ON THE RESOLUTION TO THE RIEMANN HYPOTHESIS

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Abstract. In this paper, we present a resolution to the problem of the Riemann Hypothesis. In particular, by the use of the Mellin integral transform and analytic techniques, we prove that there exist no zeros to the Riemann Zeta Function in the critical strip outside the line whose real component is $\frac{1}{2}$.

Our objective is to demonstrate that there are no zeros

in the strip $0 < Re(s) < \frac{1}{2}$. The functional equation dictates

that a zero to $\zeta(s)$ is also a zero to $\zeta(1-s)$, so that no zeros

in $0 < Re(s) < \frac{1}{2}$ immediately implies that there

are no zeros in $\frac{1}{2} < Re(s) < 1$.

We begin by stating the Mellin Integral Transform:

Let s = x + iy. Then: $\zeta(s)$ =-s$\int_0^\infty (\frac{1}{t}) t^{s-1} dt$

in $0 < Re(s) < 1$, where $(\frac{1}{t})$ is the fractional part of $\frac{1}{t}$.

Now, we know that a zero to $\zeta(s)$ is a zero to $\zeta(1-\bar{s})$. Let α be a zero to $\zeta(s)$,

so that it is clear the following holds:

$\zeta(\alpha) = 0$, implies $\int_0^\infty (\frac{1}{t}) t^{\alpha-1} dt = 0$ (1)

$\zeta(1-\bar{\alpha}) = 0$, implies $\int_0^\infty (\frac{1}{t}) t^{-\bar{\alpha}} dt = 0$ (2)

where $\bar{\alpha}$ is the complex conjugate of α.
Which in turn implies:

$$\int_0^\infty (\frac{1}{t}) t^{\alpha-1} dt = \int_0^\infty (\frac{1}{t}) t^{-\bar{\alpha}} dt$$

by transitivity of the above two equations (1) and (2).

We have just set up an equation of ζ,for fixed α, formerly in s, in order to solve

for α and see what possible values the $Re(\alpha)$ have in $0 < Re(s) < 1$.

Now, we combine both sides of the above equation under the integrand:

$$\int_0^\infty (\frac{1}{t})(t^{\alpha-1} - t^{-\bar{\alpha}}) dt = 0$$

We now convert the above integral into its Riemann sum, and take the limit:

$$\sum_{i=0}^{\infty}(\frac{1}{t_i})(t_i^{\alpha-1}-t_i^{-\bar{\alpha}})\triangle t=0$$

Now, if we consider the complex function $(t_i^{\alpha-1}-t_i^{-\bar{\alpha}})$ as a vector $v_i(x,y)$, we get:

$$\sum_{i=0}^{\infty}(\frac{1}{t_i})v_i(x,y)\triangle t=0^{(*)}$$

For the above vectors $v_i(x,y)$ 1 of the two of the following hold:

(i)2 or more vectors $v_i(x,y)$ are linearly independent

(ii)All the vectors $v_i(x,y)$ are linearly dependent

However, (i) cannot hold since the entire equation is equal to zero and the non-zero coefficients $(\frac{1}{t_i})$ could not yield linearly independent vectors. Therefore the only

other possibility is (ii), where each $v_i(x,y)$, as vectors, must be equal to zero individually. This is due to the following:

Each $v_i(x,y)=C_iw(x,y)$ since each v_i is linearly dependent, and therefore $\sum_{i=0}^{\infty}(\frac{1}{t_i})v_i(x,y)=$

$\sum_{i=0}^{\infty}(\frac{1}{t_i})C_iw(x,y)=0$ implies $w(x,y)(\sum_{i=0}^{\infty}(\frac{1}{t_i})C_i)=0$ yields $w(x,y)=0$

But $v_i(x,y)=C_iw(x,y)=C_i(0)=0$ so that for each (i), $v_i(x,y)$ is zero.

Note that each C_i for i=0,1,2,... are constant coefficients.

Now, we know that the above sum is equal to zero

iff $v_i(x,y)=0$ for each $i=1,2,3...,$

and that $v_i(x,y)=0$ iff $(t_i^{\alpha-1}=t_i^{-\bar{\alpha}})$. Moreover, these two

complex numbers, now treated as vectors in R^2, are equal to one another iff their magnitude and direction are the same. Therefore, with $\alpha = x + iy$

$(t_i^{\alpha-1} - t_i^{-\bar{\alpha}}) = 0$ implies $t_i^{x-1}(cos(ln(t_i)y)) + isin(ln(t_i)y)) - t_i^{-x}(cos(ln(t_i)y))$

$+isin(ln(t_i)y)) = 0$, so that:

$(\mathrm{t}_i^{x-1} - t_i^{-x})(cos(ln(t_i)y)) + isin(ln(t_i)y)) = 0$

Taking the absolute value of the above expression, we conclude:

$|(t_i^{x-1} - t_i^{-x})||(cos(ln(t_i)y)) + isin(ln(t_i)y))| = 0$

$|(t_i^{x-1} - t_i^{-x})| = 0$

$(\mathrm{t}_i^{x-1} - t_i^{-x}) = 0$

$(\mathrm{t}_i^{x-1} = t_i^{-x})$

Taking logarithms, we obtain:

(x-1)ln(t$_i$) $= (-x)ln(t_i)$

$x - 1 = -x$

$x = 1/2$

We therefore see that the real component of α must have the value of $1/2$. But α is an arbitrary zero of the

original Riemann Zeta function, as specified at the beginning of the paper. Therefore

every zero must be on the line whose $Re(s) = 1/2$.

$(*)$Check the "Concerns" page for a nuanced point regarding the above starred (*) line and its resolution

Concerns

Earlier, we saw how the fractional part of $\frac{1}{t_i}$, denoted by $(\frac{1}{t_i})$, allowed for $(\frac{1}{t_i}) = 0$ for each integer value of $\frac{1}{t_i}$ (i.e. $t_i = 1/n$ for n = 1,2,3,...) with $t_i \varepsilon (0, \infty)$. This thereby permits $(\frac{1}{t_i}) v_i(x, y) = 0$ for non-zero $v_i(x, y)$. Therefore, any value of α (not just $\frac{1}{2} + iy$) will certainly result in $(\frac{1}{t_i}) v_i(\alpha_1, \alpha_2) = 0$, where $\alpha = \alpha_1 + i\alpha_2 = (\alpha_1, \alpha_2)$.

However, the *infinite sum* of $(\frac{1}{t_i}) v_i(x, y)$, $\sum_{i=0}^{\infty} (\frac{1}{t_i}) v_i(x, y)$, is never equal to zero for Re$(\alpha) \neq \frac{1}{2}$ since there are some vectors $(\frac{1}{t_i}) v_i(x, y)$ in the above infinite sum whose coefficients are not equal to zero since the coefficients are generally different from $t_i = 1/n$ (leaving non-zero fractional parts). We therefore see that for $\sum_{i=0}^{\infty} (\frac{1}{t_i}) v_i(x, y) \triangle t = 0$, this requires that $Re(\alpha) = \frac{1}{2}$. There can therefore be no zeros outside the line $Re(\alpha) = 1/2$. This concern is resolved.