L'Hospital's Rule

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Abstract. We give a short proof of l'Hospital's Rule.

Warning: We use Bourbaki's notation $]a, b[$ for open intervals.

Theorem. Suppose f and g are real and differentiable in $]a, b[$, and $g'(x)$ is not 0 for all x in $]a, b[$, where $-\infty \le a < b \le +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to b.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to b$, or if $g(x) \to +\infty$ as $x \to b$, then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to b.$$

(Statement taken word for word from Rudin's **Principles of Analysis**.)

Lemma. Let a, b and A be as above, let T be the triangle

$$T := \{(x, y) \mid x, y \in \,]a, b[\, , \ x < y\},$$

let $u : T \to \mathbb{R}$ be a function such that $u(x, y)$ tends to A as (x, y) tends to (b, b) (while remaining in T), and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $]a, b[$ converging to b. If $u(x, y_n)$ tends to $v(x) \in [-\infty, +\infty]$ for all x, then $v(x) \to A$ as $x \to b$.

Proof of the Lemma. If N is a closed neighborhood of A in $[-\infty, +\infty]$, then there is a c in $]a, b[$ such that $c \le x < y < b$ implies $u(x, y) \in N$, and thus $v(x) \in N$.

Proof of the Theorem. For $(x, y) \in T$ put

$$u(x, y) := \frac{f(x) - f(y)}{g(x) - g(y)} \quad .$$

By Cauchy's Mean Value Theorem (or Extended Mean Value Theorem), there is, for each (x, y) in T, a t in $]x, y[$ such that

$$u(x, y) = \frac{f'(t)}{g'(t)} \quad .$$

This implies that $u(x, y)$ tends to A as (x, y) tends to (b, b). Let (y_n) be a sequence in $]a, b[$ converging to b such that $f(y_n)/g(y_n)$ tends to some B in $[-\infty, +\infty]$, let $v(x)$ be equal to $f(x)/g(x)$ if $f(x) \to 0$ and $g(x) \to 0$ as $x \to b$, and to B if $g(x) \to +\infty$ as $x \to b$, and use the Lemma.

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