Article 25:

Hierarchy of Theories of Unified Gravity and Dynamics at the neighborhood of Several Gravitational Field Sources. Part I.

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The two-theory approach to gravitation at the second stage of evolutions of spacetime/intrinsic spacetime and parameters/intrinsic parameters in a gravitational field of arbitrary strength, comprising of the theory of gravitational relativity/intrinsic theory of gravitational relativity (TGR/ϕTGR) on flat spacetime/flat intrinsic spacetime and the metric theory of absolute intrinsic gravity (ϕMAG) on curved absolute intrinsic spacetime, isolated at the neighborhood of one gravitational field source in the earlier articles, is advanced to the situations where two, three and several gravitational field sources are scattered in the Euclidean 3-space about a location where the theories are formulated. Gravitational time dilation, gravitational length contraction and assemblage of parameter transformations in the context of TGR, are extended to the neighborhood of several gravitational field sources. Extension of TGR to the situation where a number N of gravitational field sources are interacting (the N-body problem), is accomplished for $N = 2$ and $N = 3$ and shown to admit of straight forward extension to larger values of N, except that it becomes increasingly cumbersome as N increases beyond 4. On the other hand, ϕMAG admits of easy and straight forward extension to the N-body problem for any value of N. Einstein's principle of equivalence is validated in the context of TGR at the neighborhood of any number of gravitational field sources, from which its universal validity follows.

1 Intrinsic theory of gravitational relativity at the neighborhoods of two and several isolated gravitational field sources

1.1 Deriving intrinsic gravitational local Lorentz transformation and establishing intrinsic gravitational local Lorentz invariance at the neighborhood two and several isolated gravitational field sources

Let us start with two gravitational field sources of inertial masses M_1 and M_2 in the relativistic Euclidean 3-space Σ of the theory of gravitational relativity (TGR), whose centers are at radial distances r_1 and r_2 respectively from a point P in Σ. This implies that the centers of the rest masses M_{01} and M_{02} of the field sources are at radial distances r'_1 and r'_2 respectively from the corresponding point P′ in the proper Euclidean 3-space Σ' (at the end of the first stage of evolutions of spacetime/intrinsic spacetime prior to the second stage). The centers of the field sources may be collinear with the point P as illustrated in Fig. 1a, or not, as illustrated in Fig. 1b.

Let different spherical coordinate systems $(r_1, r_1\theta_1, r_1 \sin\theta_1\varphi_1)$ and $(r_2, r_2\theta_2, r_2 \sin\theta_2\varphi_2)$ of the Euclidean 3-space Σ, originate from the centers of the inertial masses M_1 and M_2 respectively of the gravitational field sources. By adding the time coordinates ct_1 and ct_2 to the spherical coordinates of the Euclidean 3-space originating from the centers of M_1 and M_2 respectively, we obtain the four-dimensional space-time coordinate systems $(ct_1, r_1, r_1\theta_1, r_1 \sin\theta_1\varphi_1)$ and $(ct_2, r_2, r_2\theta_2, r_2 \sin\theta_2\varphi_2)$ associated with the gravitational field sources on flat four-dimensional relativistic spacetime (Σ, ct) in the context of the theory of gravitational relativity (TGR).

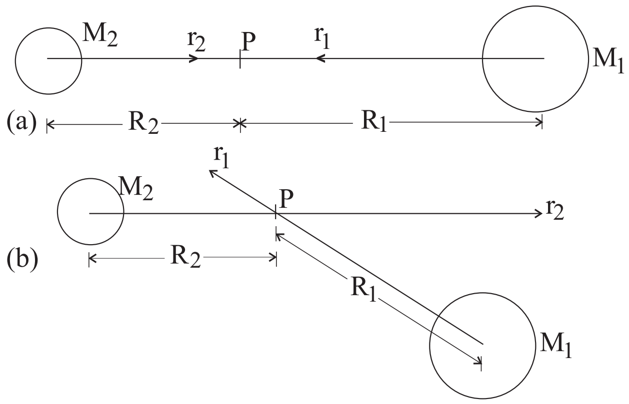

Fig. 1: The inertial masses of two gravitational field sources in the relativistic Euclidean 3-pace with **a** their centers collinear with a point P in 3-space and **b** their centers not collinear with the point P in 3-space.

In the context of the two-dimensional theory of gravitational relativity (ϕTGR) due to the field sources on flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, on the other hand, the point P in Σ is at 'distances' ϕr_1 from the base of the intrinsic inertial mass ϕM_1 and at 'distance' ϕr_2 from the base of ϕM_2. The intrinsic inertial masses ϕM_1 and ϕM_2 can be considered to be aligned along the singular isotropic universal relativistic intrinsic space $\phi\rho$, irrespective of how their inertial masses are arranged in the Euclidean 3-space Σ. Hence the intrinsic space coordinates ϕr_1 and ϕr_2 from the bases of ϕM_1 and ϕM_2 to point P respectively, both lie along the singular isotropic intrinsic space $\phi\rho$, in both the situations where the centers of M_1 and M_2 are collinear with point P as in Fig. 1a, and not, as in Fig. 1b.

In other words, the diagrams in the context of ϕTGR that correspond to Fig. 1a or Fig. 1b is Fig. 2, where only the first and second quadrants of the full two-world diagrams involving four quadrants are shown and the curved 'two-dimensional' absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$ of the metric theory of absolute intrinsic gravity (ϕMAG) is also hidden. The curved proper intrinsic space $\phi\rho'$ containing the intrinsic rest mass ϕM_{02} at its origin is curved relative to the curved proper intrinsic space $\phi\rho''$ containing the intrinsic rest mass ϕM_{01} at its origin in Fig. 2. This corresponds to a situation where a gravitational field source of mass M_2 is contained within the gravitational field of a gravitational field source of larger mass M_1.

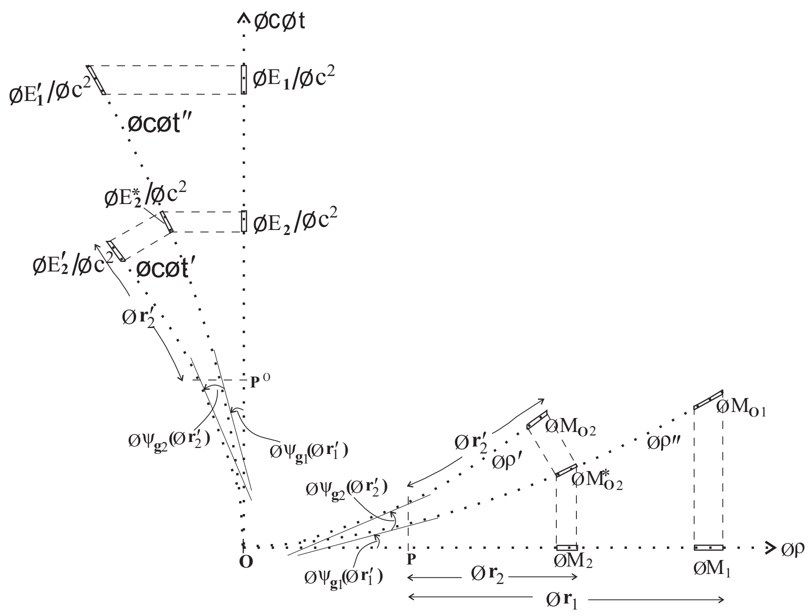

Fig. 2:

Let us temporarily disregard the presence of ϕM_1 in Fig. 1. We must also consider the proper intrinsic space $\phi\rho''$ to temporarily lie along the horizontal in the place of $\phi\rho$, so that the curved $\phi\rho'$ containing ϕM_{02} at its origin is temporarily curved relative to straight line $\phi\rho''$ along the horizontal, and ϕM_{02} in the curved $\phi\rho'$ 'projects' an intermediate intrinsic inertial mass ϕM_2^* into $\phi\rho''$ along the horizontal, as illustrated in the temporary diagram of Fig. 3a.

