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ORIGINAL ARTICLE

Calculation of radar signal delays in the vicinity of the Sun due to the contribution of a Yukawa correction term in the gravitational potential

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Abstract There has been a renewed interest in the recent years in the possibility of deviations from the predictions of Newton's "inverse-square law" of universal gravitation. One of the reasons for renewing this interest lies in various theoretical attempts to construct a unified elementary particle theory, in which there is a natural prediction of new forces over macroscopic distances. Therefore the existence of such a force would only coexist with gravity, and in principle could only be detected as a deviation from the inverse square law, or in the "universality of free fall" experiments. New experimental techniques such that of Sagnac interferometry can help explore the range of the Yukawa correction $\lambda \geq 10^{14}$ m where such forces might be present. It may be, that future space missions might be operating in this range which has been unexplored for very long time. To study the effect of the Yukawa correction to the gravitational potential and its corresponding signal delay in the vicinity of the Sun, we use a spherically symmetric modified space time metric where the Yukawa correction its added to the gravitational potential. Next, the Yukawa correction contribution to the signal delay is evaluated. In the case where the distance of closest approach is much less than the range λ, it results to a signal time delay that satisfies the relation $t(b < \lambda) \cong 37.7t(b = \lambda)$.

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1 Introduction

Although the accuracy and prediction power of Newton's gravitational theory has been proved in a lot of cases, observations do not always match the inverse square force law. Therefore, many of the present theories of gravitation predict forces coupled to gravitation. These forces are expected to be significant for pairs of celestial bodies that lie in a mutual distance greater than 10^{10} m, or for elementary massive particles separated by distances in a submillimeter range. While some experimental work has already been developed for the second case (see, e.g., Chiaverini et al. 2003; Hoyle et al. 2004; Smullin et al. 2005), the first domain remains almost unexplored (Camacho 2004). Experimental techniques, such as the Sagnac interferometry, and future space missions may be of much help in exploring this range.

The above mentioned non-Newtonian forces are expressed by an additional term, which must be incorporated into the original $1/r$ Newtonian potential. For instance, Newton himself (in his unpublished work, Portsmouth Collection, 1888) has proposed a modified potential of the form $a/r + b/r^2$ with a, b positive constants. Maneff (1924) has considered a potential of the same kind, but based on physical arguments, specifying the expressions of the constants a, b. Mücket and Treder (1977) have introduced a logarithmic correction to the Newtonian gravitational potential. Of course, we can mention here the empirical models proposed by Bertrand, or Hall and Newcomb, as well as Seeliger's model of potential with an exponential correction. In Celestial Mechanics, a Yukawa-type potential is very often proposed to modify the Newtonian one (see, e.g., D'Olivo

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and Ryan 1987; Iorio 2002; Brownstein and Moffat 2006; Haranas and Ragos 2011).

In this paper, the Yukawa-type correction is applied to the potential of the primary body, i.e. the Sun, in order to calculate possible radar signal delays.

Section 2 introduces the gravitational potential with the Yukawa-type correction term, as well as the corresponding force. A few considerations on the values of coupling constants in connection with the interaction range are given.

Section 3 presents the main analytical results of our endeavour. The modified line interval is established and an analytical result for the total time delay is derired. Next, we only evaluate the integral that corresponds to the Yukawa correction and we examine two different cases.

In Sect. 4 we numerically evaluate the integrals using a formula derived by Moffat and Toth (2009), thus calculating the strength and the range of this interaction.

Section 5 surveys this contribution and formulates some concluding remarks.

2 The theory of non-Newtonian gravity

The non-Newtonian effects in the gravitational theory are conventionally described in terms of the potential energy. For a closed system of a primary and a secondary body of mass M and m, respectively, that interact in a Newtonian field with a Yukawa correction, this energy is (Fischbach et al. 1991)

$$V(r) = -\frac{GMm}{r}\left(1+\alpha e^{-\frac{r}{\lambda}}\right) = V_N(r) + V_{Yk}(r). \qquad (1)$$

In this equation, G represents the Newton's gravitational constant, r is the distance between these bodies, $\alpha = \frac{kK}{GMm}$, where k and K are the coupling constants of the new force to the bodies relative to gravity (Ciufolini and Wheeler 1995), and λ is the range of this interaction. The range λ of the Yukawa potential characterizes the distance scale beyond which the effects of this potential become unimportant (Fischbach and Talmadge 1999). $V_{Yk}(r)$ is the Yukawa correction to the Newtonian potential $V_N(r)$. The corresponding force can be written as:

$$\begin{aligned}\boldsymbol{F}(r) &= -\frac{GMm}{r^2}\left[1+\alpha\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\right]\boldsymbol{r} \\ &= [F_N(r) + F_{Yk}(r)]\boldsymbol{r}. \end{aligned} \qquad (2)$$

For Earth orbiting satellites, i.e. LAGEOS, $\alpha_{\min} = 1.38 \times 10^{-11}$ and $\lambda = 6.081 \times 10^6$ (Kolosnitsyn and Melnikov 2004). Pitjeva (1999) has estimated by using a radar that random Mercury perihelion motions are of the order of $0.052''$/cy, which implies that a minimum value of the Yukawa coupling constant is $\alpha_{\min} = 3.57 \times 10^{-10}$ for $\lambda = 2.89 \times 10^{10}$ m.

