R(g) in general.

We can on R(g) define infinitely many groupoids called the groupoid of dual numbers.

Let $x, y \in R(g)$ define $x * y = \sqrt{3}\,x + 5y$.
Take $x = 3g$ and $y = 7$.

$$x * y = \sqrt{3} \times 3g + 5.7$$
$$= 3\sqrt{3}\,g + 35$$

$$y * x = \sqrt{3}\,y + 5x$$
$$= \sqrt{3} \times 7 + 5 \times 3g$$
$$= 7\sqrt{3} + 15g.$$

Thus $(R(g), (\sqrt{3}, 5), *)$ is a groupoid of dual numbers of dual numbers of infinite order.

We can instead of R use Q,
$Q(g)=\{a + bg \mid a, b \in Q$, g is the new element such that $g^2 = 0\}$.
Define for $x, y \in Q(g)$; $x * y = 7x + 2y$.

Let $x = 3 - g$ and $y = 5g + 2$ be in Q(g).
$x * y = 7\,(3-g) + 2(5g + 2) = 21 - 7g + 10g + 4 = 3g + 25$.

Thus $(Q(g), *, (7, 2))$ is again an infinite groupoid of dual numbers.

Clearly we can using Q(g) build infinite number of groupoids of dual numbers given by (Q(g), *, (m, n)) where m, n $\in$ Q.

We can also replace Q by Z and
Z (g) = {a + bg | a, b $\in$ Z, g a new element such that $g^2 = 0$}.

Consider

S = {Z(g), *, (m, n) | m, n $\in$ Z; x, y $\in$ Z(g), x * y = mx + ny}. S is a dual integer number groupoid of infinite order. We can get infinite number of them as we vary the pair (m, n) in Z $\times$ Z.

Apart from this we can also get infinite order groupoids by the following methods.

Let M = {($a_1, a_2, \ldots, a_p$) | $a_i \in$ Z(g), g a new element such that $g^2 = 0$; $1 \leq i \leq p$} and for x, y $\in$ M define x * y = sx + ry for s, r $\in$ Z. s $\neq$ r.

That is if x = ($a_1, a_2, \ldots, a_p$) and y = ($b_1, b_2, \ldots, b_p$) are in M then

$$\begin{aligned} x * y &= (a_1, a_2, \ldots, a_p) * (b_1, b_2, \ldots, b_p) \\ &= (a_1 * b_1, a_2 * b_2, \ldots, a_p * b_p) \\ &= (sa_1 + rb_1, sa_2 + rb_2, \ldots, sa_p + rb_p). \end{aligned}$$

Clearly x * y $\in$ M, thus (M, (s, r), *) is a groupoid of row matrix of dual numbers. M is a commutative groupoid of infinite order.

Now if we take

$$N = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \; a_i \in Q(g) \text{ g is a new element; } g^2 = 0; \; 1 \leq i \leq n \right\}$$

to be collection of all n × 1 column matrices whose entries are dual numbers. Define * on N as follows, for x, y ∈ N define x * y = tx + sy (t, s ∈ Q, t ≠ s, once the pair is chosen it is fixed).

$$\text{That is if } x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ then}$$

$$x \times_n y = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} * \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 * b_1 \\ a_2 * b_2 \\ \vdots \\ a_n * b_n \end{bmatrix} = \begin{bmatrix} ta_1 + sb_1 \\ ta_2 + sb_2 \\ \vdots \\ ta_n + sb_n \end{bmatrix} \in N.$$

So (N, (t, s), *) is a groupoid known as the dual number groupoid of column matrices.

If we take $S = \{A = (a_{ij})_{m \times n} \mid m \neq n, a_{ij} \in R(g); 1 \leq i \leq m$ and $1 \leq j \leq n$ with g a new element such that $g^2 = 0\}$ then S is a collection of dual number of m × n matrices. Define a binary non associative operation * on S as follows:

For A, B $\in$ S define A * B
= tA + sB (t, s $\in$ R)
= $(ta_{ij}) + (sb_{ij}) = (c_{ij}) \in$ S.

Thus (S, (t, s), *) is a dual number groupoid of rectangular (or m $\times$ n) matrices.

$$\text{Suppose } A = \begin{pmatrix} 2+g & 0 & 7g & 12 \\ 5+2g & 4+9g & 5-9 & 0 \\ 3-g & -3g & 11 & 1+g \end{pmatrix}$$

$$\text{and } B = \begin{pmatrix} 0 & 3+4g & 8g & 7 \\ 4+5g & 6g & 9 & 0 \\ 8 & 0 & 11+2g & 3-4g \end{pmatrix}$$

be two 3 $\times$ 4 matrices with dual number entries.

Let s = 3 and t = –4 we define A * B = 3A * (–4B)

$$= \begin{bmatrix} 3(2+g)+0 & 0+(-4(3+4g)) & 3\times 7g+(-4\times 8g) & 3\times 12-4\times 7 \\ 3(5+2g)+(-4(4+5g)) & 3(4+9g)-4\times 6g & 3(5-g)+-4\times 9 & 0 \\ 3(3-g)+(-4\times 8) & 3\times(-3g) & 3\times 11-4(11+2g) & 3(1+g)-4(3-4g) \end{bmatrix}$$

$$= \begin{bmatrix} 6+3g & -12-16g & -11g & 8 \\ -1-14g & 12-9g & -21-3g & 0 \\ -23-3g & -9g & -11-8g & -9+19g \end{bmatrix}$$

is in the collection of dual number 3 $\times$ 4 matrices. This is the way the operation * is performed on m $\times$ n matrices with dual number entries.

Finally consider P = {A = (m_{ij}) | A is a n × n matrix with $m_{ij} \in Z(g)$; $1 \leq i, j \leq n$, g a new element such that $g^2 = 0$}, P the collection of all n × n matrices with dual number entries. We define a non associative binary operation on P as follows:

For A, B ∈ P, A * B = pA + qB where p, q ∈ Z.

We will just illustrate this by a simple example.

Let P = {all 3 × 3 matrices with entries from Q(g), where g is a new element such that $g^2 = 0$}. S = (P, (3, 1), *) is a groupoid of square matrices of dual numbers.

$$\text{Suppose } A = \begin{pmatrix} 3+g & 2g & 3g+8 \\ 9g-1 & 7g-1 & -9g \\ 4g+2 & 2+g & 0 \end{pmatrix}$$

$$\text{and } B = \begin{pmatrix} 9+2g & 5-g & 0 \\ 0 & 8+4g & 7g \\ 2+g & -8 & 3+6g \end{pmatrix} \text{ are in P.}$$

Now A * B = 3A + B

$$= \left\{ \begin{pmatrix} 3+g & 2g & 3g+8 \\ 9g-1 & 7g-1 & -9g \\ 4g+2 & 2+g & 0 \end{pmatrix} + \begin{pmatrix} 9+2g & 5-g & 0 \\ 0 & 8+4g & 7g \\ 2+g & -8 & 3+6g \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 15+5g & 5+5g & 9g+24 \\ 27g-3 & 25g+5 & -20g \\ 13g+8 & 3g-2 & 3+6g \end{pmatrix} \text{ is in P.}$$

Thus S = (P, (3, 1), *) is a groupoid of infinite order.

Now we can have like groupoid of matrices of dual numbers the notion of polynomial groupoid of dual numbers.

Let S = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, \right.$ $a_i \in Z(g)$; g a new element such that $g^2 = 0$ and $a_i = t_i + s_i g$, $t_i, s_i \in Z\}$ be the set of polynomials with dual number coefficients from Z(g).

Let $p(x) = (3+5g) + (2g+1)x + 5gx^3 + 7x^4$ and
$q(x) = 3 + (8+g)x^2 + (7–4g)x^3 + 10gx^5 + (11g+1)x^6$

be two polynomials in S. Now we define a binary operation * on S as follows: for any $p(x), q(x) \in S$.

$$\begin{aligned} p(x) * q(x) &= 7\,((3+5g) + (2g+1)x + 5gx^3 + 7x^4) + \\ &\quad 2(3+(8+g)x^2 + (7–4g)x^3 + 10gx^5 + (1+11g)x^6) \\ &= 21 + 35g + (14 + 7)x + 35gx^3 + 49x^4 + 6 + \\ &\quad (16 + 2g)x^2 + (14 – 8g)x^3 + 20gx^5 + (x+22g)x^6 \\ &= (27 + 35g) + (14 + 7)x + (16 + 2g)x^2 + \\ &\quad (14 + 27g)x^3 + 49x^4 + 20gx^5 + (2 + 22g)x^6 \in S. \end{aligned}$$

Thus (S, *, (7, 2)) is defined as the polynomial groupoid of dual numbers.

We can get infinite number of groupoids by varying this (7, 2) in $Z \times Z$. All these groupoids are also of infinite order.

One can solve polynomial equations p(x) = 0 and solutions if it exists should be in Z(g).

Further one can replace Z by R or Q i.e. the dual number can take its entries from R(g) or Q(g).

This task of solving equations of polynomials with dual number coefficients is left as an exercise to the reader.

Now we proceed onto give construction of finite dual number groupoids.

Consider Z_n, let $Z_n(g) = \{a + bg \mid a, b \in Z_n$ g a new element, with $g^2 = 0\}$. Define on $Z_n(g)$ a non associative binary operation * such that for x, y $\in Z_n(g)$;

$x * y = tx + sy$ ($t, s \in Z_n$). Clearly $\{Z_n(g), *, (t, s)\}$ is a groupoid will be known as the modulo integer finite groupoid of dual numbers.

We will illustrate this by some examples.

Example 4.1: Let $G = \{Z_8(g) \mid g = 6 \in Z_{12}, *, (3, 5)\}$ be a finite groupoid of finite modulo integers of dual numbers.

If $x = 3 + 2g$ and $y = 1 + 5g$.

$$\begin{aligned} x * y &= 3x + 5y \\ &= 3\,(3+2g) + 5\,(1 + 5g) \\ &= 9 + 6g + 5 + 25g \\ &= 14 + 31g \\ &= 6 + 7g \in G. \end{aligned}$$

Take $z = g$ then $(x * y) * z$

$$\begin{aligned} &= (6 + 7g) * g = 3\,(6 + 7g) + 5g \\ &= 18 + 21g + 5g \\ &= 2 + 8g. \end{aligned}$$

$$\begin{aligned} \text{Now } x * (y * z) &= x * (3y + 5z) \\ &= x * (3 + 15g + 5h) \\ &= x * (3 + 2g) \\ &= 3x + 5\,(3 + 2g) \\ &= 9 + 6g + 15 + 10g \\ &= 0 \end{aligned}$$

Clearly $x * (y * z) \neq (x * y) * z$ in G. Thus the binary operation * on G is non associative in general.

Example 4.2: Let
$\{M, (8, 2), *\} = \{Z_{12}(g) = a + bg$ where $a, b \in Z_{12}$, $g = 4 \in Z_{16}$, $*, (8, 2)\}$ be the groupoid of dual numbers of finite order.

Example 4.3: Let $S = \{(a_1, a_2, a_3, \ldots, a_{12}) \mid a_i \in Z_7(g) = \{a + bg \mid a, b \in Z_7\ g = 2 \in Z_4\}; 1 \leq i \leq 12, *, (3, 1)\}$ be a groupoid of dual numbers.

If in these groupoids the pair (p, q) are not taken from Z_n but for $x, y \in Z_n(g)$ we define $px + qy \pmod n$ we call these dual number groupoids as new special groupoids.

We will illustrate this concept by some examples.

Example 4.4: Let

$$P = \{Z_{10}(g) = a + bg \text{ where } a, b \in Z_{10}, g = 6 \in Z_{36}, *, (12, 5)\}$$

be the new special groupoid of dual numbers.

Let $x = 3 + 2g$ and $y = 5 + 7g$ be in P.

$$\begin{aligned} x * y &= 12x + 5y \pmod{10} \\ &= 36 + 24g + 25 + 35g \pmod{10} \\ &= 1 + 9g \pmod{10}. \end{aligned}$$

Example 4.5: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{pmatrix} \middle| \; a_i \in Z_{23}(g); 1 \leq i \leq 12, \right.$$

$$g = 5 \in Z_{25}, *, (9, 16)\}$$

be the new special groupoid of dual numbers.

Example 4.6: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i = x_i + y_i g \in Z_{16}(g), x_i, y_i \in Z_{16}; \right.$$

$$\left. 1 \leq i \leq 30, g = 3 \in Z_9, *, (17, 43) \right\}$$

be the new special groupoid of dual numbers.

We can also have infinite groupoid of dual numbers using $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ or C(complex numbers) and $C(Z_n)$ complex modulo integers. Thus groupoids of dual numbers finite or infinite is aboundant in literature that also generated in a natural way.

Example 4.7: Let S = {C(g) | g = 4 $\in Z_8$, a + bg $\in$ C(g) with a, b $\in$ C (complex numbers) define *, (t, s) where t and s $\in$ C} be a complex groupoid of dual numbers of infinite order.

Take t = 3 + I and s = 2 + 4i.

For x = (2 + 3i) + (7–i)g and y = (1+i) + 3ig in C(g)

We have x * y = tx + sy

= (3 + i) [2 + 3i + (7–i)g] + (2 + 4i) ((1 + i) + 3ig)

= (3 + i) (2 + 3i) + (3–i) (7–i) g + (2 + 4i) (1 + i) + (2 + 4i) 3ig

= 6 + 2i + 9i – 3 + (21 – 7i – 3i + 1)g + (2 + 4i + 2i – 4) + (6i – 12)g

= 1 + 17i + (10 – 4i)g $\in$ C(g).

Suppose z = 7 then

$(x * y) * z = ((1 + 17i) + (10–4i)g) * 7$
$= (3 + i) ((1 + 17i) + (10–4i)g) + 7 (2+4i)$
$= (3 + i) (1 + 17i) + (3+i) (10–4i)g + 14 + 28i$
$= 3 + 3i + 51i – 17 + (30 + 10i – 12i + 3) \times g + 14 + 28i$
$= 82i + (34 – 2i)g$ I

Consider $x * (y * z)$
$= x * ((3 + i) (1+i + 3ig) + (2 + 4i)7)$
$= x * [(3 + 3i + I – I + 9ig – 3g + 14 + 28i]$
$= x * (16 + 22i + (9i – 3) g)$
$= (3 + i) (2 + 3i + ((7–i)g) + (2 + 4i) (16 + 22i + (9i – 3)g)$

$= (3 + i) (2 + 3i) + (3 + i) (7 – i)g + (2 + 4i) (16 + 22i) +$
$(2 + 4i) (9i – 3)g)$

$= 6 + 2i + 9i – 3 + (21 + 1 + 7i – 3i) g + (32 + 64i +$
$44i – 88) + (18i – 36 – 6 – 12i)g$

$= –53 + 119i + (–20 + 10i)g$ II

Clearly II and I are not equal so S is a complex groupoid of dual numbers of infinite order.

Example 4.8: Let $M = \{C(Z_9) (g) = \{a + bi_F + (c + di_F)g \mid a + bi_F$ and $c + di_F \in C(Z_9)$ and $g = 7 \in Z_{49}, *, (2 + i_F, 4 + 3i_F)\}$ is the complex modulo integer groupoid of finite order.

If $x = (3 + 2i_F) + (7 + i_F)g$ and $y = 3i_F + 2g \in M$;
then $x * y = (2 + i_F) [3 + 2i_F + (7 + i_F)g] + (4 + 3i_F) (3i_F + 2g)$

$= (2 + i_F) (3 + 2i_F) + (2+i_F) (7 + i_F)g + (4 + 3i_F)3i_F + (4 + 3i_F) 2g$

$= 6 + 3i_F + 4i_F + 2 \times 8 + (14 + 7i_F + 2i_F + 8) g + 12i_F +$
$9 \times 8 + 8g + 6i_Fg$

$= (4 + i_F) + (3 + 6i_F)g \in M.$

Example 4.9: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in C(Z_5)(g) \right.$$

$= \{a + bi_F + (c + di_F)\, g \mid a, b, c, d \in Z_5,\ i_F^2 = 4, g = 3 \in Z_9\}, *, (3i_F, 2); 1 \leq i \leq 6\}$ be the complex modulo integer groupoid of dual numbers.

Example 4.10: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \; a_i \in C(Z_{12})(g) \right.$$

$= \{a + bi_F + (d + ci_F)g \mid a, b, c, d \in Z_{12},\ i_F^2 = 11, g = 5 \in Z_{25}\}, (3 + 2i_F, 4i_F), *\}$ be the complex modulo integer groupoid of dual number of infinite order.

We have seen only groupoid of dual numbers. Now on similar lines we can build groupoid of special dual like numbers.

We will illustrate this situation by some examples.

Example 4.11: Let

$S = \{R(g), *, (3, 76); R(g) = \{a + bg \mid a, b \in R, g = 3 \in Z_6\}\}$ be the groupoid of special dual like numbers of infinite order.

It is pertinent to mention we need not say whether R(g) is a dual number collection or a special dual like number collection, from g one can easily understand; if $g^2 = 0$ it is a dual number collection and if $g^2 = g$ it is a special dual like number collection.

Example 4.12: Let $M = \{Z(g), *, (-7, 2)$ where $g = 5 \in Z_{10}\}$ be the groupoid of special dual like numbers of infinite order.

Example 4.13: Let
P = {Q (g), *, (3/2, –1) where g = (3, 3, 3) with 3 $\in Z_6$} be the groupoid of special like numbers of infinite order.

Example 4.13: Let
P = {Z_9 (g) = {a + bg | a, b $\in Z_9$ and g = 4 $\in Z_{12}$}, (3, 2), *} be the groupoid of special dual like numbers of finite order.

Example 4.14: Let
M = {($a_1, a_2, a_3, \ldots, a_{15}$) | $a_i \in Z_{25}$ (g); $1 \leq i \leq 15$, g = 7 $\in Z_{42}$, *, (20, 4)} be the groupoid of special dual like number of finite order.