The following intrinsic gravitational local Lorentz transformation (ϕGLLT) of elementary interval of intrinsic spacetime coordinates $d\phi\rho'$ and $\phi cd\phi t'$ of the curved $(\phi\rho', \phi c\phi t')$ into the projective elementary interval of intrinsic spacetime coordinates $d\phi\rho''$ and $\phi cd\phi t''$) of the flat $(\phi\rho'', \phi c\phi t'')$, at point P in the complete form of Fig. 3a, and its inverse, in the context of ϕTGR, derived originally with the complete diagrams in [1, 2], arise due to the presence of ϕM_{02} along the curved $\phi\rho'$ in the partial

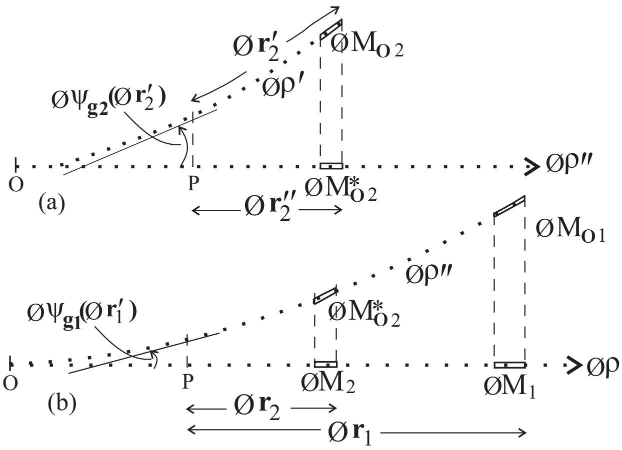

Fig. 3:

temporary diagram of Fig. 3a.

$$\left.\begin{aligned} d\phi t' &= \phi\gamma_{g2}(\phi r_2')(d\phi t'' - \frac{\phi V_{g2}'(\phi r_2')}{\phi c_g^2} d\phi\rho'') \ ; \\ d\phi\rho' &= \phi\gamma_{g2}(\phi r_2')(d\phi\rho'' - \phi V_{g2}'(\phi r_2') d\phi t'') \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} d\phi t'' &= \phi\gamma_{g2}(\phi r_2')(d\phi t' + \frac{\phi V_{g2}'(\phi r_2')}{\phi c_g^2} d\phi\rho') \ ; \\ d\phi\rho'' &= \phi\gamma_{g2}(\phi r_2')(d\phi\rho' + \phi V_{g2}'(\phi r_2') d\phi t') \end{aligned}\right\} \tag{2}$$

where

$$\phi\gamma_{g2}(\phi r_2') = (1 - \frac{\phi V_{g2}'(\phi r_2')^2}{\phi c_g^2})^{-1/2} = (1 - \frac{2G\phi M_{0\mathrm{a}2}}{\phi r_2' \phi c_g^2})^{-1/2} \tag{3a}$$

$$\phi\beta_{g2}(\phi r_2') = \frac{\phi V_{g2}'(\phi r_2')}{\phi c_g} = (\frac{2G\phi M_{0\mathrm{a}2}}{\phi r_2' \phi c_g^2})^{1/2} \tag{3b}$$

System (1) or (2) yields intermediate intrinsic gravitational local Lorentz invariance (ϕGLLI) (with the complete form of the intermediate geometry of Fig. 3a) at point P and at every point in spacetime in all the finite neighborhood of M_2, in the context of ϕTGR namely,

$$\phi c^2 d\phi t''^2 - d\phi\rho''^2 = \phi c^2 d\phi t'^2 - d\phi\rho'^2 \tag{4}$$

This invariance implies that the field source M_2 (with the assumed absence of M_1), prescribes an intermediate flat 2-dimensional intrinsic spacetime $(\phi\rho'', \phi c\phi t'')$ (with constant Lorentzian metric tensor) in the context of ϕTGR in all its finite neighborhood.

Thus when the field source M_1 is allowed to be in place, so that ϕM_{01} appears in the straight line $\phi\rho''$ along the horizontal, at 'distance' $\phi r'_1$ from point P in Fig. 3a, it will cause the curvature of $\phi\rho''$ containing both ϕM^*_{02} and ϕM_{01}, and thereby prescribe partial geometry of Fig. 3b, in which $\phi\rho''$ containing ϕM^*_{02} and ϕM_{01}, is curved relative its projective relativistic intrinsic space $\phi\rho$ along the horizontal. In other words, by bringing M_1 in place, the following intrinsic gravitational local Lorentz transformation and its inverse will arise from the complete form of Fig. 3b,

$$\left.\begin{aligned} d\phi t'' &= \phi\gamma_{g1}(\phi r'_1)(d\phi t - \frac{\phi V'_{g1}(\phi r'_1)}{\phi c_g^2} d\phi\rho) \ ; \\ d\phi\rho' &= \phi\gamma_{g1}(\phi r'_1)(d\phi\rho - \phi V'_{g1}(\phi r'_1) d\phi t) \end{aligned}\right\} \tag{5}$$

and

$$\left.\begin{aligned} d\phi t &= \phi\gamma_{g1}(\phi r'_1)(d\phi t'' + \frac{\phi V'_{g1}(\phi r'_1)}{\phi c_g^2} d\phi\rho'') \ ; \\ d\phi\rho &= \phi\gamma_{g1}(\phi r'_1)(d\phi\rho'' + \phi V'_{g1}(\phi r'_1) d\phi t'') \end{aligned}\right\} \tag{6}$$

where

$$\phi\gamma_{g1}(\phi r'_1) = (1 - \frac{\phi V_{g1}(\phi r'_1)^2}{\phi c_g^2})^{-1/2} = (1 - \frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2})^{-1/2} \tag{7a}$$

$$\phi\beta_{g1}(\phi r'_1) = \frac{\phi V'_{g1}(\phi r'_1)}{\phi c_g} = (\frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2})^{1/2} \tag{7b}$$

The presence of the intrinsic rest mass ϕM_{02} solely, transforms the curved proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$ into intermediate flat intrinsic $(\phi\rho'', \phi c\phi t'')$ in all its finite neighborhood in the intermediate diagram of Fig. 3a. The intrinsic rest mass ϕM_{02} along the curved $\phi\rho'$ is likewise transformed into intermediate intrinsic inertial mass ϕM^*_{02} in $\phi\rho''$ along the horizontal in Fig. 3a. And when the intrinsic rest mass ϕM_{01} of the other field source is brought into the $\phi\rho''$ along the horizontal, it causes the curvature of $\phi\rho''$ containing ϕM^*_{02} and ϕM_{01} in Fig. 3b. The resulting curved $(\phi\rho'', \phi c\phi t'')$ in the complete form of Fig. 3b is then transformed into the final flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, and the intrinsic rest masses ϕM^*_{02} and ϕM_{01} in the curved $\phi\rho''$ are transformed into the final intrinsic inertial masses ϕM_{02} and ϕM_{01} respectively at their respective positions in $\phi\rho$ along the horizontal.

The partial two-world diagram of Fig. 2 is the resultant of Figs. 3a and 3b with the respective curved proper intrinsic time dimensions included.

Again system (5) or (6) yields intrinsic gravitational local Lorentz invariance (ϕGLLT) at point P an at every point in space in all finite neighborhood of M_1 namely,

$$\phi c^2 d\phi t^2 - d\phi\rho^2 = \phi c^2 d\phi t''^2 - d\phi\rho''^2 \tag{8}$$

Thus the simultaneous presence of the gravitational field sources M_1 and M_2 prescribe intrinsic gravitational local Lorentz invariance and consequently the Lorentzian metric tensor of ϕTGR on the resultant relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ at point P and at every point in spacetime in all their finite neighborhood.

It is actually the intrinsic spacetime intervals $d\phi\rho''$ and $\phi cd\phi t''$, which the corresponding intervals $d\phi\rho'$ and $\phi cd\phi t'$ in the curved $(\phi\rho', \phi c\phi t')$, projects into the curved proper intrinsic spacetime $(\phi\rho'', \phi c\phi t'')$ at point P in the complete two-world form of the intermediate Fig. 3a, that becomes curved along with the curved $\phi\rho''$ and $\phi c\phi t''$ and project intervals $d\phi\rho$ and $\phi cd\phi t$ respectively into the flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ at point P in the complete two-world form of Fig. 3b or in the complete two-world form of the resultant Fig. 2. Thus the intrinsic gravitational local Lorentz invariance (8) can therefore be combined with the intrinsic gravitational local Lorentz invariance (4) to have

$$\phi c^2 d\phi t^2 - d\phi\rho^2 = \phi c^2 d\phi t''^2 - d\phi\rho''^2 = \phi c^2 d\phi t'^2 - d\phi\rho'^2 \tag{9}$$

Equation (9) states formally intrinsic gravitational local Lorentz invariance (ϕGLLI) in the context of the intrinsic theory of gravitational relativity (ϕTGR), in terms of the intrinsic spacetime coordinate intervals of the co-existing curved proper intrinsic spacetimes $(\phi\rho', \phi\phi t')$ and $(\phi\rho'', \phi\phi t'')$ and their underlying flat relativistic intrinsic spacetime $(\phi\rho, \phi\phi t)$, at the neighborhood of two gravitational field sources of inertial masses M_1 and M_2. The curved proper intrinsic spacetime $(\phi\rho', \phi\phi t')$ due to the presence of M_2, is curved relative to the curved proper intrinsic spacetime $(\phi\rho'', \phi\phi t'')$ due to the presence of M_1, and $(\phi\rho'', \phi\phi t'')$ is curved relative to the underlying flat relativistic intrinsic spacetime $(\phi\rho, \phi\phi t)$, as illustrated partially in Fig. 2. The ϕGLLI (9) states the flatness of the underlying relativistic intrinsic spacetime everywhere at the neighborhood of M_1 and M_2.

