Electrical conductivity of metals: a new look at this subject

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ABSTRACT

Various parameters tied to the electrical conductivity of typical metals are estimated and are expressed in terms of universal constants. It happens that they are close to those found in metallic copper at room temperature. The fact that the realization of the model occurs at room temperature is explained by using the Landauer's erasure principle. The averaged collision time of the electron of conduction is also thought as a particle lifetime. Finally an analogy is established between the motion of the electron of conduction and the cosmological constant problem, where a spherical surface of radius equal to the electron mean free path has been thought as a surface horizon for the charge carriers.

1 - INTRODUCTION

Highly purified water is a bad electrical conductor. However, the addition of small amounts of sodium chloride (NaCl) to this liquid, can increase its electrical conductivity in a substantial way. At the ambient temperature (295K), the water's dielectric constant of 80, permits the Na^+ and Cl^- ions to move freely through the liquid and this feature can account for the change in its conductive behavior. It seems that the concentration of free charge carriers has the most relevant role in determining the electrical conductivity of the substances.

But what to say about electrical conductivity in metals?

Isolated metallic atoms have their inner electrons belonging to closed shells and hence tightly bounded to their corresponding atomic nucleus. However the electrons of the outer most shell are weakly bonded to its respective nucleus. When arranged in a crystal lattice structure, the bond weakness of these outer electrons is enhanced due to the interactions among neighbor atoms of the lattice, so that the electrons of conduction are free to travel through the whole crystal. Resistance to their motion is due to the thermal vibrations (phonons) and defects provoked by the presence of impurities and lattice dislocations. In a perfect crystal at zero absolute temperature, these free electrons can be described by using the quantum mechanical formalism of the Bloch waves [1,2]. The concentration of free electrons plays an important role in the description of the electrical conductivity in metals.

2 – EVALUATION OF TYPICAL PARAMETERS TIED TO THE ELECTRIC CONDUCTIVITY OF METALS

A possible way to estimate the concentration of conduction electrons in a typical good metal will be next presented. An alternative way to estimate the Casimir force between two parallel uncharged metallic plates separated by a close distance d was developed in reference [3]. There, we considered the cutting of a cubic cavity of edge d in a metallic block. We imagined that the free electrons in metal as a gas of non-relativistic particles confined by the vacuum pressure in the interior of a cubic box of edge equal to d. On the other hand as was pointed out by Jaffe [12], the Casimir force can be calculated without reference to the vacuum fluctuations, and like other observable effects in QED, it vanishes as the fine structure constant α goes to zero.

In reference [3], we treated a non-relativistic Fermi gas confined by the vacuum pressure B and found the relation

$$B\, d^3 = (2/5)\, E_{av}, \qquad (1)$$

where E_{av} stands for the averaged energy of the gas. Meanwhile it is convenient to consider that an equivalent way to treat the problem is by taking in account the electromagnetic interaction through the dependence of the energy levels of the system on the fine structure constant α. We reproduce here some steps of the reasons outlined in reference [3].

One of the simplest models which exhibits energy levels dependence on the fine structure α is the Bohr atom, namely

$$E_n = -\,\tfrac{1}{2}\,(\alpha^2\, m\, c^2)/n^2 = -\,E_1/n^2. \qquad (2)$$

By taking the maximum occupied energy level equal to N/2, we get the Fermi energy (E_F) of the system

$$E_F = -\,4\,E_1/N^2. \qquad (3)$$

The averaged energy could be estimated as

$$E_{av} = (2/N) \int_o^{N/2} (-E_1\, n^2\, dn) = (2/N)\, E_1\, [(2-N)/N]. \qquad (4)$$

In the limit, as N>>1, we have

$$E_{av} = -\,2\,E_1/N. \qquad (5)$$

Now let us estimate the vacuum pressure. We have

$$B\, d^3 = -\,(2/5)\,(\alpha^2\, m\, c^2)/N = (2/5)\, E_{av}. \qquad (6)$$

By taking $p_o = \tfrac{1}{2}\,\alpha\, m\, c$ and $\lambda_o = h/p_o = 2h/(\alpha\, m\, c)$, it is possible to make the choice

$$N = d/\lambda_o = (\alpha\, m\, c\, d)/2h. \qquad (7)$$

Inserting (7) into (6), we obtain

$$B = - [8/(5\pi)] (\alpha \pi^2 \hbar c) (1/d^4). \quad (8)$$

Therefore we notice that by making the choice indicated by (7), the explicit dependence of B on the electron mass m and on the maximum quantum number N has disappeared.