3 The modified metric and the derivation of the signal delay

To deal with the Yukawa effect on the propagation of electromagnetic signals in the vicinity of the Sun, we incorporate an additional Yukawa term in the Schwarzschild spacetime metric coefficients of the line element. We analogously modify the Schwarzschild metric used in the solar system when general relativistic effects are taken into account. Therefore, if r, θ, ϕ are the polar coordinates of any point along the signal's path, and Ω is the corresponding solid angle, the line element takes the form:

$$\begin{aligned} ds^2 &= c^2\left[1-\frac{2GM}{rc^2}\left(1+\alpha e^{-r/\lambda}\right)\right]dt^2 \\ &\quad - \left[1-\frac{2GM}{rc^2}\left(1+\alpha e^{-r/\lambda}\right)\right]^{-1} dr^2 - r^2 d\Omega^2, \end{aligned} \qquad (3)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Next, the photon transmission time can be written as follows:

$$dt' = \frac{ds}{c}, \qquad (4)$$

and

$$\begin{aligned} dt &= \left[1+\frac{2GM}{rc^2}\left(1+\alpha e^{-\frac{r}{\lambda}}\right)\right]dt' \\ &= \left[1+\frac{2GM}{rc^2}\left(1+\alpha e^{-\frac{r}{\lambda}}\right)\right]\frac{ds}{c}. \end{aligned} \qquad (5)$$

We have that $r = b/\cos\theta$, where b is the distance of the closest signal approach, so that

$$dr = -\frac{b\sin\theta}{\cos^2\theta}d\theta. \qquad (6)$$

Also $ds = (dr^2 + r^2 d\theta^2)^{1/2}$. After substitution and simplification, (5) becomes:

$$dt = \left[\frac{b}{c\cos\theta} + \frac{GM}{c^3}\left(1+\alpha \mathrm{e}^{-\frac{b\cos\theta}{\lambda}}\right)\right]\sec\theta d\theta, \qquad (7)$$

so that:

$$\begin{aligned} t &= \frac{2b}{c}\int_{-\pi/2}^{+\pi/2}\sec^2\theta d\theta + \frac{4GM}{c^3}\int_{-\pi/2}^{+\pi/2}\sec\theta d\theta \\ &\quad + \frac{4GM\alpha}{c^3}\int_{-\pi/2}^{+\pi/2}\sec\theta e^{-\frac{b}{\lambda}\sec\theta}d\theta. \end{aligned} \qquad (8)$$

In this paper, we deal only with the Yukawa correction to the time delay which is represented by the third term of the rhs of (8). Therefore, by integrating this term, we have that this delay is:

$$t = \frac{4GM\alpha}{c^3}\int_{-\pi/2}^{\pi/2}\sec\theta e^{-\frac{b}{\lambda}\sec\theta}d\theta = \frac{8GM\alpha}{c^3}K_0\left(\frac{b}{\lambda}\right), \qquad (9)$$

where $K_0(x)$ denotes the modified Bessel function of the second kind of order $n=0$. $K_0(x)$ can be expressed in the following way (Spiegel 1968):

$$K_0(x) = \left(-\gamma + \ln\frac{2}{x}\right) + \sum_{i=1}^{\infty}\left[\left(\prod_{j=1}^{i}(2j)^2\right)^{-1}\left(\left(\sum_{j=1}^{i}\frac{1}{j}\right) - \gamma + \ln\frac{2}{x}\right)\right]x^{2i}, \tag{10}$$

where γ is the Euler's constant. Therefore the general expression for the excess distance travelled can be written as follows:

$$d_{\mathrm{ex}} = ct = \frac{8GM\alpha}{c^2}\left[\left(-\gamma + \ln\frac{2}{x}\right) + \sum_{i=1}^{\infty}\left[\left(\prod_{j=1}^{i}(2j)^2\right)^{-1}\left(\left(\sum_{j=1}^{i}\frac{1}{j}\right) - \gamma + \ln\frac{2}{x}\right)\right]x^{2i}\right] \tag{11}$$

which can be also written as:

$$d_{\mathrm{ex}} = ct = 4R_{\mathrm{Sch}}\alpha\left[\left(-\gamma + \ln\frac{2}{x}\right) + \sum_{i=1}^{\infty}\left[\left(\prod_{j=1}^{i}(2j)^2\right)^{-1}\left(\left(\sum_{j=1}^{i}\frac{1}{j}\right) - \gamma + \ln\frac{2}{x}\right)\right]x^{2i}\right], \tag{12}$$

where $R_{\mathrm{Sch}} = \frac{2GM}{c^2}$ is the Schwarzschild radius of the sun.