The procedure used to establish the ϕGLLI (9) at the neighborhood of two isolated gravitational field sources above, admits of straight forward extension to the neighborhood of three isolated gravitational field sources. If a third gravitational field source of inertial mass M_3 is brought into the neigborhood of M_1 and M_1 in Fig. 1a or 1b, such that M_3 is located at radial distance r_3 from the point P, where it shall be assumed that M_3 is contained within the gravitational field of M_2, which, in turn, is contained in the gravitational field of M_1, then we must let M_3 establish a third curved proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$; M_2 to establish curved proper intrinsic spacetime $\phi\rho'', \phi c\phi t'')$ and M_1 to establish curved proper intrinsic spacetime $(\phi\rho''', \phi c\phi t''')$, such that the curved $(\phi\rho', \phi c\phi t')$ due to M_3 is curved relative to the curved $(\phi\rho'', \phi c\phi t'')$ due to M_2, which, in turn, is curved relative to the curved $(\phi\rho''', \phi c\phi t''')$ due to M_1 that is curved relative to resultant flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ in modified form of the partial diagram of Fig. 2.

The tandem of intrinsic gravitational local Lorentz transformation at two levels above must then be extended to three levels. The elementary intrinsic coordinate intervals $(d\phi\rho'$ and $\phi cd\phi t'$ of the upper most curved intrinsic spacetime $(\phi\rho', \phi c\phi t')$, must be transformed into $(d\phi\rho''$ and $\phi cd\phi t''$ of the underlying curved intrinsic spacetime $(\phi\rho'', \phi c\phi t'')$ in the gravitational field of isolated M_3 at the first level, followed by transformation of $(d\phi\rho''$ and $\phi cd\phi t''$ into $(d\phi\rho'''$ and $\phi cd\phi t'''$ in the gravitational field of isolated M_2 at the second level, and followed by the transformation of $(d\phi\rho'''$ and $\phi cd\phi t'''$ into $(d\phi\rho$ and $\phi cd\phi t$ in the gravitational field of isolated M_1 at the third level. Doing these will lead to the following ϕGLLI

$$\phi c^2 d\phi t^2 - d\phi\rho^2 = \phi c^2 d\phi t'''^2 - d\phi\rho'''^2 = \phi c^2 d\phi t''^2 - d\phi\rho''^2 = \phi c^2 d\phi t'^2 - d\phi\rho'^2 \tag{10}$$

The ϕGLLI (10) again says that the resultant relativistic intrinsic spacetime of ϕTGR is flat everywhere in all finite the neighborhood of three isolated gravitational field sources of inertial masses M_1 M_2 and M_3 in the relativistic Euclidean 3-space Σ, irrespective of how they are scattered in Σ.

The procedure used to derive Eq. (9) for two isolated gravitational field sources and Eq. (10) for three isolated gravitational field sources, admits of extension to four, five and any number N of isolated gravitational field sources in Σ. We conclude from this that the relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ of the intrinsic theory of gravitational relativity (ϕTGR) is everywhere flat in all finite neighborhood of any number N of gravitational field sources that are scattered in arbitrary manner in 3-space.

1.2 The resultant intrinsic gravitational time dilation and resultant intrinsic gravitational length contraction at the neighborhood of two and several gravitational field sources

The intrinsic time dilation and intrinsic length contraction formulae implied by systems (1) and (2), as derived in [2, 3], are the following

$$d\phi t'' = \phi\gamma_{g2}(\phi r_2')d\phi t' = \left(1 - \frac{2G\phi M_{0a2}}{\phi r_2'\phi c_g^2}\right)^{-1/2} d\phi t' \tag{11}$$

$$d\phi\rho'' = \phi\gamma_{g2}(\phi r_2')^{-1} d\phi\rho' = \left(1 - \frac{2G\phi M_{0a2}}{\phi r_2'\phi c_g^2}\right)^{1/2} d\phi\rho' \tag{12}$$

The intrinsic time dilation and intrinsic length contraction formulae implied by systems (5) and (6) are likewise given as follows

$$d\phi t = \phi\gamma_{g1}(\phi r_1')d\phi t'' = (1 - \frac{2G\phi M_{0a1}}{\phi r_1'\phi c_g^2})^{-1/2} d\phi t'' \quad (13)$$

$$d\phi\rho = \phi\gamma_{g1}(\phi r_1')^{-1}d\phi\rho'' = (1 - \frac{2G\phi M_{0a1}}{\phi r_1'\phi c_g^2})^{1/2} d\phi\rho'' \quad (14)$$

By eliminating $d\phi t''$ between Eqs. (11) and (13) and by eliminating $d\phi\rho''$ between Eqs. (12) and (14) we obtain the resultant intrinsic gravitational time dilation and resultant intrinsic gravitational length contraction respectively as follows

$$\begin{aligned} d\phi t &= \phi\gamma_{g1}(\phi r_1')\phi\gamma_{g2}(\phi r_2')d\phi t' \\ &= (1 - \frac{2G\phi M_{0a1}}{\phi r_1'\phi c_g^2})^{-1/2}(1 - \frac{2G\phi M_{0a2}}{\phi r_2'\phi c_g^2})^{-1/2} d\phi t' \quad (15) \end{aligned}$$

$$\begin{aligned} d\phi\rho &= \phi\gamma_{g1}(\phi r_1')^{-1}\phi\gamma_{g2}(\phi r_2')^{-1}d\phi\rho' \\ &= (1 - \frac{2G\phi M_{0a1}}{\phi r_1'\phi c_g^2})^{1/2}(1 - \frac{2G\phi M_{0a2}}{\phi r_2'\phi c_g^2})^{1/2} d\phi\rho' \quad (16) \end{aligned}$$

As follows naturally (by induction) from the above, the resultant intrinsic gravitational time dilation and the resultant intrinsic gravitational length contraction of the elementary intrinsic spacetime intervals $d\phi\rho'$ and $d\phi t'$ of the (upper most) Nth curved proper intrinsic spacetime $(\phi\rho', \phi c\phi t')$, of a tandem of N curved proper intrinsic spacetimes namely, a curved $(\phi\rho', \phi c\phi t')$ above a curved $(\phi\rho'', \phi c\phi t'')$, above a curved $(\phi\rho''', \phi c\phi t''')$, above a curved $(\phi\rho'''', \phi c\phi t'''')$, ... , above a curved $(\phi\rho^{n\prime}, \phi c\phi\phi t^{n\prime})$ above a flat $(\phi\rho, \phi c\phi t)$, as in Fig. 2a for N=2, within a local Lorentz frame at a point P in the Euclidean 3-space Σ, at the neighborhood of the N gravitational field sources scattered arbitrarily in Σ about this point, are given respectively as follows

$$\begin{aligned} d\phi t &= \phi\gamma_{g1}(\phi r_1')\phi\gamma_{g2}(\phi r_2')\phi\gamma_{g3}(\phi r_3')\ldots\phi\gamma_{gN}(\phi r_N')d\phi t' \\ &= (1 - \frac{2G\phi M_{0a1}}{\phi r_1'\phi c_g^2})^{-1/2}(1 - \frac{2G\phi M_{0a2}}{\phi r_2'\phi c_g^2})^{-1/2}\ldots(1 - \frac{2G\phi M_{0aN}}{\phi r_N'\phi c_g^2})^{-1/2} d\phi t' \end{aligned} \quad (17)$$

or

$$d\phi t = \prod_{k=1}^{N}\phi\gamma_{gk}(\phi r_k')d\phi t' = \prod_{k=1}^{N}(1 - \frac{2G\phi M_{0ak}}{\phi r_k'\phi c_g^2})^{-1/2} d\phi t' \quad (18)$$

and

$$\begin{aligned} d\phi\rho &= \phi\gamma_{g1}(\phi r_1')^{-1}\phi\gamma_{g2}(\phi r_2')^{-1}\phi\gamma_{g3}(\phi r_3')^{-1}\ldots\phi\gamma_{gN}(\phi r_N')^{-1}d\phi\rho' \\ &= (1-\frac{2G\phi M_{0\mathrm{a}1}}{\phi r_1'\phi c_g^2})^{1/2}(1-\frac{2G\phi M_{0\mathrm{a}2}}{\phi r_2'\phi c_g^2})^{1/2}\ldots(1-\frac{2G\phi M_{0\mathrm{a}N}}{\phi r_N'\phi c_g^2})^{1/2}d\phi\rho' \end{aligned} \tag{19}$$

or

$$d\phi\rho = \prod_{k=1}^{N}\gamma_{gk}(\phi r_k')^{-1}d\phi\rho' = \prod_{k=1}^{N}(1-\frac{2G\phi M_{0\mathrm{a}k}}{\phi r_k'\phi c_g^2})^{1/2}d\phi\rho' \tag{20}$$

The $d\phi\rho$ and $d\phi t$ in Eqs. (17) – (20) are the relativistic intrinsic space and relativistic intrinsic time coordinate intervals (in the resultant underlying flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ at the neighborhood of the N gravitational field sources, and $\phi r_k'$ is the 'distance' from the base of the intrinsic rest mass M_{0k} to point P in the proper intrinsic space $\phi\rho'$, which corresponds to 'distance' ϕr_k from the base of the intrinsic inertial mass ϕM_k to point P in the relativistic intrinsic space $\phi\rho$, of the kth gravitational field source of intrinsic inertial mass ϕM_k. Equations (17) through (20) have been written at point P along $\phi\rho$, which corresponds to point P in the relativistic Euclidean 3-space Σ about which the N gravitational field sources are scattered.