Example 4.15: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \right| a_i \in Z_{45}\,(g)$$

$$= \{x + yg \mid x, y \in 45, g = 10 \in Z_{30}\}, 1 \leq i \leq 8, *, (10, 0)\}$$

be the groupoid of special dual like numbers of finite order. Clearly P is a non commutative groupoid.

Example 4.16: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \right| a_i \in Z_{17}(g)$$

$$= \{a + bg \mid a, b \in Z_{17}, g = 6 \in Z_{30}\}, 1 \leq i \leq 40, *, (10, 2)\}$$

be the groupoid of special dual like numbers. S is a non commutative finite groupoid.

Example 4.17: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in Z(g) \right.$$

= {a + bg | a, b ∈ Z, g = (1, 1, 1, 1, 1, 1), g^2 = (1, 1, 1, 1, 1, 1) = g}, *, (3, –2), 1 ≤ i ≤ 9} be the special dual like number groupoid of infinite order.

Example 4.18: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{34} & a_{35} & a_{36} \end{bmatrix} \middle| \; a_i = R(g) = \{a + bg \mid a, b \in R, \right.$$

$$g = 5 \in Z_{20}\},\ 1 \leq i \leq 36;\ *,\ (\sqrt{3}, -2)\}$$

be the special groupoid of infinite order.

Example 4.19: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \; a_i = Z_{19}(g) \right.$$

$$= \{x + yg \mid x, y \in Z_{19};\ g = 16 \in Z_{40}\};\ *,\ (3, 2)\}$$

be a special dual number groupoid of infinite order.

Take $p(x) = 2 + 5x + 3x^3 + 7x^4$
and $q(x) = 8 + 7x + 18x^2 + 5x^3$ in T,
$p(x) * q(x) = 3p(x) + 21q(x)$
$= 6 + 15x + 9x^3 + 21x^4 + 16 + 14x + 36x^2 + 10x^3$
$= 3 + 10x + 17x^2 + 9x^3 + 2x^4 \in T.$

Example 4.20: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R(g) \right.$$

$$= \{a + bg \mid a, b \in R, g = 4 \in Z_{12}\}, (\sqrt{41}, -\sqrt{13}), *\}$$

be the special dual like number groupoid of polynomials.

Example 4.21: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{11}(g) \right.$$

$$= \{a + bg \mid a, b \in Z_{11}, g = 3 \in Z_6\}, 0 \leq i \leq 4, (3, 7), *\}$$

be a polynomial groupoid of special dual like numbers of finite order.

Example 4.22: Let

$$S = \left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in Z_{13}(g) \right.$$

$$= \{a + bg \mid a, b \in Z_{13}, g^2 = 5 \in Z_{20}\}, *, (3, 0), 0 \leq i \leq 7\}$$

be the polynomial groupoid of special dual like numbers of finite order.

It is pertinent to mention here that all neutrosophic number like $\langle Z \cup I\rangle$, $\langle Q \cup I\rangle$, $\langle R \cup I\rangle$, $\langle Z^+ \cup \{0\} \cup I\rangle$, $\langle Q^+ \cup \{0\} \cup I\rangle$, $\langle R^+ \cup \{0\} \cup I\rangle$ and $\langle Z_n \cup I\rangle$ can be made into neutrosophic groupoids of special dual like numbers.

We will give one or two examples before we proceed onto define mixed dual numbers.

Example 4.23: Let
$S = \{a + bI \mid a, b \in R, a + bI \in \langle R \cup I\rangle$ with $I^2 = I, *, (\sqrt{7}, -3)\}$ be a special dual like number neutrosophic groupoid of infinite order.

Example 4.24: Let
$T = \{a + bI \mid a + bI \in \langle Q \cup I\rangle, I^2 = I, *, (-7/3, 8/11)\}$ be a special dual like number neutrosophic groupoid of infinite order.

Example 4.25: Let
$M = \{d = a + bI \mid d \in \langle Z_{25} \cup I\rangle, *, (20, 7)\}$ be a special dual like number neutrosophic groupoid of finite order.

Example 4.26: Let
$S = \{(a_1, a_2, \ldots, a_7) \mid a_i \in \langle Z \cup I\rangle, 1 \leq i \leq 7, *, (-11, 0)\}$ be a special dual like number neutrosophic groupoid of infinite order.

Example 4.27: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle|\ a_i \in \langle Q^+ \cup I \cup \{0\}\rangle, 1 \leq i \leq 12, *, (12, 17)\right\}$$

be a special dual like number neutrosophic groupoid of infinite order. Clearly M is non commutative.

If $x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}$ and $y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{12} \end{bmatrix}$ are in M then

$$x * y = 12x + 17y$$

$$= \begin{bmatrix} 12a_1 + 17b_1 \\ 12a_2 + 17b_2 \\ \vdots \\ 12a_{12} + 17b_{12} \end{bmatrix} \in M.$$

Example 4.28: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \middle| \, a_i \in \langle Z_{17} \cup I\rangle, 1 \leq i \leq 40, (7, 10), * \right\}$$

be the special dual like number neutrosophic groupoid of finite order.

Example 4.29: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \, a_i \in (Z_{10} \cup \{0\}); \right.$$

$$\left. 1 \leq i \leq 16, (35, 2), * \right\}$$

be the special dual like number neutrosophic groupoid of finite order.

Example 4.30: Let

$$S = \left\{ \sum_{i=0}^{9} a_i x^i \,\middle|\, a_i \in \langle Z_8 \cup I\rangle, 0 \leq i \leq 9, (7, 1), * \right\}$$

be the finite special dual like number groupoid of polynomials.

If $p(x) = 2 + 6x + 3x^3 + 7x^4$ and
$q(x) = 4 + x^2 + 7x^3 + 6x^4 + x^5$ are in S.
$p(x) * q(x) = 7p(x) + 1q(x)$

$= 14 + 42x + 21x^3 + 49x^4 + 4 + x^2 + 7x^3 + 6x^4 + x^5$
$= 2 + 2x + x^2 + 4x^3 + 7x^4 + x^5 \in S.$

This is the way * operation is performed. By performing * operation we see the degree of the polynomial does not increase.

Example 4.31: Let

$$T = \left\{ \sum_{i=0}^{20} a_i x^i \;\middle|\; a_i \in \langle Q \cup I\rangle;\ 0 \leq i \leq 20, (8, -3), * \right\}$$

be the infinite polynomial neutrosophic groupoid of special dual like numbers.

Example 4.32: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_{31} \cup I\rangle, (14, 0), * \right\}$$

be an infinite polynomial neutrosophic groupoid of special dual like numbers.

Example 4.33: Let

$$T = \left\{ \sum_{i=0}^{5} a_i x^i \;\middle|\; a_i \in \langle Z_{15} \cup I\rangle, 0 \leq i \leq 15, (2, 3), * \right\}$$

be a finite polynomial neutrosophic groupoid of special dual like numbers.

Now having seen examples of special dual like number groupoids we proceed onto give examples of mixed dual number groupoids.

Example 4.34: Let M = $\{a + bg + cg_1$, $g = 6$ and $g_1 = 4 \in Z_{12}$, $g^2 = 0$ and $g_1^2 = g_1$, $gg_1 = g_1g = 0$, a, b, c $\in$ Z, (3, –2), $*\}$ be a mixed dual number groupoid of infinite order.

If $x = 5 + 3g - 4g_1$ and $y = 3 - 2g + g_1$ are in M

$$\begin{aligned} x * y &= 3x - 2y \\ &= 3(5 + 3g - 4g_1) - 2(3 - 2g + g_1) \\ &= 15 + 9g - 12g_1 - 6 + 4g - 2g_1 \\ &= 9 + 13g - 14g_1 \in M. \end{aligned}$$

Example 4.35: Let T = $\{a + bg + cg_1 \mid a, b, c \in Z_{12}$, $g = 6$ and $g_1 = 9 \in Z_{18}$, $g^2 = 36 \equiv 0 \pmod{18}$ $g_1^2 = 81 = 9 \pmod{18}$ and $gg_1 = 54 = 0 \pmod{18}$, (4, 2), $*\}$ be a finite mixed dual number groupoid.

For if $x = 3 + g + 6g_1$ and $y = 5 + 3g + g_1$ are in T, then $x * y = 4x + 2g$

$$= 4(3 + g + 6g_1) + 2(5 + 3g + g_1)$$

$$= 12 + 4g + 24g_1 + 10 + 6g + 2g_1$$

$$= 10g + 2g_1 + 10 \in T.$$

Example 4.36: Let
S = $\{(a_1, a_2, a_3)$ where $a_i = x_1 + x_2g + x_3g_2$ with $x_j \in Z_{40}$, $g = (2, 2, 2, 2, 0, 0, 2, 0)$ and $g_1 = (1, 1, 1, 1, 0, 1, 0)$, 0, 1, 2 $\in Z_4$, $g^2 = (0, 0, 0, 0, 0, 0, 0, 0)$, $g_1^2 = (1, 1, 1, 1, 0, 1, 0) = g_1$, $*$, (8, 19)$\}$ be the groupoid of mixed dual numbers of finite order.

Example 4.37: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_i = x_1 + x_2g + x_3g_1;\ 1 \le i \le 8, \right.$$

$x_j \in Q$, $1 \le j \le 3$, $g = 5$ and $g_1 = 10$, $g^2 = 5 \pmod{20}$, $g_1^2 = 0 \pmod{20}$, $5, 10 \in Z_{20}$, $(3/7, 10/7)$, $*\}$ be a mixed dual number groupoid of infinite order.

Example 4.38: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i = x_1 + x_2g + x_3g_1,\ 1 \le i \le 9, \right.$$

$$x_j \in Z^+ \cup \{0\},\ 1 \le j \le 3,\ g = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \text{ and } g_1 = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} \text{ with}$$

$$g \times_n g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \pmod{12},\ 6, 9 \in Z_{12};\ g_1 \times_n g_1 = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \\ 9 \end{bmatrix},$$

$$g_1 \times_n g = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}, (3, 9), *\}$$

be a mixed dual number groupoid of infinite order.

Example 4.39: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \end{pmatrix} \middle| a_i = x_1 + x_2g + x_3g_1, \right.$$

$1 \leq i \leq 20$, $x_j \in Z_{14}$, $1 \leq j \leq 3$, $g = 3$ and $g_1 = 3 + 3i_F \in C(Z_6)$, $g^2 = 3 \pmod 6$ and $g_1^2 = 9 + 9 \times 5 + 18i_F = 0 \pmod 6$.
$gg_1 = 3 + 3i_F = g_1$; *, (7, 7)} be a mixed dual number groupoid of finite order.

Example 4.40: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i = x_1 + x_2g + x_3g_1, x_j \in Q; \right.$$

$1 \leq j \leq 3$, $g = I$, $g_1 = 3I \in \langle Z_9 \cup I \rangle$, $g^2 = g$, $g_1^2 = 0$, $gg_1 = 3I = g_1$, *, (7, 13/2)} be a mixed dual number neutrosophic groupoid of infinite order.

Example 4.41: Let

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| a_i = x + x_1g + x_2g_1, x, x_1, x_2 \in Q, g = 4 \right.$$

and $g_1 = 6 \in Z_{12}$, $g^2 = g \pmod{12}$, $g_1^2 = 0 \pmod{12}$, $gg_1 = 0 \pmod{12}$, $(-17, 3/11)$, $*\}$ be a polynomial groupoid of mixed dual numbers of infinite order.

Example 4.42: Let

$$S = \left\{ \sum_{i=0}^{7} a_i x^i \;\middle|\; a_i = x_1 + x_2 g_1 + x_3 g_1;\ 0 \le i \le 7, \right.$$

$g = 16$, $g_1 = 20 \in Z_{40}$, $g^2 = 16 = g \pmod{40}$, $g_1^2 = 0 \pmod{40}$, $g_1 g = 0 \pmod{40}$, $x_j \in R$, $1 \le j \le 3$, $(-\sqrt{7}, 17)$, $*)$ be a polynomial groupoid of mixed dual numbers of infinite order.

Example 4.43: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = x_1 + x_2 g + x_3 g_1,\ x_j \in Z_6, \right.$$

$$1 \le j \le 3,\ g = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix},\ g_1 = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix};\ 6, 9 \in Z_{12};$$

$$g \times_n g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ g_1 \times_n g_1 = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$$

$$g_1 \times_n g = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix}, *, (2, 4)\}$$

be the polynomial groupoid of mixed dual numbers of infinite order.

Example 4.44: Let

$$M = \left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1;\ 0 \le i \le 7,\ x_j \in Z_{40}, \right.$$

$$1 \le j \le 3,\ g = \begin{bmatrix} 6+6I \\ 6+6I \\ 6+6I \\ 6+6I \\ 6+6I \\ 6+6I \end{bmatrix},\ g_1 = \begin{bmatrix} 9I \\ 9I \\ 9I \\ 9I \\ 9I \\ 9I \end{bmatrix},\ 9I,\ 6+6I \in C(Z_{12});$$

$$g \times_n g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ g_1 \times_n g_1 = \begin{bmatrix} 9I \\ 9I \\ 9I \\ 9I \\ 9I \\ 9I \end{bmatrix},\ g \times_n g_1 = \begin{bmatrix} 6+6I \\ 6+6I \\ 6+6I \\ 6+6I \\ 6+6I \\ 6+6I \end{bmatrix},\ (10, 16), *)\}$$

be the polynomial groupoid of mixed dual numbers of finite order.

Now we proceed onto give examples of groupoid of special quasi dual numbers.

Example 4.45: Let
M = {a + bg | a, b ∈ Q, 3 = g ∈ Z_{12}, g^2 = 9 = –3 = –g, *, (3, –7/11)} be the special quasi dual number groupoid.

Example 4.46: Let
P = {a + bg + cg_1 | a, b, c ∈ Z^+ ∪ {0}, g = 2, g^2 = 4 = g_1, (7, 20)} be the special quasi dual number groupoid of infinite order.

Example 4.47: Let
M = {a + bg | a, b ∈ Z_{10}, g = 8, g^2 = 64 = –g (mod 12), 8 ∈ Z_{12}, *, (2, 7)} be the special quasi dual numbers groupoid of finite order.

Consider x = 2 + 3g, y = 7 + g ∈ M;
x * y = 2x + 7y = 2 (2+3g) + 7 (7+g)
= 4 + 6g + 49 + 7g = 3 + 3g ∈ M. For

$$\begin{aligned} z = 1 + 8g \in M; (x * y) * z &= (3 + 3g) * z \\ &= 2\,(3 + 3g) + 7\,(1+8g) \\ &= 6 + 6g + 7 + 56g \\ &= 3 + 2g \qquad \text{I} \end{aligned}$$

$$\begin{aligned} x * (y * z) &= x * [2y + 7\,(1+ 8g)] \\ &= x * [14 + 2g + 7 + 56g] \\ &= x * (1 + 8g) \\ &= 2x + 7\,(1 + 8g) \\ &= 2(2+3g) + 7\,(1 + 8g) \\ &= 4 + 6g + 7 + 56g \\ &= 1 + 2g \qquad \text{II} \end{aligned}$$

The equations I and II are not equal;

x * (y * z) ≠ (x * y) * z.

Example 4.48: Let P = $\{a + bg + cg_1 \mid a, b, c \in R^+ \cup \{0\}$, $g = 24 \in Z_{40}$, $g^2 = 16 = g_1 = -g \pmod{40}$, $g \times g_1 = g$; $(\sqrt{7}+1, 5+\sqrt{3})$, $*\}$ be the special quasi dual number groupoid.

Take $x = 3 + g + 5g_1$ and $y = 2 + 5g + g_1 \in P$,
$x * y = (\sqrt{7}+1)x + (5+\sqrt{3})y$

$= (\sqrt{7}+1)(3 + g + 5g_1) + (5+\sqrt{3})(2 + 5g + g_1)$

$= 3\sqrt{7} + \sqrt{7}g + 5\sqrt{7}g_1 + 3 + g + 5g_1 + 10 + 25g + 5g_1 + 2\sqrt{3} + 5\sqrt{3}g + \sqrt{3}g_1$

$= (3\sqrt{7} + 2\sqrt{3} + 3 + 10) + (\sqrt{7} + 1 + 25 + 5\sqrt{3})g + (5\sqrt{7} + 10 + \sqrt{3})g_1$

$= (3\sqrt{7} + 2\sqrt{3} + 13) + (26 + \sqrt{7} + 5\sqrt{3})g + (10 + \sqrt{3} + 5\sqrt{7})g_1 \in P.$

This is the way '*' operation is performed on P.

Example 4.49: Let T = $\{(a_1, a_2, a_3, a_4) \mid a_i = x + x_1g;\ 1 \leq i \leq 4$, $x, x_1 \in Z_{11}$, $g = 14 \in Z_{21}$, $g^2 = 7 = -g \pmod{21}$, $(5, 6)$, $*\}$ be the special quasi dual number groupoid of finite order.

Example 4.50: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle|\ a_i = x_1 + x_2g + x_3g_1;\ 1 \leq i \leq 10,\ x_j \in Z^+ \cup \{0\}, \right.$$

$1 \leq j \leq 3$, $g = 20 \in Z_{30}$, $g^2 = 20^2 = -g = 10 = g_1 \pmod{30}$, (7, 8), *} be the special quasi dual number groupoid of infinite order.

Example 4.51: Let

$$P = \left\{ \left(\begin{matrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \end{matrix} \right) \middle| \, a_i = x_1 + x_2g + x_3g_1;\ 1 \leq i \leq 20, \right.$$

$$x_1, x_2, x_3 \in Z_{15},\ g = 3 \in Z_{12},\ g^2 = 9 = -g_1 \pmod{12},\ (7, 2), *\}$$

be a special quasi dual number groupoid of finite order.