4 Numerical results

The numerical values of the constants used to complete the calculations are

$\gamma = 0.5772156649$,

$G = 6.67428 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$,

$c = 299792458\ \mathrm{ms^{-1}}$ $\quad M_\odot = 1.98892 \times 10^{30}$ kg,

$R_\odot = 6.95508 \times 10^{8}$ m,

where $M_\odot$, $R_\odot$ denote the mass and the radius of the Sun.

In the vicinity of the Sun, a reasonable assumption is that $\lambda \approx b$. In this case, (9) results that the Yukawa signal delay is equal to:

$$t = \frac{8GM_\odot\alpha}{c^3}K_0(1). \tag{13}$$

In order to compute $K_0(1)$ within accuracy of 10 significant digits, seven terms of the series are required to obtain that:

$$K_0(1) = 0.4210244382. \tag{14}$$

Therefore, when $b \approx \lambda$, the delay time of a radar signal due to the Yukawa contribution is:

$$t = 3.3691955056\frac{GM_\odot\alpha}{c^3}. \tag{15}$$

According to Moffat and Toth (2009), the strength α of the Yukawa field in the vicinity of the Sun is:

$$\alpha = \frac{M_\odot}{(\sqrt{M_\odot} + C_1')^2}\left(\frac{G_\infty}{G_N} - 1\right), \tag{16}$$

where

$$G_\infty \cong 20G_N, \tag{17}$$

$$C_1' \cong 25000\sqrt{M_\odot}. \tag{18}$$

Therefore:

$$\alpha = \frac{19M_\odot}{(\sqrt{M_\odot} + 25000\sqrt{M_\odot})^2} = \frac{19}{(25001)^2} = 3.0397568146 \times 10^{-8}. \tag{19}$$

Substituting the values of α and $K_0(1)$ in (9), we obtain that the Yukawa contribution to the radar signal delay is:

$$t = 0.0005044\ \mathrm{ns}. \tag{20}$$

Next, we consider the case that $b < \lambda$. Moffat and Toth (2009) propose for the range λ of the Yukawa potential that:

$$\lambda = \frac{\sqrt{M_\odot}}{C_2'}, \tag{21}$$

where

$$C_2' \cong 6250\sqrt{M_\odot}\ \mathrm{kpc^{-1}}. \tag{22}$$

Substituting the mass of the Sun, we obtain that

$$\lambda \cong 4.937084512 \times 10^{15}\ \mathrm{m}. \tag{23}$$

The distance of the closest signal approach b can be taken to be equal the radius of the Sun, i.e. $b = R_\odot$. Comparing the values of b and λ, we see that, indeed, $b < \lambda$. S, In this case,

$$K_0\left(\frac{b}{\lambda}\right) = 15.8913298160 \tag{24}$$

and (9) gives the following numerical result:

$$t = 0.0190392 \text{ ns}. \tag{25}$$

Next using (11) we obtain the following numerical values for the excess length that the signal travels in the cases where $b \approx \lambda$ and $b < \lambda$ respectively to be:

$$d_{\mathrm{ex}} = 1.51215 \times 10^{-4} \text{ m}, \tag{26}$$

$$d_{\mathrm{ex}} = 5.7078 \times 10^{-4} \text{ m}. \tag{27}$$

With reference to our numerical results given by (20) and (25) we can see that the time delays due to the Yukawa correction in the gravitational field of the Sun are of the order of nanosecond fractions. To get an idea of today's radar systems, somebody could talk about the sensitivity of a radar, a property that is related to the power of the transmitting radar. Since we are interested in signal time delays and in order to substantiate our finding we will referer to today's radar resolution instead something that is related to the detectable times. Quoting Shapiro (1968, 1999), we say that fractional system errors of echo time delays in solar system experiments can be up to 1 part in 10^{10} or smaller. Given today's technological progress it might be possible that such effects will be detected in the years to come.

5 Conclusions

We have added a Yukawa correction to the line element used in solar system general relativistic calculations, in order to calculate the time signal delay in the vicinity of the Sun. In particular, out of the three terms included in our derivation, only the third term is related to the Yukawa correction, and this is the term that we will deal with. First, we have considered the $b = \lambda$ case and calculated that the Yukawa signal delay contribution is expressed in terms of the modified Bessel function of the second kind of zero order and argument one, multiplied by the strength of the potential α related to the mass of the Sun which was also calculated using an equation derived by Moffat and Toth (2009). Next, in the case where $b < \lambda$, the same integral was calculated for the same value of α, with the only difference that the range of the Yukawa interaction λ related to the Sun had to be calculated, using the formula given by Moffat and Toth. In this case, the Yukawa contribution depends again on the zero order modified Bessel functions of the second kind of argument b/λ multiplied by the strength of the potential α. It appears that in the case where $\lambda = 10^7 b \approx 10^7 R_\odot$ the corresponding time delays are related as follows: $t(b < \lambda) \cong 37.7t(b = \lambda)$. Finally, we conclude that in both cases the calculated signal time delays can be probably measured by the technology of the decay to come.

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