Let us denote the resultant factor $\phi\gamma_g$ in the resultant intrinsic length contraction and resultant intrinsic time dilation formulae above by $\phi\bar{\gamma}_g$. Then as follows from Eqs. (18) and (20), $\phi\bar{\gamma}_g$ is given as follows

$$\phi\bar{\gamma}_g = \prod_{k=1}^{N}\phi\gamma_{gk}(\phi r_k') = \prod_{k=1}^{N}(1-\frac{2G\phi M_{0\mathrm{a}k}}{\phi r_k'\phi c_g^2})^{-1/2} \tag{21}$$

For $N = 2$,

$$\begin{aligned} \phi\bar{\gamma}_g &= (1-\frac{2G\phi M_{0\mathrm{a}1}}{\phi r_1'\phi c_g^2})^{-1/2}(-\frac{2G\phi M_{0\mathrm{a}2}}{\phi r_2'\phi c_g^2})^{-1/2} \\ &= (1-\frac{2G\phi M_{0\mathrm{a}1}}{\phi r_1'\phi c_g^2}-\frac{2G\phi M_{0\mathrm{a}2}}{\phi r_2'\phi c_g^2}+\frac{4G^2\phi M_{0\mathrm{a}1}\phi M_{0\mathrm{a}2}}{\phi r_1'\phi r_2'\phi c_g^4})^{-1/2} \end{aligned} \tag{22}$$

1.3 Intrinsic mass relation and intrinsic kinetic energy relation at the neighborhood of two and several isolated gravitational field sources

Intrinsic mass relation in the context of the intrinsic theory of gravitational relativity (ϕRTG) at the neighborhood of a singular gravitational field source derived formally in [2] is

$$\phi m = \phi m_0 \phi\gamma_g(\phi r')^{-2} = \phi m_0(1 - \frac{2G\phi M_{0a}}{\phi r' \phi c_g^2}) \tag{23}$$

We must simply replaced $\phi\gamma_g(\phi r')^{-2}$ by the resultant factor $\phi\overline{\gamma}_g^{-2}$ in this relation to have intrinsic mass relation at the neighborhood of isolated N gravitational field sources that are scattered arbitrarily in the Euclidean 3-space Σ as follows

$$\phi m = \phi m_0(\phi\overline{\gamma}_g^t)^{-2} = \phi m_0 \prod_{k=1}^{N}(1 - \frac{2G\phi M_{0ak}}{\phi r'_k \phi c_g^2}) \tag{24}$$

For $N = 2$,

$$\begin{aligned} \phi m &= \phi m_0(1 - \frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2})(-\frac{2G\phi M_{0a2}}{\phi r'_2 \phi c_g^2}) \\ &= \phi m_0(1 - \frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2} - \frac{2G\phi M_{0a2}}{\phi r'_2 \phi c_g^2} + \frac{4G^2\phi M_{0a1}\phi M_{0a2}}{\phi r'_1 \phi r'_2 \phi c_g^4}) \end{aligned} \tag{25}$$

The corresponding intrinsic special relativistic kinetic energy expression in the context of ϕTGR at the neighborhood of the isolated N gravitational field sources is

$$\begin{aligned} \overline{T} &= \phi m_0 \phi c_\gamma^2 \phi\overline{\gamma}_g^{-2}[\gamma(\phi v) - 1] \\ &= \phi m_0 \phi c_\gamma^2 \prod_{k=1}^{N}(1 - \frac{2G\phi M_{0ak}}{\phi r'_k \phi c_g^2})[(1 - \frac{\phi v^2}{\phi c_\gamma^2})^{-1/2} - 1] \end{aligned} \tag{26}$$

For N=2,

$$\overline{T} = \phi m_0 \phi c_\gamma^2(1 - \frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2} - \frac{2G\phi M_{0a2}}{\phi r'_2 \phi c_g^2} + \frac{4G^2\phi M_{0a1}\phi M_{0a2}}{\phi r'_1 \phi r'_2 \phi c_g^4})[(1 - \frac{\phi v^2}{\phi c_\gamma^2})^{-1/2} - 1] \tag{27}$$

The transformation of every other intrinsic parameter in the context of ϕTGR at the neighborhood of N isolated gravitational field source, can likewise be written by simply replacing the factor $\phi\gamma_g(\phi r')$ in the expression at the neighborhood of one gravitational field source by $\phi\overline{\gamma}_g$.

2 Theory of gravitational relativity at the neighborhood of two and several isolated gravitational field sources

2.1 *Deriving gravitational local Lorentz transformation and establishing gravitational local Lorentz invariance at the neighborhood of two and several isolated gravitational field sources*

Let us consider Fig. 1a and 1b again. In deriving the gravitational local Lorentz transformation GLLT and its inverse at point P due to both gravitational field sources

M_1 and M_2, we must let M_1 to be temporarily absent and derive GLLT and its inverse due to M_2 solely on an intermediate spacetime (Σ'', ct''). The GLLT and its inverse to be derived for M_2 assuming M_1 is temporarily absent, are the outward manifestations on the intermediate four-dimensional spacetime (Σ'', ct'') of the ϕGLLT and its inverse of systems (1) and (2) on intermediate flat intrinsic spacetime $(\phi\rho'', \phi c\phi t'')$ established by ϕM_{02}, assuming the temporary absence of ϕM_{01} in Fig. 2 (with the partial geometry of Fig. 3a). They are given as follows

$$\left.\begin{aligned} dt'_2 &= \gamma_{g2}(dt''_2 - \frac{V'_{g2}(r'_2)}{c_g^2} dr''_2) \; ; \\ dr'_2 &= \gamma_{g2}(dr''_2 - V_{g2}(r'_2) dt''_2) \; ; \; r'_2 d\theta'_2 = r''_2 d\theta''_2 \; ; \\ & \text{and } \; r'_2 \sin\theta'_2 d\varphi'_2 = r''_2 \sin\theta''_2 d\varphi''_2 \end{aligned}\right\} \tag{28}$$

and

$$\left.\begin{aligned} dt''_2 &= \gamma_{g2}(dt'_2 + \frac{V'_{g2}(r'_2)}{c_g^2} dr'_2) \; ; \\ dr''_2 &= \gamma_{g2}(dr'_2 + V_{g2}(r'_2) dt'_2) \; ; \; r''_2 d\theta''_2 = r'_2 d\theta'_2 \; ; \\ & \text{and } \; r''_2 \sin\theta''_2 d\varphi''_2 = r'_2 \sin\theta'_2 d\varphi'_2 \end{aligned}\right\} \tag{29}$$

Either the GLLT (28) or its inverse (29) yields gravitational local Lorentz invariance at point P on the intermediate spacettime (Σ'', ct''),

$$c^2 dt''^2_2 - dr''^2_2 - r''^2_2(d\theta''^2_2 + \sin\theta''^2_2 d\varphi''^2_2) = c^2 dt'^2_2 - dr'^2_2 - r'^2_2(d\theta'^2_2 + \sin\theta'^2_2 d\varphi'^2_2) \tag{30}$$

This GLLT guarantees the flatness at point P and everywhere of the intermediate flat four-dimensional spacetime (Σ'', ct'') that evolves due to the presence of M_2 solely, with the assumed temporary absence of M_1.

Then let M_1 be brought in place at radial distance r_1 from the point P on the intermediate flat spacetime (Σ'', ct'') prescribed by M_2 solely. The intermediate flat (Σ'', ct'') will be transformed into the final flat relativistic spacetime (Σ, ct) due to the presence of M_1. The following GLLT and its inverse, which are the outward manifestations on the flat relativistic spacetime (Σ, ct) of the ϕGLLT and its inverse of systems (5) and (6) obtain,

$$\left.\begin{aligned} dt''_1 &= \gamma_{g1}(dt_1 - \frac{V'_{g1}(r'_1)}{c_g^2} dr_1) \; ; \\ dr''_1 &= \gamma_{g1}(dr_1 - V_{g1}(r'_1) dt_1) \; ; \; r''_1 d\theta''_1 = r_1 d\theta_1 \; ; \\ & \text{and } \; r''_1 \sin\theta''_1 d\varphi''_1 = r_1 \sin\theta_1 d\varphi_1 \end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned} dt_1 &= \gamma_{g1}(dt_1'' + \frac{V_{g1}'(r_1')}{c_g^2}dr_1'') ; \\ dr_1 &= \gamma_{g1}(dr_1'' + V_{g1}(r_1')dt_1'') ;\ r_1 d\theta_1 = r_1'' d\theta_1'' ; \\ &\text{and }\ r_1 \sin\theta_1 d\varphi_1 = r_1'' \sin\theta_1'' d\varphi_1'' \end{aligned}\right\} \quad (32)$$

Either the GLLT (31) or its inverse (32) yields gravitational local Lorentz invariance on the final (or rfesultant) relativistic spacettime (Σ, ct),

$$c^2dt_1^2 - dr_1^2 - r_1^2(d\theta_1^2 + \sin\theta_1^2 d\varphi_1^2) = c^2dt_1''^2 - dr_1''^2 - r_1''^2(d\theta_1''^2 + \sin\theta_1''^2 d\varphi_1''^2) \quad (33)$$

This GLLT guarantees the flatness at point P and everywhere of the resultant relativistic four-dimensional spacetime (Σ, ct) that evolves due to the presence of both M_2 and M_1. As a matter of fact, the flatness of the resultant relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ in all finite neighborhood of M_1 and M_2 guaranteed by the intrinsic gravitational local Lorentz invariance (ϕGLLI) of Eq. (9) or (10), already implies the flatness of (Σ, ct) in all finite neighborhood of M_1 and M_1, since (Σ, ct) is the outward manifestation of $(\phi\rho, \phi c\phi t)$.