Example 4.52: Let

$$S = \left\{ \sum_{i=0}^{11} a_i x^i \,\middle|\, a_i = x_1 + x_2g + x_3g_1;\ 0 \leq i \leq 11,\ x_j \in Z_7, \right.$$

$$1 \leq j \leq 3,\ g = 2 \in Z_6,\ g^2 = 4 = -g = g_1 \in Z_6,\ *,\ (3, 1)\}$$

be the polynomial groupoid of special quasi dual numbers of finite order.

Let $p(x) = 3 + 2x + 5x^2 + 2x^7$ and

$q(x) = 2 + 5x + 3x^2 + 4x^5 + 2x^6$ be in S.

$p(x) * q(x) = 3p(x) + q(x)$

$= 9 + 6x = 15x^2 + 6x^7 + 2 + 5x + 3x^2 + 4x^5 + 2x^6$

$= 4 + 4x + 4x^2 + 4x^5 + 2x^6 + 6x^7 \in S.$

Example 4.53: Let

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1;\ x_j \in Q;\ g = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 0 \\ 24 \\ 0 \end{bmatrix}, \right.$$

$$24 \in Z_{40},\ g^2 = \begin{bmatrix} 16 \\ 16 \\ 16 \\ 0 \\ 16 \\ 0 \end{bmatrix} = -g = \begin{bmatrix} -24 \\ -24 \\ -24 \\ 0 \\ -24 \\ 0 \end{bmatrix} = g_1 \pmod{40},$$

$$\left. gg_1 = g \pmod{40};\ 1 \le j \le 3,\ (1/7, 8/13), * \right\}$$

be a polynomial special quasi dual number groupoid of infinite order.

Example 4.54: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1,\ x_j \in Z_{12}, \right.$$

$$1 \le j \le 3,\ g = \begin{bmatrix} 8 & 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 & 8 \end{bmatrix};\ 8 \in Z_{12},$$

$$g^2 = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix} \pmod{12}$$

$$= -g = \begin{bmatrix} -8 & -8 & -8 & -8 & -8 & -8 \\ -8 & -8 & -8 & -8 & -8 & -8 \\ -8 & -8 & -8 & -8 & -8 & -8 \end{bmatrix} \pmod{12}$$

$= g_1, 4 \in Z_{12}, (3, 7), *\}$

be the polynomial groupoid of special quasi dual number.

Now we proceed into give examples of groupoids of mixed special quasi like dual numbers, strong mixed dual numbers and mixed quasi dual numbers.

Example 4.55: Let P = $\{a + bg + cg_1 \mid a, b, c \in R, g = 24$ and $g_1 = 20 \in Z_{40}, (\sqrt{3}, 7 - \sqrt{5}), *,$ (Here $g_1^2 = 0 \pmod{40}$ and $g^2 = -g \pmod{40})\}$ be the groupoid of mixed special quasi dual number. P is an infinite groupoid.

Example 4.56: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i = x_1 + x_2g + x_3g_1; 1 \leq i \leq 4, x_j \in Z_{15}, \right.$$

$1 \leq j \leq 3, g = 6\ g_1 = 3 \in Z_{30}, g^2 = 0 \pmod{12},$

$g_1 = 9 = -3 \pmod{12}, (10, 5), *\}$

be the mixed special quasi dual groupoid.

$$\text{Let } x = \begin{bmatrix} 3+g_1 \\ 2+g+2g_1 \\ g_1 \\ 5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2g_1 \\ 5g \\ 1+2g \\ 3+4g+g_1 \end{bmatrix} \text{ be in M.}$$

x* y = 10x + 5y

$$= \begin{bmatrix} 30+10g_1 \\ 20+10g+10g_1 \\ 10g_1 \\ 50 \end{bmatrix} + \begin{bmatrix} 10g_1 \\ 25g \\ 5+10g \\ 15+20g+5g_1 \end{bmatrix}$$

$$= \begin{bmatrix} 5g_1 \\ 5+5g+10g_1 \\ 5+10g+10g_1 \\ 5+5g+5g_1 \end{bmatrix} \in M.$$

Example 4.57: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_1 + x_2g + x_3g_1, 1 \le i \le 9, x_j \in Q, \right.$$

$1 \le j \le 3, g = 20$, $g_1 = 15 \in Z_{40}$, $g^2 = 0 \pmod{40}$, $g_1^2 = -g_1 \pmod{40}$, $gg_1 = 20 = g \pmod{40}$, *, (7/3, 1/2)} be the mixed special quasi dual groupoid of infinite order.

$$\text{If } x = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \text{ are in P.}$$

$$x*y = \begin{bmatrix} 7/3a_1+1/2b_1 & 7/3a_2+1/2b_2 & 7/3a_3+1/2b_3 \\ 7/3a_4+1/2b_4 & 7/3a_5+1/2b_5 & 7/3a_6+1/2b_6 \\ 7/3a_7+1/2b_7 & 7/3a_8+1/2b_8 & 7/3a_4+1/2b_4 \end{bmatrix} \in P.$$

Example 4.58: Let

$$T = \left\{ \sum_{i=0}^{5} a_i x^i \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1, x_j \in Z_{16}, 0 \leq i \leq 5, \right.$$

$1 \leq j \leq 3$, $g = (6, 6, 6)$, $g_1 = (8, 8, 8)$, $6, 8 \in Z_{12}$, $g_2 = (0, 0, 0)$, $g_1 = (4, 4, 4) = -g_1$, $(0, 8),*)\}$ be a polynomial groupoid with mixed special quasi dual coefficients.

Next we proceed onto give examples of mixed special quasi dual like numbers groupoid.

Example 4.59: Let $P = \{a + bg + cg_1 \mid a, b, c \in R$, $g = 4$ and $g_1 = 3 \in Z_{12}$, $g^2 = g \pmod{12}$ and $g_1^2 = -g_1 \pmod{12}$ $(\sqrt{7} - 1,\ 5 + \sqrt{13}),\ *)\}$ be a mixed special quasi dual like number groupoid of infinite order.

Example 4.60: Let
$W = \{(a_1, a_2, a_3, \ldots, a_{10}) \mid a_i = x_1 + x_2 g + x_3 g_1$, $1 \leq i \leq 10$, $x_j \in Z_{46}$, $1 \leq j \leq 3$, $g = 16$ and $g_1 = 15 \in Z_{40}$, $g^2 = g$ and $g_1^2 = -g_1 \pmod{40}$, $(8, 23)$, $*\}$ be a mixed special quasi dual like number groupoid of finite order.

Example 4.61: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \,\middle|\, a_i = x_1 + x_2 g + x_3 g_1, x_j \in Z_{20}; 1 \leq j \leq 3, \right.$$

$g = (6, 6, 6)$, $g_1 = (7, 7, 7)$; $6, 7 \in Z_{42}$, $g_2 = (36, 36, 36) = (-6, -6, -6) = -g$ and $g_1^2 = (49, 49, 49) \pmod{42} = (7, 7, 7) \pmod{42} = g_1 \pmod{42}$; $gg_1 = (0, 0, 0)$, $(7, 8)$, $*\}$ be the mixed special quasi dual like number groupoid.

Example 4.62: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_1 + x_2 g + x_3 g_1,\ 1 \leq i \leq 9, \right.$$

$$x_j \in Z^+ \cup \{0\},\ 1 \leq j \leq 3,\ g = \begin{bmatrix} 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \end{bmatrix};\ 14, 15 \in Z_{21},\ g^2 = \begin{bmatrix} 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \end{bmatrix}$$

$$\left. = -g,\ g_1^2 = \begin{bmatrix} 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \end{bmatrix} = g_1, (8, 7), * \right\}$$

be the mixed special quasi dual like number groupoid.

We can also build mixed special quasi like dual number coefficient polynomial groupoid of finite as well as infinite order. It is left as an exercise to the reader.

Now we proceed onto give examples of strong special mixed dual number groupoid.

Example 4.63: Let $S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_j \in Q,\ 1 \leq j \leq 4,\ g_1 = 6,\ g^2 = 3$ and $g_3 = 4 \in Z_{12},\ g_1^2 = 0 \pmod{12}$, $g_2^2 = -g_2 \pmod{12},\ g_3^2 = g_3 \pmod{12},\ (3/7, -8/11), *\}$ be the strong special mixed dual number groupoid of infinite order.

Example 4.64: Let $S = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3;\ 1 \leq i \leq 6,\ x_j \in Z_{25},\ 1 \leq j \leq 4, (5, 20), *$ where $g_1 = 8$,

$g_2 = 9$ and $g_3 = 6 \in Z_{12}$, $g_1^2 = -g_1$, $g_2^2 = 9$ and $g_3^2 = 0$, $g_1g_2 = 0$, $g_1g_3 = 0$ and $g_2g_3 = 6 = g_3 \pmod{12}\}$ be the strong special mixed dual number groupoid of finite order.

Example 4.65: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3; \right.$$

$1 \leq i \leq 18$, $x_j \in Z^+ \cup \{0\}$, $1 \leq j \leq 4$; $g_1 = 20$, $g^2 = 16$ and $g_3 = 15 \in Z_{40}$, $g_1^2 = 0 \pmod{40}$, $g_2^2 = 16 \pmod{40}$ and $g_3 = -g_3 \pmod{40}$, (7, 8), $*\}$ be the strong special mixed dual number groupoid of infinite order.

Example 4.66: Let

$$S = \left\{ \sum_{i=0}^{8} a_i x^i \,\middle|\, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3; x_j \in Z_7, 0 \leq i \leq 8, \right.$$

$1 \leq j \leq 4$, $g_1 = 6$, $g^2 = 9$ and $g_3 = 8 \in Z_{36}$, $g_1^2 = 0 \pmod{36}$, $g_2^2 = 81 = g_2 \pmod{36}$, $g_3^2 = 28 \pmod{36} = 8 \pmod{36}$, $g_1g_2 = g_3$, $g_1g_3 = 0 \pmod{36}$, $g_3g_2 = 0 \pmod{36}$, (3, 2), $*\}$ be a polynomial strong special mixed dual number groupoid of finite order.

The task of studying, analyzing and describing higher dimensional dual number groupoids, higher dimensional special dual like number groupoids and higher dimensional special quasi dual number groupoid is left to the reader.

Further the reader is expected to study and describe the four types of mixed groupoids of higher dimension.

Now we proceed onto define three types of non associative rings using dual number groupoids of all types and rings and dual number rings.

DEFINITION 4.1: *Let S = {Z(g) | g is a new element and a + bg is a dual number with a, b ∈ Z, $g^2 = 0$} be the general ring of dual numbers. L be a loop. SL be the loop ring of the loop L over the ring S. SL is a non associative dual number ring.*

If Z(g) is replaced by R(g) or Q(g) or $Z_n(g)$ still the result hold good.

We will give some examples of this concept.

Example 4.67: Let S = Q(g) = {a + bg | a, b ∈ Q, g = 5 ∈ Z_{25}, $g^2 = 0$ (mod 25) be the new element} be the general ring of dual numbers.

L = $L_7(3)$ = {e, 1, 2, 3, 4, 5, 6, 7} be a loop given by the following table.

o	e	1	2	3	4	5	6	7
e	e	1	2	3	4	5	6	7
1	1	e	4	7	3	6	2	5
2	2	6	e	5	1	4	7	3
3	3	4	7	e	6	2	5	1
4	4	2	5	1	e	7	3	6
5	5	7	3	6	2	e	1	4
6	6	5	1	4	7	3	e	2
7	7	3	6	2	5	1	4	e

SL is a non associative loop ring of general dual numbers.

Example 4.68: Let

M = {R(g) | a + bg, a, b ∈ R, g is a new element such that $g^2 = 0$} be the dual number general ring. L = L_5 (2) = {e, 1, 2, 3, 4, 5} given by the composition table.

o	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

ML is the loop ring (non associative) of dual numbers of infinite order.

Example 4.69: Let
S = {Z_5 (g) = a + bg; a, b ∈ Z_5, g a new element such that $g^2 = 0$} be the dual number ring, L be the loop given by the following table.

o	e	a	b	c	d	g
e	e	a	b	c	d	g
a	a	e	d	b	g	c
b	b	d	e	g	c	a
c	c	b	g	e	a	d
d	d	g	c	a	e	b
e	g	c	a	d	b	e

SL is the loop ring of dual number of finite order and is also commutative.

All properties of non associative rings can be derived in case of loop rings of dual numbers. This task is left as an exercise to the reader.

At this stage it is important to note that we cannot construct loops of dual numbers for loops too like groups should have identity and inverse under product. So loops of dual numbers or loops of special dual like numbers or loop of special quasi dual number or loop of mixed dual numbers is an impossibility under product. So we cannot use loop and ring (not dual number rings) to get non associative ring of dual numbers. However to get non associative dual numbers we make use of loops and general ring of dual numbers.

Further if the ring of dual numbers is replaced by special dual like number ring R we use loops L can construct loop rings RL which will be the non associative ring of special dual like numbers. We can have R to be Z(g) or R(g) or C(g) or $Z_n(g)$ or Q(g) where g is a new element such that $g^2 = g$ and
$Q(g) = \{a + bg \mid a, b \in Q \text{ and } g^2 = g\}$. This task of constructing and studying special dual like numbers non associative loop ring using any loop L is also left as an exercise to the reader.

Further to construct non associative ring of special quasi dual numbers also one can use a loop L and a special quasi dual number ring Q(g) (or R(g) or C(g) or Z(g) or
$Z_n(g)) = \{a + bg \mid a, b \in Q$ with g a new element; $g^2 = -g\}$. Q(g)L (or R(g)L or $Z_n(g)$L or C(g)L or Z(g)L) will be a non associative loop ring of special quasi dual numbers.

This work is also a matter of routine and hence this task is left as an exercise to the reader.

It is pertinent to note the following.

Suppose $R(g, g_1, g_2) = \{a + bg + cg_1 + dg_2 \mid a, g_1$ and g_2 are new elements that that $g_2 = 0$, $g_1^2 = g_1$ and $g_2^2 = -g_2$ with $gg_1 = g_2$ (or g or 0 or g_1), $g_1g_2 = g_1$ (or g_2 or g or 0) and $gg_2 = g$

(or g_1 or g_2 or 0); a, b, c, d $\in$ R} be the strong mixed special dual number ring.

(R reals can be replaced by Q or Z or Z_n or C and all results hold good).

Clearly $R(g, g_1) \subseteq R(g, g_1, g_2)$,
$R(g, g_2) \subseteq R(g, g_1, g_2)$ and
$R(g_1, g_2) \subseteq R(g_1, g_2, g)$.
$R(g) \subseteq R(g, g_1)\ (R(g, g_2)) \subseteq R(g, g_1, g_2)$.

$R(g_1) \subseteq R(g, g_1)$ (or $R(g_1, g_2)$) $\subseteq R(g, g_1, g_2)$,
$R(g_2) \subseteq R(g, g_2)$ (or $R(g_1, g_2)$)
$= R(g, g_1, g_2)$.

So if we study $R(g, g_1, g_2)$ all other six subrings are contained properly in $R(g, g_1, g_2)$.

We give examples of a non associative mixed ring using a loop and the reader is expected to develop all other related properties.

Example 4.70: Let $S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in R, 1 \leq i \leq 4,\ g_1^2 = 0,\ g_2^2 = g_2,\ g_3^2 = -g_3$ where $g_1 = 6$, $g_2 = 4$ and $g_3 = 3$ are in Z_{12}; $g_1g_2 = 0$, $g_1g_3 = g_1$ and $g_2g_3 = 0\}$ be the ring of strong mixed special dual numbers. L be a loop given by the following table:

o	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	5	4	3	2
2	2	3	e	1	5	4
3	3	5	4	e	2	1
4	4	2	1	5	e	3
5	5	4	3	2	1	e

SL is the loop ring called the non associative strong mixed special dual number ring.

Clearly S contain all the six types of subrings of dual numbers.

Example 4.71: Let P = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in Z_7; 1 \leq i \leq 4; g_1 = 20, g^2 = 16$ and $g_3 = 15 \in Z_{40}$, $g_1^2 = 0 \pmod{40}$, $g_2^2 = g_2 \pmod{40}$ and $g_3^2 = -g_3 \pmod{40}$, $g_1g_2 = 0 \pmod{40}$, $g_3g_1 = g_1 \pmod{40}$; $g_2g_3 = 0 \pmod{40}\}$ be the strong mixed dual number ring of finite order. Let L be a loop given by the following table.

$$L = \{e, 1, 2, \ldots, 9\}$$

o	e	1	2	3	4	5	6	7	8	9
e	e	1	2	3	4	5	6	7	8	9
1	1	e	9	8	7	6	5	4	3	2
2	2	3	e	1	9	8	7	6	5	4
3	3	5	4	e	2	1	9	8	7	6
4	4	7	6	5	e	3	2	1	9	8
5	5	9	8	7	6	e	4	3	2	1
6	6	2	1	9	8	7	e	5	4	3
7	7	4	3	2	1	9	8	e	6	5
8	8	6	5	4	3	2	1	9	e	7
9	9	8	7	6	5	4	3	2	1	e

be the loop of order 10. PL be the loop ring, PL is a non associative general ring of strong mixed dual numbers of finite order.

Now we can construct groupoids G and using these dual number rings or mixed dual number ring or special dual like number ring or special quasi dual number ring and their mixed combinations of dual ring we can build non associative dual number rings.

We will illustrate by some examples.

Example 4.72: Let
$S = R(g) = \{a + bg \mid a, b \in R, g^2 = 0, g$ a new element$\}$ be the dual number ring. Let $G = (Z_{26}, *, (3, 2))$ be the groupoid of order 26. SG be the groupoid ring of the groupoid G over S. SG is a non associative dual number ring.

Example 4.73: Let $M = Z_{20}(g, g_1) = \{a + bg + cg_1 \mid a, b, c \in Z_{20}, g = 6, g_1 = 4 \in Z_{12}; g^2 = 0, g_1^2 = 4, g_1g = 0 \pmod{12}\}$ be the dual number ring. $G = \{(C(Z_{19}), *, (3, 4i_F)\}$ be the groupoid. MG the groupoid ring of G over M. MG is a non associative mixed dual number ring.