The alternative way we have used in order to treat the Casimir force problem, permit us to calculate a typical density of charge carriers in good metals. Let us write

$$n\, d^3 = (4\pi/3)\, N^3\, (3!) = 8\, \pi\, N^3. \quad (9)$$

In (9), we have considered the volume of a sphere in the N-space, and the possible number of permutations among the N_x,N_y and N_z quantum numbers. Putting (7) into (9) we obtain

$$n = \pi\, [(\alpha\, m\, c)/\, h]^3. \quad (10)$$

Numerical evaluation of (10) gives $n = 8.56 \times 10^{28}\ m^{-3}$, which could be compared with $8.45 \times 10^{28}\ m^{-3}$ the density of charge carriers in metallic copper [1,2].

Meanwhile the Fermi energy of metals could be expressed as [1,4]

$$E_F = [h^2/(8\, m)]\, (3/\pi)^{2/3}\, n^{2/3}. \quad (11)$$

Inserting (10) into (11), we get

$$E_F = [(3^{2/3})/8]\, \alpha^2\, m\, c^2. \quad (12)$$

Numerical estimate of (12) gives $E_F = 7.07$ eV, which naturally is very close to the value found in metallic copper.

In order to proceed further, let us compute the electrical conductivity of a typical good metal. To do this we first suppose that we have n scatters per unit of volume and by considering a prism shaped tube having longitudinal size equal to the electron mean free path ℓ, width ℓ_F equal to half of the Fermi wavelength of the electron, and thickness ℓ_C equal to half of its Compton wavelength. If we consider that the electrical conductivity always happens in a regime of charge neutrality, the number of scatters per unit of volume will be equal to the number density of charge carriers, and we can write

$$n\, \ell_F\, \ell_C\, \ell = n\, [h/(2\, m\, v_F)]\, [h/(2\, m\, c)]\, v_F\, \tau = 1. \quad (13)$$

In (13), ℓ_C stands for the wavelength of a photon with a momentum related to the creation of a electron-positron pair and this corresponds to a minimum thickness of the prism, which also implies in a maximum τ, the averaged time between collisions. From (13) we obtain the relation

$$n\,\tau = m^2\,c/(\pi^2\,\hbar^2). \qquad (14)$$

Now, Drude formula for the electrical conductivity σ is given by (please see Kittel [1])

$$\sigma = e^2\,n\,\tau/m. \qquad (15)$$

Inserting $n\tau$ of (14) into (15), we obtain

$$\sigma = (e^2\,m\,c)/(\pi^2\,\hbar^2). \qquad (16)$$

Numerical estimate of the electrical resistivity ρ, gives
$\rho = 1/\sigma = 1.57 \times 10^{-8}\ \Omega.m$ and can be compared with the resistivity of the metallic copper measured at the temperature of 295K, namely
$\rho(copper) = 1.70 \times 10^{-8}\ \Omega.m$.

From (10) and (14) we also obtain the averaged time between collisions

$$\tau = (1/\alpha^3)\,[4\,h/(\pi\,m\,c^2)]. \qquad (17)$$

Numerical estimate of (17) gives $\tau = 2.65 \times 10^{-14}$ s. This number must be compared with the value estimated of $\tau(copper) = 2.5 \times 10^{-14}$ s, for copper at the room temperature as quoted by Allen [2].

It is also interesting to write formulas for the Fermi velocity v_F and the electron mean free path ℓ. We have

$$v_F = (2\,E_F/m)^{1/2} = (3^{1/3}/2)\,\alpha\,c, \qquad (18)$$

and

$$\ell = v_F\,\tau = (3^{1/3}\,2\,h)/(\alpha^2\,\pi\,m\,c). \qquad (19)$$

These relations for the quantities associated to the electrical conduction in typical metals are exhibited in table 1, as well their respective numerical estimates and are also compared with the corresponding ones quoted for copper at the room temperature.

Table 1 – Formulas related to the electrical conductivity of typical metals, in terms of universal constants (This work). Numerical estimates of then are compared with those quoted for Copper at room temperature.