The tandem of gravitational local Lorentz transformation (GLLT) and its inverse done to two levels due to the presence of two isolated gravitational field sources above, admits of straight forward extension to the third level, due to the presence of three isolated gravitational field sources; to the fourth level, due to the presence of four isolated gravitational field sources; . . . ; and to the Nth level, due to the presence of N isolated gravitational field sources. It then follows that the final (or resultant) relativistic spacetime (Σ, ct) that evolves in the tandem of GLLT is flat everywhere in all finite neighborhood of any number N of gravitational field sources, whose inertial masses M_1, M_2, M_3, . . . , M_N are scattered arbitrarily in the relativistic Euclidean 3-space Σ.

2.2 The resultant gravitational time dilation at the neighborhood of two and several isolated gravitational field sources

The gravitational time dilation formula implied by systems (28) and (29), as derived in [2, 3], is the following

$$dt_2'' = \gamma_{g2}(r_2')dt_2' = (1 - \frac{2GM_{0a2}}{r_2' c_g^2})^{-1/2} dt_2' \quad (34)$$

The gravitational time dilation formula implied by systems (31) and (32) is likewise given as

$$dt_1 = \gamma_{g1}(r_1')dt_1'' = (1 - \frac{2GM_{0a1}}{r_1' c_g^2})^{-1/2} dt_1'' \quad (35)$$

The proper spacetime coordinate system $(ct_2', r_2', r_2'\theta', r_2' \sin\theta_2'\varphi_2')$ attached to the rest mass M_{02} of the second gravitational field source is contained within the gravitational field of the first gravitational field source M_1. Consequently the time interval dt_2'' in Eq. (34) at point P is further dilated to dt_2 by the gravitational field of M_1 as follows

$$dt_2 = \gamma_{g1}(r_1')dt_2'' = (1 - \frac{2GM_{0\mathrm{a}1}}{r_1'c_g^2})^{-1/2}dt_2'' \tag{36}$$

On the other hand, the proper spacetime coordinate system $(ct_1'', r_1'', r_1''\theta_1'', r_1'' \sin\theta_1''\varphi_1'')$ attached to M_{01} is not contained within the gravitational field of M_2. Consequently dt_1 in Eq. (35) at point P is not further dilated by the gravitational field of M_{02}.

Eliminating dt_2'' between Eqs. (34) and (36) we have

$$dt_2 = \gamma_{g1}(r_1')\gamma_{g2}(r_2')dt_2' = (1 - \frac{2GM_{0\mathrm{a}1}}{r_1'c_g^2})^{-1/2}(1 - \frac{2GM_{0\mathrm{a}2}}{r_2'c_g^2})^{-1/2}dt_2' \tag{37}$$

Equation (37) shall be written in terms of unsubscripted time intervals dt and dt' at the neighborhood of the gravitational field M_2 as follows

$$dt = \gamma_{g1}(r_1')\gamma_{g2}(r_2')dt' = (1 - \frac{2GM_{0\mathrm{a}1}}{r_1'c_g^2})^{-1/2}(1 - \frac{2GM_{0\mathrm{a}2}}{r_2'c_g^2})^{-1/2}dt' \tag{38}$$

The procedure used to derive Eq. (38) in the case of two isolated gravitational field sources, admits of direct extension to the case of three isolated gravitational field sources; four isolated gravitational field sources; ... and any number N of isolated gravitational field sources. Thus the resultant time dilation formula in the case of N gravitational field sources whose inertial masses M_1, M_2, M_3, ... , M_N are scattered arbitrarily in the relativistic Euclidean 3-space Σ is the following

$$\begin{aligned} dt &= \gamma_{g1}(r_1')\gamma_{g2}(r_2')\gamma_{g3}(r_3')\ldots\gamma_{gN}(r_N')dt' \\ &= (1 - \frac{2GM_{0\mathrm{a}1}}{r_1'c_g^2})^{-1/2}(1 - \frac{2GM_{0\mathrm{a}2}}{r_2'c_g^2})^{-1/2}\ldots(1 - \frac{2GM_{0\mathrm{a}N}}{r_N'c_g^2})^{-1/2}dt' \end{aligned} \tag{39}$$

or

$$dt = \prod_{k=1}^{N}\gamma_{gk}(r_k')dt' = \prod_{k=1}^{N}(1 - \frac{2GM_{0\mathrm{a}k}}{r_k'c_g^2})^{-1/2}dt' \tag{40}$$

Equations (39) and (40) are the outward manifestations on flat spacetime (Σ, ct) in the context of TGR of Eqs. (17) and (18) on flat intrinsic spacetime $(\phi\rho, \phi c\phi t)$ in the context of ϕTGR.

We shall denote the resultant factor γ_g in the resultant gravitational time dilation formula (40) in the context of TGR by $\overline{\gamma}_g^{\,t}$ and write,

$$\overline{\gamma}_g^{\,t} = \prod_{i=1}^{N} \gamma_{gk}(r'_k) = \prod_{k=1}^{N}(1 - \frac{2GM_{0\mathrm{a}k}}{r'_k c_g^2})^{-1/2} \tag{41}$$

For $N = 2$, Eq. (41) simplifies as follows

$$\begin{aligned} \overline{\gamma}_g^{\,t} &= (1 - \frac{2GM_{0\mathrm{a}1}}{r'_1 c_g^2})^{-1/2}(1 - \frac{2GM_{0\mathrm{a}2}}{r'_2 c_g^2})^{-1/2} \\ &= (1 - \frac{2GM_{0\mathrm{a}1}}{r'_1 c_g^2} - \frac{2GM_{0\mathrm{a}2}}{r'_2 c_g^2} + \frac{4G^2 M_{0\mathrm{a}1} M_{0\mathrm{a}2}}{r'_1 r'_2 c_g^4})^{-1/2} \end{aligned} \tag{42}$$

2.3 Resultant gravitational length contraction at the neighborhood of two and several isolated gravitational field sources

While the manner in which the gravitational field sources are scattered in 3-space about point P does not count in the formulae for resultant intrinsic gravitational length contraction and resultant intrinsic time dilation in the context of ϕTGR, as well as in resultant time dilation formula in the context of TGR, as found too this point, it does in the resultant gravitational length contraction in the context of TGR.

Let us consider first the case of two gravitational field sources whose centers are collinear with a point P of interest in Σ, as illustrated in Fig. 1a. In this situation, the radial coordinate r_1 of the spherical coordinates $r_1, r_1\theta_1$ and $r_1 \sin\theta_1\varphi_1$ of the Euclidean 3-space Σ that originate from the center of the assumed spherical M_1, and the radial coordinate r_2 of the spherical coordinates $r_2, r_2\theta_2$ and $r_2 \sin\theta_2\varphi_2$ of Σ that originate from the center of the assumed spherical M_2, lie along the same line at point P. Consequently there is gravitational length contraction of the proper elementary radial coordinate interval dr' that lies along the straight line joining the centers of the field sources and point P at point P, due to the combined gravitational fields of M_1 and M_2, but gravitational contraction does not occur along any other direction in the 3-space Σ at point P.