Example 4.74: Let $T = \{a_1 + a_2g + a_3g_1 \mid a_i \in Z, 1 \leq i \leq 3, g = 7, g_1 = 14 \in Z_{42}, g^2 = g, g_1^2 = -g_1, gg_1 = g_1\}$ be the mixed special quasi dual number ring $G = \{Z_{72}, *, (13, 0)\}$ be the groupoid. TG be the groupoid ring of G over T. TG is a non associative mixed special quasi dual number.

Example 4.75: Let $S = \{a_1 + a_2g + a_3g_1 + a_4g_2 \mid a_i \in Z_{14}, 1 \leq i \leq 4, g = 6, g_1 = 3, g^2 = 4 \in Z_{12}, g^2 = 0, g_1^2 = -g_1, g_2^2 = g_2, gg_1 = 6 = g, gg_2 = 0, g_1g_2 = 0\}$ be the strong mixed dual number ring $G = \{Z_{15}, *, (1, 5)\}$ be the groupoid. SG be the groupoid ring of the groupoid G over the ring S. SG is a non associative strong mixed dual number ring.

Example 4.76: Let $W = \{a_1 + a_2g_2 + a_3g_1 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid a_i \in Z_7, 1 \leq i \leq 7, g_1 = 7, g_2 = 14, g_3 = 21, g_4 = 28, g_5 = 35 \in Z_{49}\}$ be the higher dimensional dual number ring.

$G = \{Z^+ \cup \{0\}, *, (7, 8)\}$ be the groupoid. WG be the groupoid ring of G over W. WG is a non associative higher dimensional ring of infinite order.

We can also build non associative dual number rings using just rings R, that is commutative rings with unit G be the dual number groupoid then RG the groupoid ring is the non associative dual number ring.

Example 4.77: Let R be the field of reals.
G = {a + bg | a, b $\in$ Z_8, g the new element; $g^2 = 0$, (3, 5), *} be the dual number groupoid. RG be the groupoid ring. RG is a non associative dual number ring.

Example 4.78: Let T = Q be the ring of rationals.
G = {a + bg | g is a new elements a, b $\in$ Z, $g^2 = 0$, (5, –3), *} be the groupoid of dual numbers. QG be the groupoid ring. QG is the non associative dual number ring.

Let x = 3 + 2(5 + 7g) + 12 (1+g) + 7/2 (2–g)
and y = –7 + (3–4g) + 5/2 (8 + 2g) be in QG.

$$x + y = -4 + 2\,(5+7g) + 12(1+g) + 7/2(2-g) + (3-4g) + 5/2\,(8+2g) \in QG.$$

$$x \times y = [3 + 2(5 + 7g) + 12\,(1+g) + 7/2\,(2-g)] \times [-7 + (3-4g) + 5/2\,(8+2g)]$$

$$= -21 - 14\,(5 + 7g) - 84\,(1+g) - 49/2\,(2-g) + 3\,(3-4g) + 12\,(1+g) * (3-4g) + (5 \times 3)/2\,(8+2g) + 5(5+7g) * (8+2g) + (12 \times 5)/2\,(1+g) * (8+2g) + 35/4\,(2-g) * (8+2g)$$

$$= -21 - 14\,(5 + 7g) - 84\,(1 + g) + -49/2\,(2-g) + 3\,(3-4g) + 2\,[5\,(5+7g) - 3\,(3-4g)] + 12\,(5\,(1+g) - 3\,(3-4g)] + 15/2\,(8+2g) + 5\,(5\,(5+7g)-3\,(8+2g)] + 30\,(5\,(1+g) - 3\,(8+2g)) + 30\,(5\,(1+g) - 3\,(8+2g)) + 35/4\,(5\,(2-g) - 3\,(8+2g))$$

$$= -21 - 14\,(5 + 7g) - 84\,(1+g) - 49/2\,(2-g) + 3(3-4g) + 2\,(16+47g) + 15/2\,(8+2g) + 12\,(-4 + 17g)\,5\,(1+2g) + 30\,(-19-g) + 35/4\,(-14 - 11g) \in QG.$$

Thus QG is a non associative dual number ring of infinite order.

Example 4.79: Let S = Z_9 be the ring of modulo integers.

G = {a + bg | $g^2 = 0$, a, b ∈ Z_7, g a new element, (2, 5), *} be the dual number groupoid SG be the groupoid ring of the groupoid G over the ring Z_9. SG is a non associative dual number ring of finite order.

Example 4.80: Let S = Z be the ring of integers. G = {a + bg_1 + cg_2 + dg_3 | a, b, c, d ∈ Z_{14}, $g_1 = 4$, $g^2 = 8$, $g_3 = 12 \in Z_{16}$, $g_1^2 = 0 \pmod{16}$, $1 \le i \le 3$} be the groupoid. ZG be the groupoid ring of the groupoid G over the ring Z. SG is the non associative ring of four dimensional dual numbers.

Example 4.81: Let S = Z_{20} be the ring of modulo integers G = {a + bg_1 + cg_2 | a, b, c ∈ Z_7, $g^2 = 3$, $g_1 = 4 \in Z_6$, $g_1^2 = g_1 \pmod 6$, $g_2^2 = g_2 \pmod 6$, $g_1g_2 \equiv 0 \pmod 6$, (2, 0), *} be the groupoid of special dual like numbers. SG be the groupoid ring. SG is a non associative special dual like number ring of finite order.

Example 4.82: Let S = Q be the field of rationals. G = {a + bg | a, b ∈ Z_{40}, g = 7 ∈ Z_{42}, $g^2 = g \pmod{42}$, *, (10, 20)} be the groupoid of special dual like numbers. SG be the groupoid ring. SG is a non associative special dual like number ring.

Example 4.83: Let G = {$a_1 + a_2g_1 + a_3g_2$ | $g_1 = 4$ and $g^2 = 3 \in Z_6$, $g_1^2 = 4 \pmod 6$, $g_2^2 = 3 \pmod 6$, $g_1g_2 = 0$, $a_j \in Z_{19}$, $1 \le j \le 3$; (7, 0), *} be the three dimensional special quasi dual groupoid. S = Z_{11} be the field of modulo integers. SG the groupoid ring. SG is the non associative special dual like number ring of finite order.

Example 4.84: Let M = Q be the ring of rationals.
G = {$a_1 + a_2g_1 + a_3g_2$ | $a_i \in Z_7$; $1 \le i \le 3$; $g_1 = 5$ and $g^2 = 6 \in Z_{10}$; $g_1^2 = g_1$, $g_2^2 = g_2$, $g_1g_2 = 0 \pmod{10}$; (3, 1), *} be the groupoid of

special dual like numbers. Z_3 be the field of integers. Z_3G be the groupoid ring of non associative special dual like numbers.

Example 4.85: Let G = $\{a_1 + a_2g + a_3g_1 \mid a_i \in Q, 1 \leq i \leq 3, g = 6$ and $g_1 = 4 \in Z_{12}$, $g^2 = 0$ and $g_1^2 = 4$ $g_1g = 0 \pmod{12}$, $(7, -3/13), * \}$ be the groupoid of mixed dual numbers. Z be the ring of integers. ZG be the groupoid ring of the groupoid G over the ring Z. ZG is a non associative mixed dual number ring of infinite order.

Example 4.86: Let G = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in Z_{19}; 1 \leq i \leq 4$, $g_1 = 20$, $g^2 = 16$ and $g_3 = 25 \in Z_{40}$, $(17, 0), *\}$ be the higher dimensional mixed dual number groupoid of finite order. Z_4 = S be the ring of modulo integers. SG be the groupoid ring of the groupoid G over the ring S. SG is a non associative higher dimensional mixed dual number ring of finite order.

Example 4.87: Let G = $\{a + bg \mid a, b \in Z_{11}, g = 3 \in Z_{12}$, $g^2 = -g \pmod{12}$, $(7, 4), *\}$ be the groupoid of special quasi dual numbers. P = Q be the field of rationals. PG be the groupoid ring of G over P. PG is the non associative special quasi dual number rings.

Example 4.88: Let
G = $\{a + bg \mid a, b \in Z; g = 15 \in Z_{40}, g^2 = -g \pmod{40}, (-7, -2), *\}$ be the groupoid of special quasi dual numbers. Z_3 be the field of modulo integers Z_3G be the groupoid ring of the groupoid G over the ring Z_3. Z_3G is the non associative special quasi dual number ring of infinite order.

Example 4.89: Let G = $\{(a_1, a_2, \ldots, a_{10}) \mid a_i = a + bg; a, b \in Z_{13}$; $g = 24 \in Z_{40}$, $g^2 = -g \pmod{40}$, $1 \leq i \leq 10$, $(7, 4), *\}$ be the groupoid of special quasi dual numbers. S = Z_{15} be the ring of modulo integers. SG be the groupoid ring is the non associative special quasi dual number ring of finite order.

Example 4.90: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i = a + bg,\ a, b \in Z_6,\ g = 8 \in Z_{12}, \right.$$

$g^2 = -g \pmod{12}$; $1 \leq i \leq 30$, $(3, 2)$, $*\}$ be the groupoid of special quasi dual numbers. $F = Z_2$ be the field of integers modulo two. FS be the groupoid of ring of S over F. FS is a non associative special quasi dual number ring of finite order.

Example 4.91: Let

$$G = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i = d_1 + d_2g_1 + d_3g_2 \text{ with } d_j \in Z_{27},\ 1 \leq j \leq 3, \right.$$

$1 \leq i \leq 5$, $g_1 = 24$ and $g^2 = 15 \in Z_{40}$, $g_1^2 = -g_1 \pmod{40}$, $g_2^2 = -g_2 \pmod{40}$; $(20, 0)$, $*\}$ be the groupoid of special quasi dual numbers. $S = Q$ be the ring of rationals. SG be the groupoid ring of the groupoid G over the ring S. SG is a non associative special quasi dual number ring of infinite order of dimension three.

Example 4.92: Let $G = \{a + bg \mid a, b \in \langle Z_{21} \cup I\rangle, g = 20 \in Z_{30}, g^2 = -g \pmod{30}, (3I, 2I+7), *\}$ be the neutrosophic groupoid. R be the ring of reals. RG be the groupoid ring. RG is a non associative special quasi dual like numbers of infinite order.

Example 4.93: Let $G = \{a + bg \mid a, b \in Z_5, g = 14 \in Z_{21}, g^2 = -g \pmod{21}, (3, 2), *\}$ be the groupoid. $S = \langle R \cup I\rangle$ be the neutrosophic ring of reals. RG be the groupoid ring. RG is a non associative neutrosophic ring of special quasi dual ring number of infinite order.

Example 4.94: Let G = $\{a + bg \mid a, b \in \langle Z_{24} \cup I\rangle, g = 20 \in Z_{30}, g^2 = -g \pmod{30}, *, (7I+3, 2+I)\}$ be the groupoid of special quasi dual numbers. S = $\langle Z_{11} \cup I\rangle$ be the neutrosophic ring of modulo integers. SP be the groupoid ring of the groupoid P over the ring S. SP is a strong neutrosophic special quasi dual number non associative ring of finite order.

Example 4.95: Let G = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid g_1 = 20, g_3 = 15 \text{ and } g_2 = 16 \in Z_{40}, a_j \in Z_{15}, 1 \leq j \leq 4, (7, 8), *\}$ be the strong mixed special dual number groupoid. S = Z_3 be the ring of modulo integers SG be the groupoid ring of G over S. SG is the non associative strong mixed special dual like ring of finite order.

Example 4.96: Let G = $\{(a_1, a_2, a_3) \mid a_i = x + yg_1 + zg_2, 1 \leq i \leq 3, x, y \in Z_{14}, g = 20 \text{ and } g^2 = 15 \in Z_{40}, g^2 = 0 \pmod{40}, g_2^2 = -g_2 \pmod{40}, *, (7, 2)\}$ be the groupoid of mixed special dual numbers. S = Z_7 be the field of modulo integers SG be the groupoid ring. SG is the non associative mixed special dual number ring of finite order.

Example 4.97: Let

$$G = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i = a + bg_1 + cg_2 + dg_3;\ 1 \leq i \leq 9;\right.$$

a, b, c, d $\in Z_{17}$, $g_1 = 10$, $g^2 = 15$ and $g_3 = 16 \in Z_{20}$, $g_1^2 = 0 \pmod{20}$, $g_2^2 = -g_2 \pmod{20}$ and $g_3^2 = g_3 \pmod{20}$, $g_1g_2 = g_1$, $g_1g_3 = 0$, $g_2g_3 = 0 \pmod{20}$, *, (8, 0)} be the strong mixed special dual number groupoid. S = $\langle Z \cup I\rangle$ be the ring of neutrosophic integers, SG be the groupoid ring. SG is the non associative strong mixed special dual number neutrosophic ring.

Now having seen all types of non associative rings we leave it as an exercise for the reader to work with special elements like idempotents, S-idempotents, units, S-units, zero divisors, S-zero divisors, subrings, S-subrings ideals and S-ideals of these rings.

Now we just illustrate a few examples of non associative semivector spaces and non associative semilinear algebras and non associative linear algebras.

Example 4.98: Let
$M = \{a + bg \mid a, b \in Z_{23}, g = 3 \in Z_9$, g a new element, $(8, 3), *\}$ be a groupoid. M is an abelian group under addition modulo 23. M is a vector space over the field Z_{23}.

Now if on M we define * M is a non associative linear algebra of dual numbers over Z_{23}.

Example 4.99: Let $M = \{(a_1, a_2, a_3, \ldots, a_{10}) \mid a_i = x_1 + x_2g_1 + x_3g_2, x_j \in Z_{19}; 1 \leq i \leq 10, 1 \leq j \leq 3, g_1 = 6, g^2 = 4 \in Z_{12}, g_1^2 = 0$ (mod 12), $g_2^2 = g_2$ (mod 12), $g_1g_2 = 0$ (mod 12), $(10, 0), *\}$ be the groupoid of mixed dual numbers. M is a mixed dual number non associative linear algebra over the field Z_{19}.

Example 4.100: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3; \right.$$

$1 \leq i \leq 15$, $x_j \in Z_{41}$, $1 \leq j \leq 4$; $g_1 = 20$, $g^2 = 16$ and $g_3 = 15$ in Z_{40}, $g_1^2 = 0$ (mod 40), $g_2^2 = 16 = g_2$ (mod 40), $g_3^2 = 25 = -g_3$ (mod 40); $g_1g_2 = 0$ (mod 40), $g_1g_3 = g_1$ (mod 40), $g_2g_3 = 0$ (mod 40), $(0, 25), *\}$ be the strong mixed special quasi dual number groupoid. P is a strong mixed special quasi dual number non associative linear algebra over the field Z_{41}.

Example 4.101 : Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2 g_1 + x_3 g_2;\ x_j \in Z_{17},\ 1 \le j \le 3, \right.$$

$g_1 = 20$, $g_3 = 15$ in Z_{40}. $g_1^2 = 0 \pmod{40}$, $g_3^2 = -g_3 \pmod{40}$ and $g_1 g_3 = g_1 \pmod{40}$, (10, 2), *} be the groupoid of mixed special quasi dual numbers. S is a non associative linear algebra of mixed special quasi dual number over the field Z_{17}.

Example 4.102: Let $S = \{a_1 + b_1 g_1 \mid a_1, b_1 \in Z^+ \cup \{0\}, g_1 = 4 \in Z_{16}$ be the new element, (3, 8), *} be a groupoid of dual numbers. S is a non associative semilinear algebra of dual numbers over the semifield $F = Z^+ \cup \{0\}$.

Example 4.103: Let
$S = \{a + bg \mid a, b \in Z^+ \cup \{0\}, g = 4 \in Z_{12}$, (3, 0), *} be the groupoid of special dual like numbers. T is a non associative semilinear algebra of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Example 4.104: Let $S = \{(a_1, a_2, a_3, \ldots, a_{16}) \mid a_i = x + yg, 1 \le i \le 16, x, y \in Q^+ \cup \{0\}; g = 3 \in Z_{12}, g^2 = -g \in Z_{12}$, (8/3, 7/11), *} be a non associative semilinear algebra of special quasi like dual numbers over the semifield $Q^+ \cup \{0\}$.

Example 4.105: Let

$$X = \left\{ \begin{bmatrix} a_1 & a_2 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \,\middle|\, a_i = d_1 + d_2 g_1 + d_3 g_2;\ 1 \le i \le 12, \right.$$

$d_j \in Q^+ \cup \{0\}$, $1 \le j \le 3$; $g_1 = 6$ and $g_2 = 4 \in Z_{12}$, (3/7, –1), *} be the non associative semilinear algebra of mixed dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 4.106: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4, \right.$$

$1 \leq i \leq 20$, $x_j \in Z^+ \cup \{0\}$, $1 \leq j \leq 5$; $g_1 = 5$, $g^2 = 6$; $g_1^2 = 10 \pmod{15} = -g_1$, $g_2^2 = g_2 \pmod{15}$, $g_3 = 9$, $g_3^2 = g_2 = -g_3 \pmod{15}$, $g_4 = 10$, $g_4^2 = 10 \pmod{15}$; $10, 5, 6, 9 \in Z_{15}$, $(8, 0)$, $*\}$ be the groupoid of special quasi dual numbers of dimension five. S is a non associative semilinear algebra of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 4.107: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3;\ 1 \leq i \leq 12, \right.$$

$x_j \in Q^+ \cup \{0\}$; $1 \leq j \leq 4$, $g_1 = 20$, $g^2 = 16$ and $g_3 = 15 \in Z_{40}$, $g_1^2 = 0 \pmod{40}$, $g_2^2 = g_2 \pmod{40}$ and $g_3^2 = -g_3 \pmod{40}$. $g_1g_2 = 0 \pmod{40}$, $g_1g_3 = g_1 \pmod{40}$, $g_2g_3 = 0 \pmod{40}$, $(7/3, 5/7)$, $*\}$ be the non associative semilinear algebra of strong mixed special dual numbers over the semifield $Z^+ \cup \{0\}$.