	Formula	Numerical Estimates	Copper at Room Temperature
Density of Charge Carriers (n)	$n = \pi[(\alpha mc)/h]^3$	$n = 8.56 \times 10^{28} m^{-3}$	$n = 8.45 \times 10^{28} m^{-3}$ [1,2]
Fermi Energy (E_F)	$E_F = = (3^{2/3}/8) \alpha^2 mc^2$	$E_F = 7.07$ eV	$E_F = 7.0$ eV [1]
Electrical Resistivity $\rho = 1/\sigma$	$\rho = = (\pi^2 \hbar^2)/(e^2 mc)$	1.57×10^{-8} Ω.m	1.70×10^{-8} Ω.m [1,2]
Time Average Between Collisions (τ)	$\tau = = [4h/(\alpha^3 \pi m c^2)]$	2.65×10^{-14} s	2.5×10^{-14} s [2, 5]
Fermi Velocity (v_F)	$v_F = (3^{1/3}/2) \alpha c$	1.6×10^6 m/s	1.6×10^6 m/s [1]
Electron Mean Free Path (ℓ)	$\ell = [(24)^{1/3} h]/(\alpha^2 \pi mc)$	419 Å	400 Å [5]

3 - REALIZATION AT THE ROOM TEMPERATURE: A POSSIBLE EXPLANATION

It is an intriguing question why a model describing the electrical conductivity of a typical good metal just realizes itself in copper crystals at room temperature. The answer to this question could be elaborated through these reasons.

i) As was pointed out by Jacobs [9], Landauer's erasure principle [8] states that: whenever a single bit of information is erased, the entropy in the environment to which the information storing system is connected must increase at least $k_B \ln 2$, where k_B is the Boltzmann's constant.

ii) A free electron in a metal travels in average a distance equal to its mean free path, with a constant velocity v_F, until to collide with the ionic vibrations (phonons). In the collision process, the free electron looses its memory.

We think that we may associate to the Fermi energy E_F, a string of length equal to its Fermi wavelength, composed by unit cells having a length equal to the Compton wavelength of the electron. Let us to introduce a quasi-particle with a mass-energy μc^2 defined as

$$\mu c^2 = E_F v_F / c. \tag{20}$$

As we can see from (20), this quasi-particle has a mass-energy equal to the Fermi energy divided by the number of cells in the string. Defining

$$\Delta F = \Delta U - T\Delta S = \tfrac{1}{2} \mu c^2 - k_B \ln 2, \tag{21}$$

And after making the requirement that

$$\Delta F|_{T=T^*} = 0, \tag{22}$$

we obtain the relation

$$E_F^3 = (k_B T^*)^2 \, 2 \, (\ln 2)^2 \, m c^2. \tag{23}$$

Putting $E_F = 7.1$ eV (please see table 1) and $m c^2 = 0.511$ MeV in (23) and solving for $k_B T^*$, we find

$$k_B T^* = 26 \text{ meV}. \tag{24}$$

The above number for the characteristic temperature T* must be compared with $k_B T_{Room} = 25$ meV.
Therefore the obtained result for the characteristic temperature given by (24) seems to make sense to the fact that the realization of the model for

the electrical conductivity of good metals to happen for copper crystals at the room temperature.

4 - THREE CHARACTERISTICS LENGTH AND THE GROW OF A POLYMER CHAIN

In a paper dealing with the cosmological constant problem [6], the time evolution of the universe world line was compared with the growing of a polymer chain by making use of a Flory-like free energy. It is possible to think the electron mean free path as the length of a polymer chain, composed by monomers of size equal to the Compton wavelength of electron. Within this analogy, the radius of gyration of the chain is identified with the Fermi wavelength of electron.

We consider as in the de Gennes derivation [7] two contributions for the Flory's free energy. The first term which goes proportional to N^2/R^d, corresponds to a repulsive-like monomer-monomer interaction.
A second term which comes from an entropic contribution, namely a logarithm of a Gaussian distribution (a signature of a random walk process) goes as $R^2/(N\,\lambda_C^2)$. We write

$$F = (N^2\,\lambda_C^d)/\,R^d \;+\; R^2/(N\,\lambda_C^2), \qquad (25)$$

where F is a Flory-like free energy, λ_C is the Compton length, N is the number of monomers in the chain, and d is the space-time dimension. Setting $\ell = N\,\lambda_C$ and minimizing (25) relative to R, we obtain for the radius of gyration R_g the relation

$$R_g = \ell^{3/(2+d)}\,\lambda_C^{(d-1)/(2+d)}. \qquad (26)$$

We identify $R_g(d=4)$ with the Fermi length of he electron, λ_F. We have

$$\lambda_F = (\ell\,\lambda_C)^{1/2}. \qquad (27)$$

We observe that equation (27) , relating the three characteristics lengths of the problem, agrees with the upper bound to the electron mean free path found in reference [13]. Please see equation (21) of the cited reference. It is worth to notice that the agreement between both calculations occurs just when the radius of gyration is evaluated in the space-time dimension d=4.

5 - HIGH TEMPERATURE BEHAVIOR OF THE COLLISION TIME

It would be interesting to evaluate a relation expressing the high temperature behavior of the collision time appearing in the Drude formula for the electrical conductivity.