If we prescribe proper elementary coordinate intervals $dr', r'd\theta'$ and $r' \sin\theta' d\varphi'$ within the local Lorentz frame at P, such that dr' lies along the straight line joining the centers of M_1 and M_2 and point P, then dr' will be contracted by virtue of the gravitational fields of both sources. The resultant gravitational length contraction at point P in this situation is the following

$$dr = \gamma_{g1}(r'_1)^{-1}\gamma_{g2}(r'_2)^{-1}dr';\ rd\theta = r'd\theta' \text{ and } r\sin\theta d\varphi = r' \sin\theta' d\varphi' \tag{43}$$

or

$$dr = (1 - \frac{2GM_{0\text{a}1}}{r_1'c_g^2})^{1/2}(1 - \frac{2GM_{0\text{a}2}}{r_2'c_g^2})^{1/2}dr' \ ; \ rd\theta = r'd\theta' \ ;$$
$$\text{and } \ r\sin\theta d\varphi = r'\sin\theta' d\varphi' \qquad (44)$$

The resultant gravitational length contraction formula (44) in the case of isolated two gravitational field sources whose centers are collinear with a point P in 3-space at which (44) is written, can be generalized to a situation where the centers of N gravitational field sources are collinear with the point P of interest. In that case, Eq. (44) becomes the following

$$dr = \prod_{k=1}^{N} \gamma_{gi}(r_k')^{-1}dr'; \ rd\theta = r'd\theta' \text{ and } r\sin\theta d\varphi = r'\sin\theta' d\varphi' \qquad (45)$$

or

$$dr = \prod_{k=1}^{N}(1 - \frac{2GM_{0\text{a}k}}{r_k'c_g^2})^{1/2}dr'; \ rd\theta = r'd\theta' \text{ and } r\sin\theta d\varphi = r'\sin\theta' d\varphi' \qquad (46)$$

Now let us consider the situation where the centers of the assumed gravitational field sources are not collinear with the point P of interest, as illustrated for two gravitational field sources in Fig. 1b. In this situation, the radial coordinate r_1 of the spherical coordinates $r_1, r_1\theta_1$ and $r_1 \sin\theta_1\varphi_1$ that originate from the center of M_1 do not lie along the same line with the radial coordinate r_2 of the spherical coordinates $r_2, r_2\theta_2$ and $r_2\sin\theta_2\varphi_2$ that originate from the center of M_2 at point P. In this situation there is no length contraction by virtue of gravitational fields of M_1 and M_2 conjointly. Rather an observer would measure gravitational length contraction, $dl = \gamma_{g1}^{-1}dl' = (1 - 2GM_{0\text{a}1}/r_1'c_g^2)^{1/2}dl'$, along the radial direction r_1 originating from the center of M_1 and $dl = \gamma_{g2}^{-1}dl' = (1 - 2GM_{0\text{a}2}/r_2'c_g^2)^{1/2}dl'$, along the radial coordinate r_2 originating from the center of M_2, and zero gravitational length contraction along every other direction in Σ at point P, in Fig. 1b. On the other hand, the resultant time dilation at P in this situation of Fig. 2b, as in the situation of Fig. 2a, is given by Eq. (40) for N=2.

Let us consider four assumed spherical gravitational field sources of masses M_1, M_2, M_3 and M_4, whose centers are at radial distances R_1, R_2, R_3 and R_4 respectively from a point P of interest in the physical Euclidean 3-space Σ, as illustrated in Fig. 4. The resultant gravitational length contractions at point P are the following:

$$dl = (1 - 2GM_{0\text{a}1}/R_1'c_g^2)^{1/2}(1 - 2GM_{0\text{a}4}/R_4'c_g^2)^{1/2}dl'; \ (\text{along } r_1 \text{ and } r_4)$$

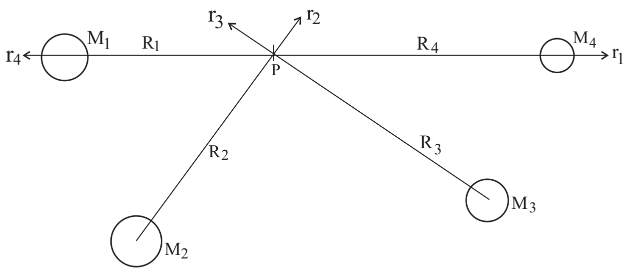

Fig. 4: The inertial masses of four gravitational field sources scattered about a point in the physical Euclidean 3-space.

$$dl = (1 - 2GM_{0a2}/R_2' c_g^2)^{1/2} dl';\ \text{(along } r_2)$$
$$dl = (1 - 2GM_{0a3}/R_3' c_g^2)^{1/2} dl';\ \text{(along } r_3)$$
$$dl = dl';\ \text{(along every other direction in } \Sigma)$$

Whereas the resultant gravitational time dilation at P in Fig. 4 is given by Eq. (40) for N=4.

2.3.1 Gravitational deformation of the shape of a solid object located in the gravitational field of several sources

A solid object located within the gravitational field of a single source will be contracted along a direction from the center of the field source through the object. The shape of the object will be altered as a consequence, as illustrated for a spherical object in Fig. 5a, while Fig. 5b illustrates the resultant shape of a spherical object located within the gravitational field of three isolated sources. The deformations of the shapes of the objects have been exaggerated in both Figs. 5a and 5b for clarity.

2.4 Transformations of mass and other physical parameters and physical constants in the context of TGR at the neighborhood of two and several isolated gravitational field sources

The mass relation derived in the context of TGR in [2, 4] at radial distance r from the center of the inertial mass M of a gravitational field source in Σ is the following

$$m = \gamma_g^{-2}(r') m_0 = m_0 (1 - \frac{2GM_{0a}}{r' c_g^2}) \tag{47}$$

where M_0 is the rest mass of the gravitational field source in the flat proper Euclidean 3-space Σ' of inertial mass M in Σ ; r' is the radial distance from the center of M_0 in Σ'; m_0 is the rest mass in Σ' of the test particle of inertial mass m in Σ.

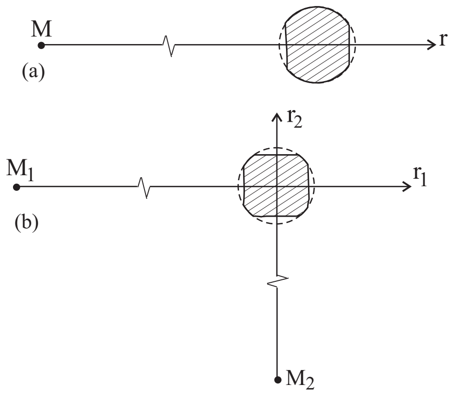

Fig. 5: The gravitationally deformed shape of a spherical object located at the neighborhood of **a** one gravitational field source and **b** two gravitational field sources.

If we now consider this test particle to be located at a point in space, which is of radial distances r_1, r_2 and r_3 from the centers of the inertial masses M_1, M_2 and M_3 respectively of gravitational field sources, then the mass relation (47) must be replaced by the following

$$m = (\overline{\gamma}_g^t)^{-2} m_0 = \gamma_{g3}(r_3')^{-2}\gamma_{g2}(r_2')^{-2}\gamma_{g1}(r_1')^{-2} m_0$$

or

$$m = (1 - \frac{2GM_{0a3}}{r_3' c_g^2})(1 - \frac{2GM_{0a2}}{r_2' c_g^2})(1 - \frac{2GM_{0a1}}{r_1' c_g^2}) m_0 \qquad (48)$$

In general, for N gravitational field sources scattered in space about a test particle we must write,

$$m = \prod_{i=1}^{N} (1 - \frac{2GM_{0a\,i}}{r_i' c_g^2}) m_0 \qquad (49)$$

It is the resultant factor $\overline{\gamma}_g^t$ in the resultant gravitational time dilation formula at a point in space at the neighborhood of N gravitational field sources, given by Eq. (41) that must appear in the mass relation as written above, and in the relations for other physical quantities and physical constants in the context of TGR, derived in [2, 4] and [5] and summarized in Table I of [5]. We must simply replaced the factors $\gamma_g(r')$, $\gamma_g(r')^{-1}$ and $\gamma_g(r')^{-2}$ that appear in those relations by $\overline{\gamma}_g^t$, $(\overline{\gamma}_g^t)^{-1}$ and $(\overline{\gamma}_g^t)^{-2}$ respectively.

For example, an inertial force $\vec{F}_{\rm i}$ impressed on the inertial mass m of a test particle located at a point at the neighborhood of the inertial masses of N gravitational field sources that are scattered about the test particle in the relativistic Euclidean 3-space Σ, is related to the proper inertial force $\vec{F}'_{\rm i}$ impressed on the rest mass m_0 of test particle at the neighborhood of the rest masses of the N gravitational field sources in the proper Euclidean 3-space Σ' as follows

$$\vec{F}_{\rm i} = \prod_{\rm i=1}^{N}(1 - \frac{2GM_{0\rm ai}}{r_{\rm i}c_g^2})\vec{F}'_{\rm i} \tag{50}$$

And the relativistic gravitational potential $\Phi_j(r'_j)$ at the location of the test particle, due to the jth gravitational field source in Σ, is related to the proper (or primed) gravitational potential $\Phi'_j(r'_j)$ of that field source in Σ' as follows

$$\Phi_j\,(r'_j,\, r'_i) = -\frac{GM_{0\rm a\it j}}{r'_j}\prod_{i=1}^{N}(1 - \frac{2GM_{0\rm a\it i}}{r'_i c_g^2})^{1/2} \tag{51}$$

The modified forms at the neighborhood of N gravitational field sources of the relations for the gravitational values in the context of TGR, of the other physical parameters in Table I of [5] can similarly be written. However for parameters such as density and current density, which involve division by volume of space occupied by matter and flow cross-sectional area, one must carefully calculate the resultant volume and area of the gravitationally deformed shape of the test particle, as illustrated in Figs. 5(a) and 5(b) in the cases of a test particle located at the neighborhoods of one and three gravitational field sources respectively.