Now having seen examples of non associative structures like linear algebras and semilinear algebras using dual numbers, special dual like numbers, special quasi dual numbers, mixed dual numbers, special mixed dual number and strong special mixed dual numbers. We can derive all properties of linear algebra and semilinear algebra as a matter of routine. This task is left as an exercise to the reader. Using dual number square matrices we can get the eigen vectors to be dual number and if we define Smarandache non associative linear algebra over Q(g) or R(g) or $Z_p(g)$ or $Z_n(g)$, Z_n a S-ring (or special dual like numbers, special quasi dual numbers), then the eigen values and eigen vector associated with them can also be dual numbers (of special dual like numbers or special quasi dual numbers) according as the S-ring which we use.

Finally if we use mixed dual number S-rings as $Q(g, g_1)$ or $Q(g, g_2)$ or $Q(g_1, g_2)$ or $Q(g, g_1, g_2)$, then also the S-linear algebra will have for its associated operator the eigen values and eigen vectors can be dual numbers, special dual like numbers, special quasi dual numbers and their mixed components.

This task is also left as exercise to the reader. However we give few examples of S-linear algebras and S-semilinear algebras.

Example 4.108: Let $R(g) = \{a + bg \mid g = 4 \in Z_{16}, a, b \in R\}$ be the ring of dual numbers.

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i = a + bg; \, 1 \le i \le 12; \right.$$

$a, b \in R$, $g = 4 \in Z_{16}$, $(\sqrt{3}, \sqrt{5} + 8)$, $*\}$ be a non associative Smarandache linear algebra of dual numbers over the S-ring $R(g)$.

Example 4.109: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_5 \\ a_6 & a_7 & \cdots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{46} & a_{47} & \cdots & a_{50} \end{bmatrix} \middle| \, a_i = x + yg; \, 1 \le i \le 50, \right.$$

$g = 6 \in Z_{36}$, $x, y \in Q(g)$, $(3/2, -2)$, $*\}$ be a non associative Smarandache linear algebra of dual numbers over the S-ring $Q(g) = \{a + bg \mid a, b \in Q; g = 6 \in Z_{36}, g^2 = 0 \pmod{36}\}$.

Example 4.110: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \end{pmatrix} \middle| \; a_i = x + yg \text{ with } x, y \in Q; \right.$$

$1 \leq i \leq 15$, $4 = g \in Z_{12}$, $g^2 = g$, $(17, 5/4)$, $*\}$ be a non associative linear algebra special dual like numbers over the S-ring $Q(g) = \{a + bg \mid a, b \in Q, g = 4 \in Z_{12}\}$.

Example 4.111: Let

$$T = \left\{ \sum_{i=0}^{9} a_i x^i \middle| \; a_i = x + yg + zg_1 \in Q(g, g_1);\; g = 6,\; g_1 = 4 \in Z_{12}, \right.$$

$x, y, z \in Q$, $*$, $(8, -1)\}$ be a S-linear algebra of mixed special dual numbers over the S-mixed special dual number ring $Q(g, g_1) = \{x + yg + zg_1 \mid x, y, z \in Q, g = 6 \text{ and } g_1 = 4 \in Z_{12}\}$.

Example 4.112: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3,\; 1 \leq i \leq 9; \right.$$

$x_j \in R$; $1 \leq j \leq 4$, $g_1 = 20$, $g_2 = 15$, $g_3 = 16 \in Z_{40}$, $(\sqrt{7}, -5)$, $*\}$ be the non associative Smarandache linear algebra of strong mixed special dual numbers over the S-ring;

$Q(g_1, g_3g_2) = \{a + bg_1 + cg_3 + dg_2 \mid a, b, c, d \in Q, g_1 = 20, g_2 = 15 \text{ and } g_3 = 16 \in Z_{40}\}$.

Example 4.113: Let $S = \{(a_1, a_2, a_3, a_4) \mid a_i = x_1 + x_2g_1 + x_3g_2;$ $1 \leq i \leq 4$; $x_j \in Q^+ \cup \{0\}$, $1 \leq j \leq 3$; $g_1 = 20$, $g^2 = 16 \in Z_{40}$, $(7/2,$

3), *} be a non associative Smarandache semilinear algebra of mixed special dual numbers over the S-semiring $(Q^+ \cup \{0\})\,(g_1, g_2) = \{x_1 + x_2g + x_3g_2 \mid x_i \in Q^+ \cup \{0\};\ 1 \leq i \leq 3,$ $g_1 = 20,\ g_2 = 16 \in Z_{40},\ g_1^2 = 0 \pmod{40},\ g_2^2 = 16 \pmod{40},$ $g_1g_2 = 20 \times 16 = 0 \pmod{40}\}$.

Example 4.114: Let

$$W = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix}\right|\ a_i \in (Z^+ \cup \{0\})\,(g_1, g_2, g_3, g_4)\right.$$

$= \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_j \in Z^+ \cup \{0\};\ 1 \leq j \leq 5;$ $g_1 = 20,\ g_2 = 16$ and $g_3 = 15,\ g_4 = 25 \in Z_{40},\ g_1^2 = 0 \pmod{4},$ $g_2^2 = g_2 \pmod{40},\ g_3^2 = g_3 \pmod{40},\ g_4^2 = g_4 \pmod{40}\}$ be the non associative S-semilinear algebra of strong mixed special dual number over the S-semiring $(Z^+ \cup \{0\})\,(g_1, g_2, g_3, g_4)$.

Now all properties can be derived and some them are given as problems in the last chapter of this book.

Chapter Five

APPLICATIONS OF SPECIAL QUASI DUAL NUMBERS AND THEIR MIXED STRUCTURES

Dual numbers find a host of applications. Authors are sure that special dual like numbers will also find lot of applications in due course of time when nilpotents elements are replaced by idempotents.

Natural sources of idempotents are lattices, matrices with entries 1 or 0.

Neutrosophic element I is an idempotent.

Further while applying one can also used mixed dual numbers $x = a + bg + cg_1$ where a, b, c are reals and $g^2 = 0$, $g_1^2 = g_1$, $gg_1 = 0$ (or g_1 or g). So using both simultaneous by one can find uses of this notion also.

Further the special quasi dual numbers $x = a + bg_1 + c(-g_1)$ are such that g_1 is a new element with $g_1^2 = -g_1$ so that $g_1^2 = -g_1$, $g_1^3 = -g_1^2 = g_1$, $g_1^4 = -g_1$, $g_1^5 = g_1$ so all even powers are negative that is $g_1^2 = g_1^4 = g_1^6 = g_1^8 = \ldots = g_1^{2n} = -g_1$ and all odd powers are positive that is $g_1^3 = g_1^5 = g_1^7 = \ldots = g_1^{2n+1} = g_1$. So this property may also find some new applications.

However the only sources of getting such new elements are $-I$, for $(-I)^2 = I^2 = I = -(-I)$ and -1, for $(-1)^2 = 1 = -(-1)$.

Further the set of modulo integers Z_n (n a composite number) happens to be a rich source of such special quasi dual number components g with $g^2 = -g \pmod{n}$.

Clearly if $n = 4m$ we are guaranteed of such elements in Z_n. The main use of Z_n is we can construct the strong mixed special dual numbers. For take Z_{12}, $g = 3 \in Z_{12}$ is such that $g^2 = 9 = -3 \pmod{12}$, $g_1 = 4 \in Z_{12}$ is such that $g_1^2 = g_1 \pmod{12}$ and $g_2 = 6 \in Z_{12}$ is such that $g_2^2 = 0 \pmod{12}$.

So $x = a + bg + cg_1 + dg_2$ is a strong mixed special dual number (a, b, c and d are all reals).

Further $gg_1 = 0 \pmod{12}$, $g.g_2 = g_2 \pmod{12}$, $g_1 g_2 = 0 \pmod{12}$.

So these strong mixed special dual numbers has all the three types of duals numbers and properties associated with them. So by suppressing one or two of them the property of the other can be studied in case of necessity.

We can also take only two dual numbers and also form the higher dimensional structures. These also will find applications in different fields.

Chapter Six

SUGGESTED PROBLEMS

In this chapter the authors introduce over 100 problems. Some of the problems are at research level and are challenging. Further as the topic dealt with this book is new these problems will enable the reader to have a better grip of this topic.

1. Obtain some special properties associated with quasi special dual numbers.

2. If $x = a + bg$, $a, b \in Q$, $g^2 = -g$ is a special quasi dual number then if $h = -g$, $(-g)^2 = h^2 = h$; prove.

3. Does Z_9 contain a g so that $x + yg$ is a quasi special dual number? $(x, y \in R)$.

4. Does Z_{16} contain a g so that $x + yg$ is a quasi special dual number?

5. Suppose $g \in Z_{15}$ is a quasi special dual number component. Find g. Can Z_{15} have more than one g?

6. Prove Z_p cannot contain any quasi special dual number component (p a prime).

7. Prove Z_{p^n} (n>1) p, a prime cannot contain any quasi special dual number component.

8. Let S = Z_{12}, find the number of special dual number components of Z_{12}.

9. Obtain some interesting applications of quasi special dual numbers.

10. Let S = {a + bg | a, b ∈ Z_{10}, g = 8 ∈ Z_{12}, g^2 = 64 = 4 (mod 12) that is g^2 = –g} be the group under '+'.
    (i) Find order of S.
    (ii) Find subgroups of S.
    (iii) What is the order of a ∈ S for every a in S?
    (iv) Is (S, ×) a semigroup?
    (v) Can (S, ×) have ideals?

11. Prove in problem (10) when S is a ring.
    (i) Can S be a field?
    (ii) Find ideals of S.
    (iii) Can S have subrings which are not ideals?

12. Let P = {a + bg | a, b ∈ Z, g = 15 ∈ Z_{40}, g^2 = 225 = 25 (mod 40) i.e., g^2 = –g} be a ring of quasi special dual numbers.
    (i) Is P a domain?
    (ii) Can P have zero divisors?
    (iii) Can P have subrings which are not ideals?
    (iv) Can P have S-idempotents?
    (v) Is P a S-ring?

13. Let M = {a + bg | a, b ∈ Z_3, g = 2 ∈ Z_6, 2^2 = 4 = –g} be the ring of special quasi dual numbers.
    (i) Find the number of elements in M.
    (ii) Is M a S-ring?
    (iii) Can M have S-idempotents?
    (iv) Can M have S-zero divisors?

14. Let
$S = \{a + bg \mid a, b \in Z_{12}, 14 = g \in Z_{21}, g^2 = 14^2 = -g = 7\}$ be the ring of special quasi dual numbers.
(i) Find o(S).
(ii) Find subrings of S which are not ideals.
(iii) Can S have S-ideals?
(iv) Can S be a S-ring?
(v) Can S have S-idempotents?

15. Let $A = \{(a_1, a_2, \ldots, a_{12}) \mid a_i = x_i + y_i g$ with $x_i, y_i \in Z_{23}, 1 \leq i \leq 12, g = 15 \in Z_{40}\}$ be the ring of special quasi dual numbers.
(i) Find order of A.
(ii) Can A have S-ideals?
(iii) Is A a S-ring?
(iv) Find the zero divisor graph of A.
(v) Can A have S-zero divisors?

16. Let $P = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{bmatrix}\right| a_i = x_i + y_i g \text{ where } x_i, y_i \in Q, 1 \leq i \leq 10,\right.$ $g = 4 \in Z_{10}\}$ be the special quasi dual number ring under the natural product $\times_n$.
(i) Find S-zero divisors if any in P.
(ii) Prove P is a S-ring.
(iii) Can P have S-subrings which are not S-ideals?

17. Let $S = Z_{252}$. Find all quasi special dual numbers component of S.

18. Let Z_n be the ring of modulo integers, n a composite number. If $S = \{\text{set of all } g \in Z_n, g^2 = -g\} \subseteq Z_n$. What is the algebraic structure enjoyed by S?

19. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \middle| \, a_i = x_i + y_i g,\ 1 \leq i \leq 20,\ x_i,\right.$ $y_i \in Q$, $g = 8 \in Z_{12}\}$ be the ring of quasi special dual like numbers.

    (i) Find ideals of M.
    (ii) Prove M has zero divisors.
    (iii) Does M contain a zero divisor which is not a S-zero divisor?
    (iv) Can M have S-idempotents?

20. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \, a_i = x_i + y_i g \text{ where } x_i, y_i \in \right.$ Z_{25}, $1 \leq i \leq 16$, $g = 2 \in Z_6\}$ be the non commutative ring of quasi special dual numbers.

    (i) Can P have right zero divisors, which are not left zero divisors?
    (ii) Can P have S-units?
    (iii) Can P have units which are not S-units?
    (iv) Find right ideals of P which are not left ideals and vice versa.

21. Let S = $\{a + bg \mid a, b \in Q, g = 15 \in Z_{40}\}$ be a vector space of special quasi dual numbers over Q.

    (i) Find a basis of S over Q.
    (ii) Write S as direct sum of subspaces over Q.
    (iii) Find Hom (S, S).
    (iv) For some $T \in$ Hom (S, S); find eigen values and eigen vector associated with that T.

22. Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_8 \end{bmatrix} \middle| \; a_i = x_i + y_i g,\ x_i, y_i \in Z_7,\ 1 \leq i \leq 8;\ g = 8 \in Z_{12} \right\}$$

be a vector space of special quasi dual numbers over the field Z_7.

(i) Find a basis of P over Z_7.
(ii) What is the basis of P over Z_7?
(iii) Find the number of elements in P.
(iv) Find the algebraic structure enjoyed by Hom(P, P).
(v) Define $f : P \rightarrow Z_7$.

23. Obtain some special properties enjoyed by vector space of special quasi dual numbers.

24. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i = x_i + y_i g,\ x_i, y_i \in Z_{12},\ 1 \leq i \leq 12,\ g = 14 \in Z_{21} \right\}$ be the Smarandache vector space of special quasi dual numbers over the S-ring Z_{12}.

(i) Find the number of elements in M.
(ii) Find dimension of M over Z_{12}.
(iii) Find a basis of M over Z_{12}.
(iv) Write M as a direct sum of S-subspaces over Z_{12}.

25. Let $P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ with } x_i, y_i \in Z_{40}, 1 \leq i \leq 5, \right.$ $g = 8 \in Z_{12}, g^2 = -g = 4 \in Z_{12}\}$ be the S-vector space of special quasi dual numbers over the S-ring Z_{40}.

(i) Find a basis of P over Z_{40}.
(ii) Can P be made into a S-linear algebra?
(iii) Find a basis of P as a S-linear algebra over Z_{40}.
(iv) Compare the basis (i) and (iii)
(v) Write P as a direct sum of S-subspaces.

26. Let $P = \{a + bg \mid a, b \in Q, g = (-1, -1, -1, -1, -1, -1)$; so that $g^2 = (1, 1, 1, 1, 1, 1) = -g\}$ be the vector space of special quasi dual numbers over the field Q.

(i) Find dimension of P over Q.
(ii) Find a basis of P over Q
(iii) Write P as a direct sum of subspaces.
(iv) Find Hom(P, P).
(v) Find the structure of L(P, Q).

27. Let $M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i = x_i + y_i g, x_i, y_i \in Z_{17}, 1 \leq i \leq 10; \right.$

$$g = \begin{pmatrix} -I & -I & -I & -I & -I \\ -I & -I & -I & -I & -I \end{pmatrix}; \; g^2 = \begin{pmatrix} I & I & I & I & I \\ I & I & I & I & I \end{pmatrix} = -g\}$$

be the vector space of special quasi dual numbers over the field Z_{17}.

(i) Find the number of elements in M over Z_{17}.

(ii) Find a basis and dimension of M over Z_{17}.
(iii) Find the cardinality of Hom (M, M).
(iv) Find the number of elements in L (M, Z_{17}).

28. Let S = $\{(a_1, a_2, a_3, a_4) \mid a_i = x_i + y_i g$ where

$$g = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \text{ such that } g^2 = -g = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$x_i, y_i \in R, 1 \leq i \leq 4\}$ be a vector space of special quasi dual numbers over the field R.

(i) Find dimension of S over R.
(ii) Find dimension of Hom (S, S) over R.
(iii) Find L (S, R).

29. Let W = $\left\{ \begin{bmatrix} a_1 & a_4 & a_5 & a_{10} \\ a_2 & a_6 & a_7 & a_{11} \\ a_3 & a_8 & a_9 & a_{12} \end{bmatrix} \,\middle|\, a_i = x_i + y_i g \right.$ where

$$g = \begin{bmatrix} -I & -I & -I \\ -I & -I & -I \\ -I & -I & -I \end{bmatrix}, g \times_n g = \begin{bmatrix} I & I & I \\ I & I & I \\ I & I & I \end{bmatrix} = -g, x_i, y_i \in Z_{11},$$

$1 \leq i \leq 12\}$ be the quasi special dual linear algebra over the field Z_{11}.

(i) Find a basis of W over Z_{11}.
(ii) What is the dimension W over Z_{11}?
(iii) Write W as a pseudo direct sum of subspaces of W over Z_{11}.

30. Let W = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, \right.$

$y_i \in Z_{19},\ 1 \leq i \leq 10,\ g = \begin{bmatrix} -I & -I & -I & -I & -I & -I \\ -I & -I & -I & -I & -I & -I \\ -I & -I & -I & -I & -I & -I \\ -I & -I & -I & -I & -I & -I \end{bmatrix},$

$g^2 = -g\}$ be the vector space of quasi special dual numbers.

(i) Can M be made into a linear algebra?
(ii) Does there exist a difference in dimension of M as a vector space over Z_{19} and as a linear algebra over M?