By considering a viscous force which depends linearly on the velocity, the power dissipated by this force can be written as

$$dE/dt = - F_{viscous}\, v = - (1/\tau)\, p\, v. \qquad (28)$$

The power dissipated by this viscous force acting on the charge carrier will appear as an increasing in the internal energy of the lattice and we write

$$dU/dt = - dE/dt = (1/\tau)\, p\, v. \qquad (29)$$

By taking

$$p = \hbar/(2\, R), \qquad \text{and} \qquad v\, dt = dR, \qquad (30)$$

where the first relation in (30) comes from the uncertainty principle, we get

$$dU = [\hbar/(2\, \tau)]\, dR/\, R. \qquad (31)$$

Performing the integration of (31) between the limits $R_0 = \hbar/(m\, c)$ and $R_1 = \hbar/(m\, v_F)$, we obtain

$$\Delta U = [\hbar/(2\, \tau)]\, \ln(c/v_F). \qquad (32)$$

Now, let us consider an entropy variation given by

$$\Delta S = k_B \ln 2^D = D\, k_B \ln 2. \qquad (33)$$

In (33), we have written an entropy variation similar to that considered in applying the Landauer's erasure principle [8], but here putting D = 4, by taking in account the four dimensions of the space-time.
Taking the extremum of the free energy, namely writing

$$\Delta F = \Delta U - T\, \Delta S = 0, \qquad (34)$$

and solving for τ, we have

$$\tau = [\hbar/(8\, k_B\, T)]\, \ln(c/v_F). \qquad (35)$$

In the case of copper ($v_F = 1.57 \times 10^6$ m/s) at the room temperature (T = 300K), we find

$$T_{copper}(300\ K) = 2.4 \times 10^{-14}\ s. \qquad (36)$$

As we can see in table 1, the result of (36) is very close to the room temperature mean collision time of the electrons of conduction in copper, as quoted in the literature.

6 - AVERAGED COLLISION TIME AS A PARTICLE LIFETIME

There are two characteristics linear momenta that we can associate to the free electrons responsible for the electrical conductivity in good metals. They are: the Fermi momentum mv_F and the Compton momentum mc. By taking into account that the free electron has a fermionic character, we will write a non-linear Dirac-like equation describing the "motion" of this particle. We have

$$\partial\Psi/\partial x - (1/c)\,\partial\Psi/\partial t = [(m\,v_F)/\hbar]\,\Psi - [(mc)/\hbar]\,|\Psi^*\Psi|\Psi. \qquad (37)$$

We see that equation (37) contains only first order derivatives of the field Ψ. Besides this, the field Ψ has not a spinorial character.

Making the two sides of eq. (37) equal to zero and solving for $|\Psi^*\Psi|$, we get

$$|\Psi^*\Psi| = v_F/c = (3^{1/3}/2)\,\alpha. \qquad (38)$$

In obtaining (38), we also used the result for v_F shown in table 1.

On the other hand in the collision process, the free electron loss its memory. We may think that this feature looks similar to the annihilation of a particle- antiparticle pair, each of mass-energy equal to E_F. Putting this thing in a form of the uncertainty principle yields

$$2\,E_F\,\Delta t = h/2 \qquad \text{or} \qquad h\,\upsilon/2 = 2\,E_F. \qquad (39)$$

Solving equation (39) for υ, we get

$$\upsilon = 1/\Delta E = 4\,E_F/h = [3^{2/3}/(2\,h)]\,\alpha^2\,m\,c^2. \qquad (40)$$

By combining the results of (38) and (40) we obtain the line width (Γ) tied to the "particle" decay

$$\Gamma = \upsilon\,|\Psi^*\Psi| = [3/(4h)]\,\alpha^3\,m\,c^2. \qquad (41)$$

Finally the "particle" lifetime (τ) is given by

$$\tau = 1/\Gamma = (4\ h/3)/(\alpha^3\ m\ c^2). \qquad (42)$$

Comparing τ giving by (42) with the time between collisions shown in table 1, we verify that the present result displays the number 3 in the denominator, instead of the number π which appears in table 1.