3 Validating Einstein's principle of equivalence at the neighborhood of several gravitational field sources

The principle of equivalence of Albert Einstein is composed of the local Lorentz invariance (LLI), the weak equivalence principle (WEP) and the strong equivalence principle (SEP). The definitions of these component principles adapted from their definitions in [6] have been presented in section 3 of [5]. The validity of LLI at the neighborhood of several gravitational field sources has been confirmed in subsection 2.1 above.

For WEP, let us multiply the mass relation (49) into the gravitational potential relation (51) to have the gravitational potential energy of the test particle located at a point P in Σ, due to the jth gravitational field source solely as follows

$$U_j = m\Phi_j\,(r'_j\,) = -\frac{GM_{0\rm a\it j}\,m_0}{r'_j}\prod_{i=1}^{N}(1 - \frac{2GM_{0\rm a\it i}}{r'_i c_g^2})^{3/2} \tag{52}$$

Then by dividing through Eq. (52) by the rest mass m_0 of the test particle, we obtain the effective gravitational potential 'seen' by the rest mass m_0 of the test particle, due to the jth gravitational field source solely, at the point P in the relativistic Euclidean 3-space Σ as follows

$$\Phi_{j\ \text{eff}} = -\frac{GM_{0\text{a}j}}{r'_j} \prod_{i=1}^{N} (1 - \frac{2GM_{0\text{a}\,i}}{r'_i c_g^2})^{3/2} \tag{53}$$

The effective gravitational acceleration suffered by the test particle towards the center of the jth gravitational field source in the relativistic Euclidean 3-space Σ is then given from definition as follows

$$\vec{g}_{j\ \text{eff}} = -\frac{\partial}{\partial r'_j}\{-\frac{GM_{0\text{a}j}}{r'_j} \prod_{i=1}^{N} (1 - \frac{2GM_{0\text{a}\,i}}{r'_i c_g^2})^{3/2}\}\frac{\vec{r}_j}{r_j} \tag{54}$$

where $\vec{r}_j/r_j$ is the unit vector of the radial coordinate from the center of the inertial mass M_j of the jth gravitational field source to the test particle in the relativistic Euclidean 3-space Σ.

The net effective gravitational force on the test particle towards the jth gravitational field source solely, in the relativistic Euclidean 3-space Σ, must be obtained by multiplying the effective acceleration (54) by the rest mass m_0 of the test particle (since $U_{j\,\text{eff}}$ in Eq. (52) has been divided by m_0 in obtaining $\Phi_{j\,,\text{eff}}$ in Eq. (53)), and summing the result over j for the N gravitational field sources as follows

$$\vec{F}_{\text{eff}} = \sum_{j}^{N} m_0 \vec{g}_{j\ \text{eff}} = \sum_{j}^{N} -\frac{\partial}{\partial r'_j}\{-\frac{GM_{0\text{a}j}\, m_0}{r'_j} \prod_{i=1}^{N} (1 - \frac{2GM_{0\text{a}\,i}}{r'_i c_g^2})^{3/2}\}\frac{\vec{r}_j}{r_j} \tag{55}$$

We find from Eq. (54) that $\vec{g}_{j\ \text{eff}}$ does not depend on any property of the test particle. Hence as long as WEP is valid in the context of the classical (or Newtonian) gravitation, as multitude of experiments have confirmed [7], WEP is valid at the neighborhood of several isolated gravitational field sources in the context of the theory of gravitational relativity.

We are thus left to demonstrate the validity of SEP at the neighborhood of several isolated gravitational field sources and, hence, in the entire universe, to validate EEP in the context of TGR. This is easy however, because (i) the validity of SEP at the neighborhood of several isolated gravitational field sources follows directly from its validity at the neighborhood of one gravitational field source already demonstrated in [5], and (ii) LLI is valid at the neighborhood of several isolated gravitational field sources, from which it follows that non-gravitational laws take on their usual classical and special-relativistic forms but in terms of gravitational-relativistic parameters and constants (i.e. their transformations in the context of TGR) at the neighborhood of several isolated gravitational field sources. All we must do then is substitute the derived expressions for (or the transformations of) physical parameters and physical constants in the context of TGR into the natural laws and check if their effect cancel out.

Since we have simply replaced the factor $\gamma_g = (1 - 2GM_{0a}/r'c_g^2)^{-1/2}$ in the derived expressions for physical parameters at the neighborhood of one gravitational field source (in Table I of [5]) by $\overline{\gamma}_g^t = \prod_{i=1}^{N}(1 - 2GM_{0ai}/r_i'c_g^2)$ of Eq. (41), at the neighborhood of several isolated gravitational field sources, as derived above, the effect of gravity will cancel out in non-gravitational laws at the neighborhood of several isolated gravitational field sources, just as it does at the neighborhood of one gravitational field source in [5]. This implies the invariance of non-gravitational laws with position in space and with time at the neighborhood of any number N of isolated gravitational field sources. Consequently the non-gravitational laws retain their usual forms everywhere and at all times in the universe, and this implies the validity of SEP. We have again validated LLI, WEP and SEP, and consequently, Einstein's principle of equivalence, in the context of the theory of gravitational relativity.

4 Modified Newton's gravitational force law in the field of isolated gravitational field sources in the context of the theory of gravitational relativity

4.1 The resultant gravitational force on a test particle (in the context of TGR) at the neighborhood of two and several isolated gravitational field sources

The resultant force on a test particle of inertial mass m at a point P in the relativistic Euclidean 3-space Σ, at the neighborhood of N gravitational field sources that are scattered in Σ about point P, is given by Eq. (55). It shall be assumed in this subsection that the particle is not a gravitational field source or is a source of negligible gravitational field. Let us consider $N = 2$, such as in Fig. 1a or 1b, in Eq. (55) then,

$$\begin{aligned}\vec{F}_{\text{eff}} &= -\frac{\partial}{\partial r_1'}\left\{-\frac{GM_{0a1}m_0}{r_1'}(1 - \frac{2GM_{0a1}}{r_1'c_g^2})^{3/2}(1 - \frac{2GM_{0a2}}{r_2'c_g^2})^{3/2}\right\}\frac{\vec{r}_1}{r_1}\\ &\quad -\frac{\partial}{\partial r_2'}\left\{-\frac{GM_{0a2}m_0}{r_2'}(1 - \frac{2GM_{0a1}}{r_1'c_g^2})^{3/2}(1 - \frac{2GM_{0a2}}{r_2'c_g^2})^{3/2}\right\}\frac{\vec{r}_2}{r_2} \qquad (56)\end{aligned}$$

$$\begin{aligned}\vec{F}_{\text{eff}} &= -\frac{GM_{0a1}m_0}{r_1'^2}(1 - \frac{2GM_{0a1}}{r_1'c_g^2})^{3/2}(1 - \frac{2GM_{0a2}}{r_2'c_g^2})^{3/2}\frac{\vec{r}_1}{r_1}\\ &\quad +\frac{3G^2M_{0a1}^2m_0}{r_1'^3c_g^2}(1 - \frac{2GM_{0a2}}{r_2'c_g^2})^{3/2}(1 - \frac{2GM_{0a1}}{r_1'c_g^2})^{1/2}\frac{\vec{r}_1}{r_1}\end{aligned}$$

$$-\frac{GM_{0a2}m_0}{r_2'^2}(1-\frac{2GM_{0a1}}{r_1'c_g^2})^{3/2}(1-\frac{2GM_{0a2}}{r_2'c_g^2})^{3/2}\frac{\vec{r}_2}{r_2}$$
$$+\frac{3G^2M_{0a2}^2m_0}{r_2'^3c_g^2}(1-\frac{2GM_{0a1}}{r_1'c_g^2})^{3/2}(1-\frac{2GM_{0a2}}{r_2'c_g^2})^{1/2}\frac{\vec{r}_2}{r_2} \quad (57)$$

In the Newtonian gravitation limit, $2GM_{0a1}/r_1'c_g^2 = 0$ and $2GM_{0a2}/r_2'c_g^2 = 0$, Eq. (57) simplifies as follows

$$\vec{F}_{\text{eff}} = -\frac{GM_{0a1}m_0}{r_1'^2}\frac{\vec{r}_1}{r_1} - \frac{GM_{0a2}m_0}{r_2'^2}\frac{\vec{r}_2}{r_2} \quad (58)$$