31. Let S = $\{a + bg + cg_1 \mid a, b, c \in Z^+ \cup \{0\}$, g = (–1, –1, –1, –1, –1, –1, –1) and g_1 = (1, 1, 1, 1, 1, 1, 1) with $g^2 = g_1$, $g_1 g = g g_1 = g$, be the semivector space of complete quasi special dual pair over the semifield $Z^+ \cup \{0\}$.

(i) Find dimension off S over $Z^+ \cup \{0\}$.
(ii) Can S have more than one basis?

32. Let V = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i = x_i + y_i g + z_i g_1 \text{ where } x_i, y_i, z_i \in \right.$

$R^+ \cup \{0\}$, $1 \leq i \leq 5$, g = 15, $g_1 = 25 \in Z_{40}\}$ be a complete quasi special dual pair semivector space over $R^+ \cup \{0\}$.

(i) What is the dimension of V over $R^+ \cup \{0\}$?

(ii) If $R^+ \cup \{0\}$ is replaced by $Q^+ \cup \{0\}$ what will be the dimension of V over $Q^+ \cup \{0\}$.
(iii) Find basis of V over $R^+ \cup \{0\}$ and over $Q^+ \cup \{0\}$. Study the difference in them.
(iv) Find Hom(V, V).

33. Let M = $\left\{\begin{pmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \end{pmatrix} \middle| \; a_i = d_i + c_ig + b_ig_1 \text{ where}\right.$

$g = 2 \in Z_6$, $g_1 = 4 \in Z_6$, $d_i, c_i, b_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 24\}$ be the complete quasi special dual pair semilinear algebra under the natural product $\times_n$ over the semifield $Z^+ \cup \{0\}$.

(i) Find a basis of M over $Z^+ \cup \{0\}$.
(ii) Can M be written as $W + W^\perp$? ($W^\perp$ the orthogonal complement of W)
(iii) Find for a $T \in$ Hom (M, M) the associated eigen values and eigen vector.

34. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_j = x_i + y_ig + z_ig_1 \text{ with } x_i, y_i, z_i\right.$

$\in R^+ \cup \{0\}$, $1 \leq i \leq 3$, $1 \leq j \leq 9$, $g = 8$ and $g_1 = 4 \in Z_{12}\}$ be the non commutative semilinear algebra of complete quasi special dual pair over the semifield $Z^+ \cup \{0\}$.

(i) Is P an infinite dimensional semilinear algebra?
(ii) Find S = Hom (P, P).
Is S finite dimensional or infinite dimensional over $Z^+ \cup \{0\}$?

35. Let S = $\{(a_1, a_2) \mid a_j = x_i + y_ig + z_ig_1$ with $x_i, y_i, z_i \in Q$; $1 \leq j \leq 2$; $g = (-1, -1, -1, -1, -1, -1)$ and $g_1 = (1, 1, 1, 1, 1, 1)\}$ be a vector space of complete quasi special dual pair numbers over the field Q.

Is S isomorphic with P = $\{(a_1, a_2)$ where $a_j = x_i + y_i g$, $x_i, y_i \in Q$, $1 \leq j \leq 2$, $g = (-1, -1, -1, -1, -1, -1)$ and $g_1 = (1, 1, 1, 1, 1, 1)\}$, P is also a vector space of quasi special dual numbers over the field Q?

36. Let $M = \left\{\begin{bmatrix} a_1 & a_4 & a_7 & a_{10} & a_{13} & a_{16} \\ a_2 & a_5 & a_8 & a_{11} & a_{14} & a_{17} \\ a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} \end{bmatrix} \middle| \; a_j = x_i + y_i g + \right.$

$z_i g_1$, $1 \leq j \leq 18$ where $g = \begin{bmatrix} -I & -I \\ -I & -I \\ -I & -I \\ -I & -I \\ -I & -I \end{bmatrix}$ and $g_1 = \begin{bmatrix} I & I \\ I & I \\ I & I \\ I & I \\ I & I \end{bmatrix}$, x_i,

$y_i, z_i \in Z\}$ be the complete quasi special dual number

ring. $N = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \middle| \; a_j = x_i + y_i g \right.$ with

$1 \leq j \leq 18$, $g = \begin{bmatrix} -I & -I \\ -I & -I \\ -I & -I \\ -I & -I \\ -I & -I \end{bmatrix}\}$ be the quasi special dual

number ring. Is M isomorphic to N as rings?

37. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \middle| a_i = x_i + y_i g + z_i g_1,\ 1 \leq i \leq 16,\ x_i, y_i,\right.$

$z_i \in Z_{16}$, $g = 2$ and $g_1 = 4 \in Z_6\}$ be the complete special quasi dual number ring.

$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \middle| a_i = x_i + y_i g\ ,\ 1 \leq i \leq 16,\ x_i, y_i \in Z_{16},\right.$

$g^2 = 2 \in Z_6$, $g^2 = -g = 4\}$ be the special quasi dual number ring.

Prove M and P are isomorphic as rings.

38. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_j = x_i + y_i g + z_i g_1 \text{ where } x_i, y_i, z_i\right.$

$\in R$, $1 \leq i \leq 9$, $g = 2$ and $g_1 = 4 \in Z_6\}$ be the non commutative ring under usual product of matrices of complete quasi special dual number pair.

$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_j = x_i + y_i g + z_i g_1 \text{ where } x_i, y_i, z_i \in\right.$

R, $1 \leq i \leq 9$, $g = 2$ and $g_1 = 4 \in Z_6\}$ be the special quasi dual number ring under the natural product $\times_n$.
Can S and P be isomorphic? Justify your claim.

39. Find all special quasi elements in Z_{96}. Does this collection form a semigroup under $\times$?

40. Let Z_{720} be a ring of modulo integers. Find the extended semigroup of associated dual numbers.

41. Find the algebraic structure enjoyed by Hom(P, P) where

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{24} \end{bmatrix} \middle| \; a_j = x_i + y_i g + z_i g_1 \text{ where } x_i, y_i, \right.$$

$z_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 24$, $g = 15$ and $g_1 = 25 \in Z_{40}\}$ is the semivector space of complete quasi special dual pairs.

42. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_j = x_i + y_i g + z_i g_1 \text{ with} \right.$

$1 \leq i \leq 16$, $x_i, y_i, z_i \in Z^+ \cup \{0\}$, $g = 14 \in Z_{21}$ and $g_1 = 7\}$ be the semilinear algebra of complete quasi special dual pairs over the semifield $Z^+ \cup \{0\}\}$. Find the algebraic structure enjoyed by $L(S, Z^+ \cup \{0\})$.

43. Let $M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_j = x_i + y_i g + z_i g_1, \; x_i, y_i, z_i \in Z_{24}, g = 2, \right.$

$g_1 = 4 \in Z_6$, $1 \leq i \leq 10\}$ be a S-vector space of complete quasi special dual number pair over the S-ring Z_{24}.

(i) Find S-dimension of M over Z_{24}.
(ii) Find S-basis of M over Z_{24}.
(iii) Find the algebraic structure enjoyed by $L(M, Z_{24})$.

44. Let $T = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \end{bmatrix} \middle| \; a_j = x_i + y_i g + z_i g_1, x_i, y_i, z_i \right.$

$\in Q$, $1 \leq i \leq 24$, $g = 8$, $g_1 = 4 \in Z_{12}\}$ be a S-vector space of complete quasi special dual pair over the S-ring $Q(g, g_1) = \{a + bg + cg_1 \mid a, b, c \in Q, g = 8 \text{ and } g_1 = 4 \in Z_{12}\}$.

(i) Find S-dimension of T over $Q(g_1, g_2)$.
(ii) Find $L(T, Q(g_1, g_2))$.

45. Obtain some interesting properties about quasi special dual number of t-dimension ($t > 2$).

46. Does there exist neutrosophic quasi special dual numbers?

47. Let $p = \{a + bg_1 + cg_2 + dg_3$ where $a, b, c \in R$; $g_1 = (-I, -I, -I, -I)$, $g_2 = (-I, -I, 0, 0)$ and $g_3 = (0, 0, -I, -I)$, $I^2 = I\}$ be the four dimensional special quasi dual like number.

(i) Is P a semigroup under $\times$?
(ii) Is P a group under $+$?
(iii) Will $(P, +, \times)$ be a ring?
(iv) Is P a S-ring?
(v) Does P contain S-ideals?

48. Give an example of a 10 dimensional neutrosophic special quasi dual number ring of finite order.

49. Let $M = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3, a_i \in Z_9, 1 \leq i \leq 4$ where

$$g_1 = \begin{pmatrix} -I & -I & -I & -I \\ -I & -I & -I & -I \end{pmatrix}, \; g_2 = \begin{pmatrix} -I & -I & 0 & 0 \\ -I & -I & 0 & 0 \end{pmatrix},$$

$g_3 = \begin{pmatrix} 0 & 0 & -I & -I \\ 0 & 0 & -I & -I \end{pmatrix}$} be a ring of four dimensional neutrosophic special quasi dual number ring.

(i) Find the number of elements in M.
(ii) Is $o(M) = 9^4$?
(iii) Does M contain S-subrings which are not ideals?
(iv) Can M have S-zero divisors?
(v) Does M contains units which are not S-units?

50. Does there exists a ring of special quasi dual numbers which is not a S-ring?

51. Enumerate the special properties associated with special quasi dual number rings.

52. Can special quasi dual number semiring be constructed of any desired dimension?

53. What will be the minimum dimension of any special quasi dual number in a semiring?

54. Is it possible to construct a two dimensional special quasi dual number semiring? Justify!

55. Let $M = \{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Z^+ \cup \{0\}, 1 \leq i \leq 3,$ $g_1 = (-I, -I, -I, -I, -I)$, $g_1^2 = (I, I, I, I, I) = -g_1$, $I^2 = I$ is the indeterminate$\}$ be a semiring of special quasi dual numbers.

(i) What is the dimension of M?
(ii) Is M a S-semiring?
(iii) Is M a strict semiring?
(iv) Can M have zero divisors?
(v) Can M have S-ideals?

56. Let $P = \left\{ \begin{bmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{bmatrix} \middle| \; a_j = x_1 + x_2g + x_3g_1 + x_4h + x_5h_1 \right.$

with $1 \leq i \leq 10$, $x_j \in Q$, $1 \leq j \leq 5$, $g = 6$, $g_1 = 15$, $h = 14$ and $h_1 = 7 \in Z_{21}\}$ be a vector space of quasi special dual pairs over the field Q. Is P a linear algebra?

$M = \left\{ \begin{bmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{bmatrix} \middle| \; a_j = x_1 + x_2g + x_3h, g = 6 \text{ and } h = 14 \right.$

$\in Z_{21}$, with $1 \leq i \leq 10$, $x_1, x_2, x_3 \in Q\}$ be a linear algebra of complete quasi special dual pairs over the field Q.

(i) Find a basis of P and M.
(ii) Is $P \cong M$? (P a linear algebra)
(iii) Find Hom(P, P) and Hom(M, M).
(iv) If P is a vector space find dimension of P over Q.
(v) Find Hom (P, P), P as a vector space.
(vi) Write P as a direct sum of sublinear algebras over Q.
(vii) Find L (P, Q) and L(M, Q).

57. Let $S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + \ldots + a_{11}g_{10}$, $g_1 = (-I, -I, -I, -I)$, $g_2 = (-I, 0, 0, 0)$ $g_3 = (0, -I, 0, 0)$, $g_4 = (0, 0, -I, 0)$, $g_5 = (0, 0, 0, -I)$, $g_6 = (I, I, I, I)$, $g_7 = (I, 0, 0, 0)$, $g_8 = (0, I, 0, 0)$, $g_9 = (0, 0, I, 0)$ and $g_{10} = (0, 0, 0, I)$, $a_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 11\}$ be the semiring of special quasi dual numbers.

(i) Is $P = \{(0, 0, 0, 0), g_1, g_2, g_3, \ldots, g_{10}\}$ a semigroup under $\times$?
(ii) Can S be a S-semiring?
(iii) Prove (S, +) is not a semigroup.
(iv) Can S be a strict semiring?
(v) Prove $(Q^+ \cup \{0\})$ (P) the semigroup semiring of the semigroup $(P, \times)$ over the semiring $Q^+ \cup \{0\}$ is isomorphic to S.

58. Let S = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4$ where $a_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 5$, $g_1 = (-1, -1)$, $g_2 = (1, 1)$, $g_3 = (-1, 0)$ and $g_4 = (1, 0)$, $g_1^2 = (1, 1) = -g_1 = g_2$; $g_3^2 = (1, 0) = -g_3 = g_4$. $g_1g_2 = g_1$, $g_1g_3 = g_4$, $g_3g_4 = g_3$, $g_2g_4 = g_4$, $g_3g_2 = g_3\}$ be the semiring of special quasi dual numbers.

    (i) Can S have zero divisors?
    (ii) Is S a semifield?
    (iii) Can S be a S-semiring?

59. Does Z_{240} contain x such that $x^2 = -x = (239)\,x$?

    (i) How many such x does Z_{240} contain?
    (ii) If $S = \{x \in Z_{240} \mid x^2 = -x\} \subseteq Z_{240}$, is $(S \cup \{0\}, \times\}$ form a semigroup?

60. Find all special quasi dual number components of Z_{48}.

61. For what values of n (n not a prime) does Z_n contain special quasi dual number component? (That is elements $x \in Z_n$ with $x^2 = -x$).

62. Let P = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in Z_{16}$, $1 \leq i \leq 4$,

$$g_1 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix},\ g_2 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix},\ g_3 = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix},$$

$g_1 \times_n g_1 = -g_1$, $g_1 \times_n g_2 = -g_2$, $g_1 \times_n g_3 = -g_3$, $g_2 \times_n g_2 = -g_2$,

$g_3 \times_n g_3 = -g_3$ and $g_2 \times_n g_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$ be the ring of special quasi dual numbers.

    (i) Find the number of elements in P.
    (ii) If $S = \{\langle (0), g_1, g_2, g_3, \times_n \rangle\}$ be the semigroup and $Z_{16}S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_4 + \ldots + a_tg_{t-1} =$

$\sum_i a_i g_i$ $a_i \in Z_{16}$, $1 \leq i \leq t = o(S)\}$ be the semigroup ring. Prove $Z_{16}S \cong P$ as rings.

(iii) Is P a S-ring?
(iv) Can P have S-ideals?
(v) Does P contain S-units?
(vi) Can P have zero divisors which are not S-zero divisors?

63. Let $T = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in Z_{19}, 1 \leq i \leq 5, g_1 = 3, g_2 = 4, g_4 = 8 \text{ and } g_3 = 9 \in Z_{12}\}$ be the general ring of complete quasi special dual number pairs.

(i) Find order of T.
(ii) Prove P is a S-ring.
(iii) Find ideals which are S-ideals in T.
(iv) Does T contain any special quasi dual element y such that $y^2 = -y$ in T?

64. Let $W = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i = 3, g_2 = 4, g_3 = 8 \text{ and } g_4 = 9 \in Z_{12}, a_i \in Z^+ \cup \{0\}, 1 \leq i \leq 5\}$ be the general quasi dual semiring.
(i) Is W a S-semiring?
(ii) Can W have S-semi ideals?
(iii) Is W a strict semiring?

65. Let $S = Q(g_1, g_2, \ldots, g_t)$ be a t-dimensional general ring of special quasi dual numbers.
Study the special features enjoyed by S.

66. What is the special feature associated with vector space of special quasi dual numbers over a field F?

67. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle| \; a_i = x_1 + x_2g + x_3k \right.$ where $g = 14$ and $k = 6 \in Z_{21}$, $x_1, x_2, x_3 \in Q\}$ be the vector space of special quasi dual numbers over the field Q.

(i) Find a basis of S over Q.
(ii) What is the dimension of S over Q?
(iii) Can S be made into a linear algebra and the natural product $\times_n$?
(iv) If Q is replaced by $Q(g, k) = \{x_1 + x_2g + x_3k \mid 14 = g, k = 6 \in Z_{21};$ $x_1, x_2, x_3 \in Q\}$. Will P be a S-vector space?
(v) What is the dimension of P as a S-vector space over Q (g, k)?

68. Let $S = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_5\ (g_1, g_2) \text{ where } g_1 = 15 \text{ and } g_2 = 24,\ 15,\ 24 \in Z_{40}\right\}$ be the linear algebra of special quasi dual numbers over the field Z_5.

(i) Find dimension of S over Z_5.
(ii) Find a basis of S over Z_5.
(iii) Can S be expressed as a direct sum of linear subalgebras over Z_5?
If Z_5 is replaced by Z_5 (g_1, g_2) study the questions (i), (ii) and (iii) with appropriate changes.

69. Let $M = \{(a_1, a_2, a_3, a_4) \mid a_j = x_1 + x_2g_1 + x_3g_2$ where

$$g_1 = \begin{bmatrix} -I \\ -I \\ -I \end{bmatrix} \text{ and } g_2 = \begin{bmatrix} I \\ I \\ I \end{bmatrix},\ x_i \in Z^+ \cup \{0\},\ 1 \le i \le 3,$$

$1 \le j \le 4\}$ be a semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

(i) What is the dimension of M over $Z^+ \cup \{0\}$?
(ii) Write M as a direct sum of subsemivector spaces.
(iii) If $Z^+ \cup \{0\}$ is replaced by $T = Z^+ \cup \{0\}$ (g_1, g_2); will M be a S-semivector space over $T = Z^+ \cup \{0\}$ $(g_1, g_2) = \{x_1 + x_2g_1 + x_3g_2 \mid$ $x_i \in Z^+ \cup \{0\},\ 1 \le i \le 3\}$.

(iv) What is dimension of M over T?

70. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2 \text{ where}\right.$

$1 \leq i \leq 9$, $x_1, x_2, x_3 \in Q$, $g_1 = 8$ and $g_2 = 3$,
$g_1^2 = -g_1 \pmod{12}$, $g_2^2 = -g_2 \pmod{12}$, $3, 8 \in Z_{12}\}$
be a vector space of special quasi dual numbers.