7 - ANALOGY WITH THE COSMOLOGICAL CONSTANT PROBLEM ($\hbar = c = k_B = 1$)

One worth point we can consider now is the analogy that can be made with the cosmological constant problem. Hsu and Zee [10] have proposed an effective action A_{eff} as a means to deal with the cosmological constant problem. They wrote

$$A_{eff} = -\ (\Lambda\ L^4 + M_P^4/\Lambda) + \text{independent of } \Lambda \text{ terms}, \qquad (43)$$

where M_P is the Planck mass, L is the radius of the event horizon of the universe and Λ is the cosmological constant.
Taking the extremum of this action they got

$$\Lambda = (M_P/L)^2. \qquad (44)$$

We could think A_{eff} above as a four-dimensional representation of a kind of free energy, where the first term plays the role of the internal energy and the second one is related to the entropy S. The absolute temperature is taken to be equal to one. We propose that

$$\Omega \sim \exp(M_P^4/\Lambda), \qquad (45)$$

with

$$S = \ln \Omega. \qquad (46)$$

On the other hand, there is a proposal [11] that the universe can be considered as a black hole with its entropy being evaluated by counting the number of cells contained in the area of its event horizon (the

holographic principle), namely

$$S_{universe} \sim (L / L_P)^2 = L^2 M_P^2. \qquad (47)$$

By considering the two equivalent ways of the entropy evaluation, from (46) and (47) relations, we can write

$$L^2 M_P^2 = M_P^4 / \Lambda, \qquad (48)$$

which reproduces the results of Hsu and Zee [10],please see equation (44).

Turning to the problem of the electrical conductivity in good metals, let us consider for instance in a copper crystal an electron of the conduction band which just suffered a collision. In the absence of an external electric field, all the directions in the space have equal probability to be chosen in a starting new free flight. Therefore if we take a sphere centered at the point where the electron has been scattered, with a radius equal to the electron mean free path, the surface of this sphere may be considered as an event horizon for the phenomena. Any electron starting from this center will be on average scattered when striking the event horizon, loosing the memory of its previous free flight. Besides this, all the lattice sites of the metallic crystal are treated on equal footing, due to the translational symmetry of the system. Based on the previous discussion and inspired on the black hole physics, let us to define the entropy related on the event horizon for the electron of conduction in metals. We write

$$S_{Metal} = 4 \pi (\ell / w)^2, \qquad (49)$$

where ℓ is the electron mean free path and w is the equivalent to the Planck length of the problem.

It is possible to write an action analogous to that of Hsu and Zee [10], in order to describe the electrical conductivity in metals. We have

$$A_{Metal} \sim [\Lambda_M \ \ell^4 + (\Lambda_M \ w^4)^{-1}]. \qquad (50)$$

Making the equality between the two ways of writing the entropy, namely equaling the entropy of a surface horizon of radius ℓ and ultra-violet cutoff w with the last term of (50), we get

$$4 \pi (\ell / w)^2 = (\Lambda_M \ w^4)^{-1}, \qquad (51)$$

which leads to

$$\Lambda_M^{-1/4} = (4 \pi)^{1/4} (\ell \ w)^{1/2}. \qquad (52)$$

Upon to identify $\Lambda_M^{-1/4}$ with the Fermi wavelength of the electron λ_F and w with its Compton wavelength λ_C, we obtain

$$\lambda_F = (4 \pi)^{1/4} (\ell \ \lambda_C)^{1/2}. \qquad (53)$$

Relation (53) must be compared with (27).

REFERENCES

[1] C. Kittel, Introduction to Solid State Physics, 5th Edition, Wiley, 1976-Ch. 6

[2] Philip B. Allen, "Electrical conductivity", The Physics Teacher **17(6),** pages 362-366, September 1979.

[3] P. R. Silva, "Casimir force: an alternative treatment", arXiv:0901.0908

[4] Paul A. Tipler, Modern Physics, Worth Pub. Inc., N. York, 1978-Ch.9

[5] David Halliday, Robert Resnick, John Merrill, Fundamentals of Physics, 3th Edition, Extended Version, Wiley, 1988- Ch. 28

[6] P. R. Silva, "Cosmological Constant and Polymer Physics", arXiv:0812.4007v1, Braz. J. Phys. **38**, 587-590, 2008.

[7] P-G de Gennes, Scaling Concepts in Polymer Physics, Cornell University Press, 1979-Ch. 1

[8] R. Landauer, IBM J. Res. Develop. **5**, 183 (1961)

[9] K. Jacobs, arXiv:quant-ph/0512105v1(Dec. 2005)

[10] S. Hsu and A. Zee, arXiv:hep-th/0406142

[11] Leonard Susskind, "The World as a Hologram",

arXiv:hep-th/9409089v2 (Sept. 1994)

[12] R. L. Jaffe, Phys. Rev.D **72**, 021301(2005)

[13] P. R. Silva, M. Sampaio, C. Nassif, M. C. Nemes, "Quantum conductance and electric resistivity", Phys. Lett. A **358**, 358-362 (2006)