(i) Let T : P → P be any linear operator on P so that T^{-1} does not exist.

(ii) Find eigen values and eigen vectors associated with S; S : P → P given by

$$S\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & a_9 \end{bmatrix}.$$

(iii) Is S an invertible operator on P?

(iv) Find ker S.

(v) Let $K_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2;\right.$

$x_j \in Q$, $1 \leq i, j \leq 3$, $g_1 = 8$ and $g_2 = 3 \in Z_{12}\} \subseteq P$,

$K_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2;\ x_j \in Q,\right.$

$1 \leq i, j \leq 3$, $g_1 = 8$ and $g_2 = 3 \in Z_{12}\} \subseteq P$ and

$$K_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2;\; x_j \in Q, \right.$$

$1 \leq i, j \leq 3$, $g_1 = 8$ and $g_2 = 3 \in Z_{12}\} \subseteq P$
be subspaces of P.
Find projection $E_j : P \to K_j$, $1 \leq j \leq 3$ such that $I = E_1 + E_2 + E_3$. Find the eigen values associated with each E_j; $1 \leq j \leq 3$.

71. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 \right.$ where $1 \leq i \leq 6$, $g_1 = 6$ and $g_2 = 14 \in Z_{21}$, $x_j \in Q$, $1 \leq j \leq 3\}$ be a vector space of special quasi dual numbers over the field Q.
    (i) Find Hom (V, V).
    (ii) Find L(V, Q).
    (iii) Find a basis for V over Q.
    (iv) What is the dimension of V over Q?

72. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 \right.$ where $1 \leq i \leq 8$, $g_1 = 24$ and $g_2 = 15 \in Z_{40}$, $x_1, x_2, x_3 \in R\}$ be the vector space of special quasi dual numbers over the field R.

    (i) Study the algebraic structures enjoyed by Hom(W, W).
    (ii) Give the algebraic structure of L(W, R).
    (iii) Write W as a pseudo direct sum.
    (iv) What is the dimension of W over R?

73. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 \text{ where } 1 \leq i \leq 6 \right.$ and $1 \leq j \leq 3$, $g_1 = 3$ and $g_2 = 8 \in Z_{12}$, $x_j \in Z_7\}$ be the vector space of special quasi dual numbers over the field Z_7.

(i) Find Hom (P, P).
(ii) Find L(P, Z_7).
(iii) Write P as a direct sum, $W_1 + W_2 + W_3 + W_4 = P$.
(iv) Now using each W_j define a projection $E_j : P \to W_j$, $1 \leq j \leq 4$.

74. Let $S = \{(a_1, a_2, \ldots, a_{10}) \mid a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$; $1 \leq i \leq 10$, $x_j \in R^+ \cup \{0\}$, $1 \leq j \leq 5$; $g_1 = 25$, $g_2 = 15$, $g_3 = 24$ and $g_4 = 16 \in Z_{40}\}$ be semivector space of special quasi dual numbers over the semifield $R^+ \cup \{0\}$.

(i) Find dimension of S over $R^+ \cup \{0\}$.
(ii) Find P = Hom(S, S). Is P a semivector space over $R^+ \cup \{0\}$?
(iii) Find dimension of L (S, $R^+ \cup \{0\}$) over $R^+ \cup \{0\}$.

75. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \; a_i = x_1 + x_2g_1 + x_3g_2,\ x_j \in R^+ \cup \{0\}, \right.$ $1 \leq j \leq 3$, $g_1 = 8$ and $g_2 = 4$ in $Z_{12}\}$ be a semivector space of special quasi dual like numbers over the semifield $S = Z^+ \cup \{0\}$.

(i) Find a basis of M over S.
(ii) Can M have more than one basis?
(iii) Find dimension of M over S.
(iv) Write M as pseudo direct sum! (Is it possible).
(v) Find L(M, $Z^+ \cup \{0\}$).

76. Obtain some special properties enjoyed by mixed special dual quasi numbers.

77. Give an example of a finite ring of mixed special dual quasi numbers.

78. Let
$P = \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Z_{43},\ 1 \leq i \leq 3,\ g_1 = 8;\ g_2 = 6 \in Z_{12}\}$ be the ring of mixed special quasi dual numbers.

(i) Find order of P.
(ii) Is P a S-ring?
(iii) Can P have S-ideals?
(iv) Can P have subrings which are not S-subrings?
(v) Does P contain S-zero divisors?
(vi) Can P contain units which are not S-units?

79. Let $S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle|\ a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5;\ 1 \leq i \leq 15,\ x_j \in Z^+ \cup \{0\},\ g_1 = 15,\ g_2 = 25,\ g_3 = 16,\ g_4 = 24 \text{ and } g_5 = 20,\ 1 \leq j \leq 6\right\}$ be a semiring of special mixed quasi dual numbers.

(i) Is S a strict semiring?
(ii) Can S have S-semi ideals?
(iii) Can S have S-units?
(iv) Can S have subsemirings which are not ideals?

80. Let S in problem (79) be a semivector space of special mixed quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

(i) Find P = Hom(S, S). Is P a semivector space?
(ii) Find a basis of S over $Z^+ \cup \{0\}$.
(iii) Can S have more than one basis?
(iv) Write W as a direct sum of semivector subspaces.
(v) Find $L(S, Z^+ \cup \{0\}) = M$, What is the algebraic structure enjoyed by M?

81. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \text{ with } g_1 = 6, g_2 = 9, g_3 = 8 \in Z_{12}, \; 1 \leq i \leq 4, x_j \in Z_{11}, 1 \leq j \leq 4 \right\}$ be a strong mixed special quasi dual number general non commutative ring.

(i) Find the number of elements in P.
(ii) Is P a S-ring?
(iii) Can P have S-ideals?
(iv) Can P have S-units?
(v) Can P have zero divisors which are not S-zero divisors?
(vi) Is $a = 4g_1$ a S-zero divisor?
(vii) Is $b = g_2$ an S-idempotent?

82. Obtain some interesting properties enjoyed by strong special mixed quasi dual numbers.

83. Is it possible to get the component of strongly mixed special quasi dual numbers from any other source other than Z_n (n an appropriate positive integer).

84. Find the component set of strong mixed special quasi dual number associated with Z_{320}.

85. Find the component set of strong mixed special quasi dual numbers of Z_{210}.

86. Let $S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + \ldots + a_{10}g_9$ with $a_i \in Z_{13}$, $1 \leq i \leq 10$ where $g_1 = (6, 6, 6, 6, 6)$, $g_2 = (0, 0, 0, 6, 6, 6)$, $g_3 = (6, 6, 6, 0, 0, 0)$, $g_4 = (9, 9, 9, 9, 9, 9)$, $g_5 = (9, 9, 9, 0, 0, 0)$, $g_6 = (0, 0, 0, 9, 9, 9)$, $g_7 = (8, 8, 8, 8, 8, 8)$, $g_8 = (0, 0, 0, 8, 8, 8)$, $g_9 = (8, 8, 8, 0, 0, 0)$; $6, 9, 8 \in Z_{12}\}$ be the ring of mixed strong special quasi dual numbers.

(i) Find the order of S.
(ii) Is S a Smarandache ring?
(iii) Can S have ideals which are not S-ideals?
(iv) Can S have units which are not S-units?
(v) Find subrings which are not ideals.

87. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + \ldots + x_{11}g_{10}\text{; where } 1 \leq I \leq 10,\; g_1 = \begin{bmatrix} 15 \\ 15 \\ 15 \\ 15 \end{bmatrix},\right.$

$g_2 = \begin{bmatrix} 15 \\ 0 \\ 15 \\ 0 \end{bmatrix}$, $g_3 = \begin{bmatrix} 16 \\ 16 \\ 16 \\ 16 \end{bmatrix}$, $g_4 = \begin{bmatrix} 16 \\ 0 \\ 16 \\ 0 \end{bmatrix}$, $g_5 = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \end{bmatrix}$ $g_6 = \begin{bmatrix} 24 \\ 0 \\ 24 \\ 0 \end{bmatrix}$,

$g_7 = \begin{bmatrix} 25 \\ 25 \\ 25 \\ 25 \end{bmatrix}$, $g_8 = \begin{bmatrix} 25 \\ 0 \\ 25 \\ 0 \end{bmatrix}$, $g_9 = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}$ and $g_{10} = \begin{bmatrix} 20 \\ 0 \\ 20 \\ 0 \end{bmatrix}$; 15, 25, 20, 16, 24 $\in Z_{40}$, $x_j \in Z_{30}$, $1 \leq j \leq 11\}$ be the strong mixed special quasi dual like number ring under natural product $\times_n$.

(i) Is M a S-ring?
(ii) Find order of M.
(iii) Can M have ideals which are not S-ideals?
(iv) Can M have idempotents which are not S-idempotents?
(v) Does M have zero divisors which are not S-zero divisors?

88. Suppose M in problem (87) is a S-vector space of mixed special strong quasi dual numbers over the S-ring of M.

    (i) Find dimension of M over the S-ring Z_{30}.
    (ii) Find a basis of in over Z_{30}.
    (iii) Find Hom(M, M) = S. Is S a S-vector space over Z_{30}?
    (iv) Find L (M, Z_{30}).
    (v) Write M as a direct sum of S-vector subspaces of M over Z_{30}.

89. Find the component semigroup of special quasi dual elements of $C(Z_{10})$.

90. Does $C(Z_{42})$ contain the component of a special quasi dual element?

91. For $C(Z_n)$ what is the condition on n so that $C(Z_n)$ has special quasi dual component-elements?

92. Let $C(Z_{40}) = \{a + bi_F \mid a, b \in Z_{40},\ i_F^2 = n - 1 = 39\}$. Find all $x \in C(Z_{40})$, (where $x = a + bi_F$, $a, b \in Z_{40} \setminus \{0\}$) such that $x^2 = (n-1)\, x = 39x$.

93. Let $A = \{(a_1, a_2, \ldots, a_6) \mid a_i = x_1 + x_2g_1$ where $g_1 = 8 + 2i_F \in C(Z_{17})$, $x_1, x_2 \in Q\}$ be the ring of complex modulo special quasi dual like number.

    (i) Prove A is a S-ring.
    (ii) Does A contains S-subrings which are not S-ideals?
    (iii) Does A contain S-units?
    (iv) Can A have zero divisors which are not S-zero divisors?

94. Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i = x_1 + x_2g \text{ where } x_1, x_2 \in Z_{11} \text{ and}\right.$ $g = 7 + 6i_F \in C(Z_{10}) = \{a + bi_F \mid a, b \in Z_{10}, i_F^2 = 9\}; g^2 = -g, 1 \leq i \leq 8\}$ be the special quasi dual number complex modulo integer general ring under $\times_n$.

(i) Find the number of elements in M.
(ii) Is M a S-ring?
(iii) Give subrings of M which are not S-ideals.

95. Prove C $(Z_p) = \{a + bi_F \mid a, b \in Z_p, i_F^2 = p-1\}$, p a prime of the form $p = m^2 + n^2$, $1 \leq m, n \leq p-1$ has always atleast one $g = a + bi_F \mid a, b \in Z_p \setminus \{0\}$ such that $g^2 = -g$.

96. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{30} \end{bmatrix} \middle| a_i = x_1 + x_2g + x_3g_1\right.$

with $x_j \in Q^+ \cup \{0\}$, $1 \leq i \leq 30$, $1 \leq j \leq 3$, $g = 2 + 4i_F$ and $g_1 = 8 + 6i_F \in C(Z_{10}) = \{a + bi_F \mid a, b \in Z_{10}, i_F^2 = 9\}, \}$ be the general semiring of complex modulo integer special quasi dual number under natural product $\times_n$.

(i) Can T be a strict semiring?
(ii) Is T a S-semiring?
(iii) Can T have semiideals?
(iv) Can T have S-idempotents?

97. If T in problem (96) is taken as a semivector space of special quasi dual numbers over the semifield $Q^+ \cup \{0\}$.

(i) Can T be finite dimensional over $Q^+ \cup \{0\}$?
(ii) Find a basis of T over $Q^+ \cup \{0\}$.

(iii) Can T have more than one basis?
(iv) Find Hom(T, T) = P, is P a semivector space over $Q^+ \cup \{0\}$?

98. Let M = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2g_1 + x_3g_2,\ x_j \in Z^+ \cup \{0\},\right.$

$1 \leq j \leq 3$ and $g_1 = \begin{bmatrix} 2+4i_F \\ 2+4i_F \\ 0 \\ 2+4i_F \end{bmatrix}$ and $g_2 = \begin{bmatrix} 8+6i_F \\ 8+6i_F \\ 0 \\ 8+6i_F \end{bmatrix}$, $2 + 4i_F$ and $8 + 6i_F \in C(Z_{10})$ and $g_1 \times_n g_1 = g_2$ and $g_1 \times_n g_2 = g_1$, $g_2 \times_n g_2 = g_2\}$ be a semivector space of special quasi dual numbers over the semifield $Z^+ \cup \{0\}$.

(i) Find a basis of M over the field $Z^+ \cup \{0\}$.
(ii) Write M as a pseudo direct sum. (Is it possible?).
(iii) What is the dimension of M over $Z^+ \cup \{0\}$?
(iv) Find Hom(M, M).
(v) Find L(M, $Z^+ \cup \{0\}$).

99. Characterize the properties enjoyed by strong mixed special quasi dual like numbers build using Z_n.

100. Does $C(Z_{148})$ contain a component semigroup which can contribute to strong mixed special quasi dual like numbers?

101. Let $C(Z_{98})$ be the complex finite modulo integer. Does $C(Z_{98}) = \{a + bi_F \mid a, b \in Z_{98},\ i_F^2 = 97\}$ contain a component semigroup which can give special quasi dual numbers?

102. Describe the properties enjoyed by groupoid of special dual like number.

103. Obtain some interesting properties enjoyed by groupoids of strong mixed dual numbers.

104. Let G = $\{a_1 + a_2g_1 + a_3g_2 + \ldots + a_7g_6 \mid a_i \in Z_{45}, 1 \leq i \leq 4, g_1 = 7, g_2 = 14, g_3 = 21, g_4 = 28, g_5 = 35$ and $g_6 = 42 \in Z_{49}, (3, 5), *\}$ be the seven dimensional groupoid of dual like numbers.

(i) Is G a S-groupoid?
(ii) Find the number of elements in G.
(iii) Can G have zero divisors?
(iv) Can G have S-subgroupoids?
(v) Is G a normal groupoid?
(vi) Show G has atleast seven distinct subgroupoids.

105. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{30} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2\right.$

$+ x_4g_3$; $1 \leq i \leq 30$, $x_j \in Z_7$, $1 \leq j \leq 4$, $g_1 = 20$, $g_2 = 16$ and $g_3 = 15 \in Z_{40}, (3, 2), *\}$ be the strong mixed dual number groupoid.

(i) Is T finite?
(ii) Can T have S-zero divisors?
(iii) Can T have normal subgroupoids?
(iv) Can T have subgroupoids which are not S-subgroupoids?

106. Let S = $\{a + bg_1 + cg_2 \mid a, b, c \in Z^+ \cup \{0\}, g_1 = 10$ and $g_2 = 5 \in Z_{20}, (8, 7), *\}$ be a groupoid.

(i) Can S be a S-groupoid?
(ii) Can S have S-idempotents?
(iii) Is P = $\{a + bg_1 \mid a, b \in 5Z^+ \cup \{0\}, g_1 = 10 \in Z_{20}, (8, 7), *\} \subseteq S$ a S-subgroupoid?
(iv) How many subgroupoids can S have?

107. Let T = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{30} \end{bmatrix}\right| a_i = x_1 + x_2g_1 + x_3g_2\right.$

$+ \ldots + x_6g_5$, $g_1 = 7$, $g_2 = 14$, $g_3 = 21$, $g_4 = 28$, $g_5 = 35 \in Z_{42}$, $g_1^2 = g_1 \pmod{42}$, $g_2^2 = g_4 \pmod{42}$, $g_3^2 = g_3 \pmod{42}$, $g_4^2 = g_4$, $g_5^2 = g_1 \pmod{42}$, $g_1g_2 = g_2 \pmod{42}$, $g_1g_4 = g_4 \pmod{42}$, $g_1g_3 = g_3 \pmod{42}$, $g_1g_5 = g_5 \pmod{42}$, $g_2g_3 = 0 \pmod{42}$, $g_2g_4 = g_2$, and so on. $x_j \in Z_{23}$, $1 \leq j \leq 6$, (7, 0), $*\}$ be the groupoid.

(i) Is T finite?
(ii) Can T have zero divisors which are not S-zero divisors?
(iii) Is T a strong mixed special dual number groupoid?

108. Let G = $\left\{\left.\sum_{i=0}^{\infty} a_i x^i \right| a_i = x_1 + x_2g_1 + x_3g_2 \text{ where } x_j \in Z_5,\right.$

$1 \leq j \leq 3$, $g_1 = 6$ and $g_2 = 9 \in Z_{36}$, $g_1^2 = 0 \pmod{36}$, $g_2^2 = g_2 \pmod{36}$, (2, 0), $*\}$ be the mixed dual number groupoid.

(i) Is G infinite?
(ii) Prove G is a S-groupoid.
(iii) Can G have a subgroupoid which is not a S-groupoid?
(iv) Can G be normal?

109. Let S = $\left\{\left.\sum_{i=0}^{6} a_i x^i \right| a_i \in Z_{12}(g) = \{a + bg \mid a, b \in Z_{12}, g = 3\right.$

$\in Z_6\}$, $0 \leq i \leq 6$, $*$, (3, 4)$\}$ be the polynomial groupoid of special dual like numbers of finite order.

(i) Find the number of elements in S.

(ii) Is S a S-groupoid?
(iii) Can S have S-subgroupoids?
(iv) Can S have zero divisors?
(v) Can S have idempotents?

110. Let M = $\left\{\sum_{i=0}^{3} a_i x^i \,\middle|\, a_i \in Z_7(g) = \{a + bg \mid a, b \in Z_7, g = 10 \in Z_{30}\}, 0 \leq i \leq 3, (2, 0), *\right\}$ be a groupoid of special dual like numbers.

(i) Find the number of elements in M.
(ii) Is M a S-groupoid?
(iii) Is M a normal groupoid?
(iv) Can M have normal subgroupoids?
(v) Can M have subgroupoids which are not S-subgroupoids?

111. Let S = $\{(a_1, a_2, \ldots, a_8) \mid a_i = x_1 + x_2g, g = 3 \in Z_9, x_1, x_2 \in Z_{89}, 1 \leq i \leq 8, (10, 8), *\}$ be a non associative linear algebra of dual numbers over the field Z_{89}.

(i) Find a basis of S over Z_{89}.
(ii) Is S finite dimensional?
(iii) Find Hom (S, S). Is Hom(S, S) a non associative linear algebra?
(iv) Write S as a direct sum of subspaces.
(v) Find T $\in$ Hom (S, S) so that T^{-1} exists.

112. Let M = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in Q, 1 \leq i \leq 4, g_1 = 3, g_2 = 6$ and $g_3 = 4 \in Z_{12}, g_1^2 = -g_1 \pmod{12}, g_2^2 = 0 \pmod{12}$ and $g_3^2 = g_3 \pmod{12}; (7/3, 4/7). *\}$ be a non associative linear algebra of strong mixed dual numbers over the field Q.

(i) What is dimension of M over Q?
(ii) For any T : M $\rightarrow$ M find the related eigen values and eigen vectors.

(iii) Are the eigen vectors associated with T strong mixed dual numbers?

113. Let N = $\left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right| a_i = x_1 + x_2g_1 + x_3g_2,\ x_i \in R;\ 1 \leq i \leq 3,\right.$

$g_1 = 20$ and $g_2 = 16 \in Z_{40}$, $(\sqrt{7}, \sqrt{13} + 4)$, *} be a non associative linear algebra of mixed dual numbers over the field R.

(i) Find dimension of N over R.
(ii) Find L(N, R). What is the algebraic structure enjoyed by L(N, R)?
(iii) Find Hom (N, N).
(iv) Is N finite dimensional?
(v) Write N as a pseudo direct sum of sublinear algebras.

114. Let S = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_8 \\ a_9 & a_{10} & a_{11} & \ldots & a_{16} \end{bmatrix}\right| a_i = x_1 + x_2g_1 + x_3g_2\,;\right.$

$1 \leq i \leq 24$, $x_j \in Z_7$; $1 \leq j \leq 3$, $g_1 = (4, 4, 4)$, $g_2 = (6, 6, 6)$, $6 \in Z_{12}$; (3, 0), *} be a non associative linear algebra of mixed dual numbers over the field Z_7.

(i) Find the number of elements in S.
(ii) Find a basis of S over Z_7.
(iii) Find dimension of Hom(S,S).
(iv) Find a basis of L(S, Z_7).

(v) If T:S → S; T = $\left(\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_8 \\ a_9 & a_{10} & a_{11} & \ldots & a_{16} \\ a_{17} & a_{18} & a_{19} & \ldots & a_{24} \end{bmatrix}\right)$

$$= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_8 \\ 0 & 0 & 0 & \dots & 0 \\ a_9 & a_{10} & a_{11} & \dots & a_{16} \end{pmatrix}.$$

Find the eigen values and eigen vectors associated with T.

115. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3; \right.$

$1 \leq i \leq 12$, $x_j \in Z^+ \cup \{0\}$ $g_1 = (20, 20)$, $g_2 = (16, 16)$, and $g_3 = (15, 15)$, $20, 16, 15 \in Z_{40}$, $1 \leq j \leq 4$, $(3, 4), *\}$ be a non associative semilinear algebra of strong mixed dual numbers over the semifield $S = Z^+ \cup \{0\}$.

(i) Find a basis of P over S.
(ii) Can P have more than one basis?
(iii) Can we say the number of linearly independent elements in S will always be less than or equal to the number of elements in a basis of P over S? (substantiate your claim!)
(iv) Find Hom(P, P). Is Hom(P, P) a non associative semilinear algebra over S?
(v) Find L (P, S). Is it a semilinear algebra over S?

116. Let $T = \{(a_1, a_2, \dots, a_{15}) \mid a_i = x_1 + x_2g_1 + x_3g_2$; $1 \leq i \leq 15$,

$x_j \in Q^+ \cup \{0\}$; $1 \leq j \leq 3$, $g_1 = \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}$ and $g_2 = \begin{bmatrix} 35 \\ 35 \\ 35 \\ 35 \\ 35 \\ 35 \end{bmatrix}$, $7, 35 \in$

Z_{42} with $g_1 \times_n g_2 = \begin{bmatrix} 35 \\ 35 \\ 35 \\ 35 \\ 35 \\ 35 \end{bmatrix}$ (mod 42), $g_1 \times_n g_1 = \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}$

(mod 42) and $g_2 \times_n g_2 = \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}$ (mod 35), (2, 0), *} be a non associative semilinear algebra of mixed special dual numbers over the semifield $S = Q^+ \cup \{0\}$.

(i) Find a basis of T over S.
(ii) Is T finite dimensional over S?
(iii) Find Hom (T, T).
(iv) Can T have more than one basis?
(v) Find L(T, S).
(vi) If $S = Q^+ \cup \{0\}$ is replaced by $F = Z^+ \cup \{0\}$ study problem (i) to (iv).

117. Let $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3;\ 1 \leq i \leq 12, \right.$

$x_j \in Z^+ \cup \{0\}$, $1 \leq j \leq 4$, $g_1 = 28$, $g_2 = 8$ and $g_3 = 7 \in Z_{56}$, $g_1^2 = 0$ (mod 56), $g_2^2 = g_2$ (mod 56) and $g_3^2 = 49 = -g_3$ (mod 56), $g_2g_3 = 0$ (mod 56), $g_1g_3 = g_1$ (mod 56), $g_1g_2 = 0$

(mod 56), (0, 2), *} be a non associative semilinear algebra of strong mixed dual number over the semifield $Z^+ \cup \{0\} = S$.

(i) Find a basis of W over S.
(ii) Is W finite dimensional over S?
(iii) Can W have more than one basis over S?
(iv) Find the algebraic structure enjoyed by Hom(W,W).
(v) If $T : W \to W$ is an invertible semilinear operator find the associated eigen values and eigen vectors associated with T.

118. Let $V = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{pmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2; \right.$

$1 \leq i \leq 20$, $x_j \in Q$; $1 \leq j \leq 3$. $g_1 = 6$ and $g_2 = 4 \in Z_{12}$, (3, –2), *} be the non associative Smarandache linear algebra of mixed dual numbers over the Smarandache ring $Q(g_1, g_2) = \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Q, 1 \leq i \leq 3, g_1 = 6$ and $g_2 = 4 \in Z_{12}\}$.

(i) Find a S-basis of V over Q (g_1, g_2).
(ii) What is the dimension of V over $Q(g_1, g_2)$?
(iii) Find Hom(V,V). Is Hom(V, V) a non associative linear algebra over $Q(g_1, g_2)$?
(iv) If $T : V \to V$, T is non invertible find the eigen values and eigen vector associated with T. Do these values belong to $Q(g_1, g_2) \setminus Q$?
(v) Suppose V is defined over $Q(g_1)$ (or $Q(g_2)$) study problems (i) to (iv).

119. Let $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \text{ with } 1 \leq i \leq 4, \right.$

$x_j \in R$, $1 \leq j \leq 4$, $g_1 = 20$, $g_2 = 16$ and $g_3 = 15 \in Z_{40}$ $(\sqrt{13} - 3, \; -\sqrt{3} + 13)$, $*\}$ be a non associative S-linear algebra over the S-ring $R(g_1, g_2, g_3) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_j \in R; 1 \leq j \leq 4, g_1 = 20, g_2 = 16 \text{ and } g_3 = 14 \in Z_{40}\}$ of strong mixed special dual numbers.

(i) Find a S-basis of S over $R(g_1, g_2, g_3)$.
(ii) If S is defined over $R(g_1, g_2)$ what is the basis of S over $R(g_1, g_2)$?
(iii) Let S be defined over $R(g_1)$ (or $R(g_2)$) study the properties of S as a non associative S-linear algebra of strong mixed dual number over the S-ring $R(g_1)$ (or $R(g_2)$).
(iv) Find Hom(S, S).
(v) Find (a) L (S, $R(g_1, g_2, g_3)$),
(b) L (S, $R(g_1, g_2)$),
(c) L (S, $R(g_2, g_3)$),
(d) L (S, $R(g_3, g_1)$),
(e) L (S, $R(g_1)$),
(f) L (S, $R(g_2)$) and
(g) L (S, $R(g_3)$).
Compare their algebraic structures and basis for the linear algebras (a) to (g).
(vi) Find a direct sum of S as sublinear algebras.
(vii) Find for atleast one $T:S \to S$ and its associated eigen values and eigen vectors.

120. Let $S = \left\{ \sum_{i=0}^{25} a_i x^i \,\middle|\, a_j = x_j + y_j g,\ 0 \leq j \leq 25,\ x_j, y_j \in Q;\ g = 4 \right.$ $\in Z_{16}$, (8, –8), *} be a non associative S-linear algebra of dual numbers over the S-ring
$Q(g) = \{a + bg \mid a, b \in Q,\ g = 4 \in Z_{16},\ g^2 = 0\}$.

(i) Find a basis of S over Q(g).
(ii) Find a linearly dependent subset of S.
(iii) Is S finite dimensional?
(iv) Can S be written as a direct sum of sublinear algebras?

121. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_8 \\ a_9 & a_{10} & a_{11} & \ldots & a_{16} \\ a_{17} & a_{18} & a_{19} & \ldots & a_{24} \end{bmatrix} \,\middle|\, a_i = x_1 + x_2 g_1 + x_3 g_2 \right.$

$+ x_4 g_3;\ 1 \leq i \leq 24,\ x_j \in Q^+ \cup \{0\},\ 1 \leq j \leq 4,\ g_1 = 20,\ g_2 = 16$ and $g_3 = 25 \in Z_{40}$, (3, 30), *} be the non associative Smarandache semilinear algebra of mixed dual numbers of four dimension over the Smarandache semiring
$F = (Q^+ \cup \{0\})(g_1, g_2, g_3) = \{x_1 + x_2 g_1 + x_3 g_2 + x_4 g_3 \mid x_i \in Q^+ \cup \{0\},\ 1 \leq i \leq 4,\ g_1 = 20,\ g_2 = 16$ and $g_3 = 25 \in Z_{40}\}$ of mixed dual numbers.

(i) What is the dimension of S over F?
(ii) Is S finite dimensional?
(iii) Find S-subsemilinear algebras of S over F.
(iv) Find Hom(S, S). Is S a finite dimensional S-semilinear algebra over F?
(v) Find L(S, F). Study the striking properties associated with L(S, F).
Is L (S, F) a S-semilinear algebra over F?

122. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \, a_i \in Z^+ \cup \{0\} \, (g_1, g_2, g_3) = \right.$

$\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_j \in Z^+ \cup \{0\}, 1 \leq j \leq 4, g_1 = 20, g_2 = 16, g_3 = 15, g_4 = 25 \in Z_{40}\}, 1 \leq i \leq 6, (8, 0), *\}$ be a non associative S-semilinear algebra of strong mixed special dual numbers over the S-semiring $F = Z^+ \cup \{0\} \, (g_1, g_2, g_3) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_j \in Z^+ \cup \{0\}, 1 \leq j \leq 4;$ g_t's mentioned above $1 \leq t \leq 3\}$ of strong mixed special dual numbers.

(i) Find dimension of P over F.
(ii) Find a basis of P over F. Can P have more than one basis?
(iii) Study Hom(P, P) and L(P, F).
(iv) If $T \in$ Hom (P, P) study the eigen values and eigen vectors associated with T.

123. Let $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \right.$ where $x_j \in Z^+ \cup \{0\}, 1 \leq j \leq 5, 0 \leq i \leq 8, g_1 = 4, g_2 = 6, g_4 = 9$ and $g_3 = 3 \in Z_{12}, (0, 2), *\}$ be a Smarandache semilinear algebra of strong mixed special dual numbers over the S-semiring $S = Z^+ \cup \{0\} \, (g_1, g_2, g_3) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_j \in Z^+ \cup \{0\}, 1 \leq j \leq 5, g_1 = 4, g_2 = 6, g_4 = 9$ and $g_3 = 3 \in Z_{12}\}$ where S is the semiring of strong special mixed dual numbers.

(i) Show eigen values and eigen vectors of any linear operator T on M can have those values to be strong mixed special dual numbers.
(ii) Will every semilinear operator T on M have those values to be some type of dual numbers?
(iii) Study the semilinear functions L(M, S).

(iv) Does any special property is enjoyed by semilinear operators which are invertible?

124. Give some nice applications of linear operators on S-linear algebras of mixed special dual numbers.

125. Suppose S is a S-linear algebra of strong mixed special dual numbers over a S-ring of special strong mixed dual numbers, is it necessary that every S-linear operator on S should have its eigen values and eigen vectors to be strong mixed special dual like numbers. Justify your claim.

126. Study the problem 125 in case of S-semilinear algebra of strong mixed dual numbers defined over a S-semiring of mixed dual numbers.

FURTHER READING

1. Ball, R.S., *Theory of screws*, Cambridge University Press, 1900.

2. Birkhoff, G., *Lattice Theory*, American Mathematical Society, 1948.

3. Clifford, W.K., *Preliminary sketch of biquaternions*, Proc. London Mathematical Society, Vol. 4, no. 64, 381-395, 1873.

4. Duffy, J., *Analysis of mechanisms and robot manipulators*, Halstead Press, 1980.

5. Fischer, I.S. and Freudensetin, F., *Internal force and moment transmission in a cardan joint with manufacturing tolerances*, ASME Journal of Mechanisms, Transmissions and Automation in Design, Vol. 106, 301-311, December 1984.

6. Fischer, I.S., *Modeling of plane joint*, ASME Journal of Mechanical Design, vol. 121. 383-386, September 1999.

7. Fischer, I.S., *Numerical analysis of displacements in a tracta coupling*, Engineering with Computers, Vol. 15, 334-344, 1999.

8. Fischer, I.S., *Velocity analysis of mechanisms with ball joints*, Mechanics Research Communications, vol. 30, 2003.

9. Gu, Y.L. and Luh, J.Y.S., *Dual number transformations and its applications to robotics*, IEEE Journal and Robotics and Automation, vol. RA-3, December 1987.

10. Pennestri, E., and Vita, L., *Mechanical efficiency analysis of a cardan joint with manufacturing tolerances*, Proc. Of the RAAD03 12th International workshop on Robotics in Alpe-Adria-Danube Region, (Cassino, Italy), Paper 053, 2003.

11. Smarandache, Florentin (Editor), *Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic set, Neutrosophic probability and statistics*, Dec. 1-3, 2001, held at Univ. of New Mexico, Published by Xiquan, Phoenix, 2002.

12. Smarandache, Florentin, *Special algebraic structures in collected papers III*, Abaddaba, Oradea, 78-81, 2000.

13. Sugimoto, K and Duffy, J., *Application of linear algebra to screw systems*, Mechanism and Machine Theory, vol. 17, no. 1, 73-83, 1982.

14. Uicker, J.J., Denavit, J., and Hartenberg, R.S., *An iterative method for the displacement analysis of spatial mechanisms*, ASME Journal of Applied Mechanics, 309-314, June 1964.

15. Vasantha Kandasamy W.B. and Smarandache Florentin, *Algebraic structures using natural class of intervals*, The Educational Publisher, Ohio, 2011.

16. Vasantha Kandasamy W.B. and Smarandache Florentin, *Finite neutrosophic complex numbers*, Zip Publishing, Ohio, 2011.

17. Vasantha Kandasamy W.B. and Smarandache, Florentin, *Natural Product* $\times_n$ *on Matrices*, Zip Publishing, Ohio, 2012.

18. Vasantha Kandasamy W.B., *Algebraic structures using natural class of intervals*, The Educational Publisher, INC, Ohio 2011.

19. Vasantha Kandasamy, W.B., *Semivector spaces over semifields*, Zeszyty Nauwoke Polilechniki, 17, 43-51, 1993.

20. Vasantha Kandasamy, W.B., *Smarandache semigroups*, American Research Press, Rehobott, 2002.

21. Vasantha Kandasamy, W.B., *Groupoids and Smarandache groupoids*, American Research Press, Rehobott, 2002.

22. Vasantha Kandasamy, W.B. and Smarandache, Florentin, *Dual Numbers*, Zip Publishing, Ohio, 2012.

23. Vasantha Kandasamy, W.B. and Smarandache, Florentin, *Neutrosophic rings*, Hexis, Phoenix, 2006.

24. Vasantha Kandasamy, W.B. and Smarandache, Florentin, *Special dual like numbers and lattices*, Zip Publishing, Ohio, 2012.

INDEX

H

L

M

N

P

Q

S

T

ABOUT THE AUTHORS

Dr.W.B.Vasantha Kandasamy is an Associate Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 68 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 72nd book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
She can be contacted at vasanthakandasamy@gmail.com
Web Site: http://mat.iitm.ac.in/home/wbv/public_html/
or http://www.vasantha.in

Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

---

A new notion of special quasi dual numbers are introduced. If a + bg is the special quasi dual number with a, b reals, g the new element is such that $g^2 = -g$. The rich source of getting new elements of the form $g^2 = -g$ is from Z_n; the ring of modulo integers. For the first time we construct non associative structures using them. We have proposed some research problems.

ISBN 9781599731926

90000 >

9 781599 731926

US $30.00

Zip Publishing