And in the post-Newtonian gravitation limit, $2GM_{0a1}/r_1'c_g^2 \approx 0$ and $2GM_{0a2}/r_2'c_g^2 \approx 0$, Eq. (57) simplifies as follows

$$\begin{aligned}\vec{F}_{\text{eff}} \quad = \quad & -\frac{GM_{0a1}m_0}{r_1'^2}(1-\frac{3GM_{0a1}}{r_1'c_g^2}-\frac{3GM_{0a2}}{r_2'c_g^2})\frac{\vec{r}_1}{r_1}\\ & +\frac{3G^2M_{0a1}^2m_0}{r_1'^3c_g^2}(1-\frac{GM_{0a1}}{r_1'c_g^2}-\frac{3GM_{0a2}}{r_2'c_g^2})\frac{\vec{r}_1}{r_1}\\ & -\frac{GM_{0a2}m_0}{r_2'^2}(1-\frac{3GM_{0a1}}{r_1'c_g^2}-\frac{3GM_{0a2}}{r_2'c_g^2})\frac{\vec{r}_2}{r_2}\\ & +\frac{3G^2M_{0a2}^2m_0}{r_2'^3c_g^2}(1-\frac{GM_{0a2}}{r_2'c_g^2}-\frac{3GM_{0a1}}{r_1'c_g^2})\frac{\vec{r}_2}{r_2}\end{aligned}$$

or

$$\begin{aligned}\vec{F}_{\text{eff}} \quad = \quad & (-\frac{GM_{0a1}m_0}{r_1'^2}+\frac{6G^2M_{0a1}^2m_0}{r_1'^3c_g^2}+\frac{3G^2M_{0a1}^2M_{0a2}^2m_0}{r_1'^2r_2'c_g^4})\frac{\vec{r}_1}{r_1}\\ & (-\frac{GM_{0a2}m_0}{r_2'^2}+\frac{6G^2M_{0a2}^2m_0}{r_2'^3c_g^2}+\frac{3G^2M_{0a2}^2M_{0a1}^2m_0}{r_2'^2r_1'c_g^4})\frac{\vec{r}_2}{r_2}\end{aligned} \quad (59)$$

It is straight forward to extend this result to N = 3, N = 4 and larger values of N in Eq. (55), although it becomes increasingly cumbersome as N increases beyond the value 3.

Then by applying the equivalence of inertial acceleration and gravitational acceleration we have

$$\frac{d^2\vec{x}}{dt^2} = \vec{g}_{\text{eff}} \quad (60)$$

where $\vec{g}_{\text{eff}}$ is give by Eq. (55) at the neighborhood of N gravitational field sources without approximation, and by Eq, (59) upon dividing through by m_0, at the neighborhood of two isolated gravitational field sources, in the post-Newtonian limit.

Eq. (59) can then be solved for the motion of the test particle within the gravitational field of the two isolated bodies on the flat four-dimensional relativistic spacetime (Σ, ct) of the theory of gravitational relativity.

One finds from Eq. (55) and from dividing Eq. (59) by m_0, that the effective gravitational acceleration does not depend on any property of the test particle (assumed not containing large quantity of non-gravitational energy). Consequently the weak equivalence principle is valid for a test particle interacting with two or any number of isolated gravitational field sources, but with the condition that it contains no large quantity of non-gravitational energy, such as energy stored in electric field or magnetic field or radiation energy. The weak equivalence principle is not valid for a test particle containing large quantity of non-gravitational energy, as found in [8].

4.2 The two-body, three-body and N-body problems in the contexts of the theory of gravitational relativity

Although the number of isolated gravitational field sources (or bodies) with which a test particle interacts in the calculations in the foregoing sub-section can be two, three, four or larger, the test particle is inherently assumed not to be a gravitational field source or a source of negligible gravitational field. On the other hand, let us consider two isolated gravitational field sources of inertial masses M_1 and M_1 (and rest masses M_{01} and M_{02}), which are separated by radial distance (from center to center) r, to interact gravitationally.

First let us consider M_1 as the gravitational field source and M_2 as the test particle. Then the gravitational potential due to M_1 at the location of M_2 in the context of TGR is,

$$\Phi_1(r') = -\frac{GM_{0\mathrm{a}1}}{r'}(1 - \frac{2GM_{0\mathrm{a}1}}{r'c_g^2})^{1/2} \tag{61}$$

Now M_2 is a gravitational field source, hence its innate inertial mass $M_{2\mathrm{i}}$ —due to its gravitational field solely—is related to its rest mass in the context of TGR as

$$M_{2\mathrm{i}} = M_{02}(1 - \frac{2GM_{0\mathrm{a}2}}{R'_2c_g^2}) \tag{62}$$

where R'_2 is the radius of the rest mass M_{02}. Since M_2 is located in the gravitational field of M_1, at radial distance r from the center of M_1, the mass relation (62) must be modified further as

$$M_2 = M_{02}(1 - \frac{2GM_{0\mathrm{a}2}}{R'_2c_g^2})(1 - \frac{2GM_{0\mathrm{a}1}}{r'c_g^2}) \tag{63}$$

The inertial mass M_2 then interacts with the gravitational potential $\Phi_1(r')$ of

Eq. (61) yielding the gravitational potential energy possessed by M_2 in the relativistic Euclidean 3-space Σ in the context of TGR as

$$\begin{aligned} U_2(r') &= -\frac{GM_{0a1}}{r'}(1-\frac{2GM_{0a1}}{r'c_g^2})_{02}^{1/2}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2}) \\ &= -\frac{GM_{0a1}M_{0a2}}{r'}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{3/2} \end{aligned} \quad (64)$$

Division through Eq. (64) by M_{02} gives the effective gravitational potential 'seen' by the rest mass M_{02} in the relativistic Euclidean 3-space Σ in the context of TGR as

$$\Phi_{2\,\mathrm{eff}}(r') = -\frac{GM_{0a1}}{r'}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{3/2} \quad (65)$$

The effective gravitational acceleration suffered by M_2 (in the gravitational field of M_1) is then given from definition as

$$\begin{aligned} \vec{g}_{21\,\mathrm{eff}}(r') &= -\frac{\mathrm{d}}{\mathrm{d}r'}\left\{-\frac{GM_{0a1}}{r'}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{3/2}\right\}\frac{\vec{r}}{r} \\ &= -\frac{GM_{0a1}}{r'^2}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{3/2}\frac{\vec{r}}{r} \\ &\quad +\frac{3G^2M_{0a1}^2}{r'^3c_g^2}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{1/2}\frac{\vec{r}}{r} \end{aligned} \quad (66)$$

where $\vec{r}/r$ is the unit vector along the radial direction from the center of M_1 to the center of M_2. The equation of motion of the body M_2 in the gravitational field of the body M_1 is then given as follows

$$\begin{aligned} \frac{\mathrm{d}^2\vec{x}_2}{\mathrm{d}t^2} &= \vec{g}_{21\,\mathrm{eff}}(r') \\ &= -\frac{GM_{0a1}}{r'^2}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{3/2}\frac{\vec{r}}{r} \\ &\quad +\frac{3G^2M_{0a1}^2}{r'^3c_g^2}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'c_g^2})^{1/2}\frac{\vec{r}}{r} \end{aligned} \quad (67)$$

On the other hand, by making M_2 the gravitational field source and M_1 the test particle, and repeating the derivation from Eq. (61), the effective gravitational acceleration suffered by M_1 in the gravitational field of M_2 is

$$\vec{g}_{12\,\mathrm{eff}}(r') = -\frac{\mathrm{d}}{\mathrm{d}r'}\left\{-\frac{GM_{0a2}}{r'}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'c_g^2})^{3/2}\right\}(-\frac{\vec{r}}{r})$$

$$= -\frac{GM_{0a2}}{r'^2}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'c_g^2})^{3/2}(-\frac{\vec{r}}{r}) + \frac{3G^2M_{0a2}^2}{r'^3c_g^2}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'c_g^2})^{1/2}(-\frac{\vec{r}}{r}) \quad (68)$$

where R'_1 is the radius of the rest mass M_{01} and $-\vec{r}/r$ is the unit vector along the radial direction from the center of M_2 to the center of M_1. The equation of motion of M_1 in the gravitational field of M_2 is then given as

$$\frac{d^2\vec{x}_1}{dt^2} = \vec{g}_{12\,\text{eff}}(r') = -\frac{GM_{0a2}}{r'^2}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'c_g^2})^{3/2}(-\frac{\vec{r}}{r}) + \frac{3G^2M_{0a2}^2}{r'^3c_g^2}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'c_g^2})^{1/2}(-\frac{\vec{r}}{r}) \quad (69)$$

The following remarks shall be made about the effective gravitational accelerations of Eqs. (66) and (68) and the equations of motion (67) and (69):

1. The effective accelerations $\vec{g}_{21\,\text{eff}}$ and $\vec{g}_{12\,\text{eff}}$ are different in magnitude and oppositely directed in the Euclidean 3-space Σ. They are equal in magnitude only if $M_{01} = M_{02}$ and $R'_1 = R'_2$.
2. The motion of one body due to the gravitational field of the other expressed by Eqs. (66) and (68) cannot be neglected in general, unless if the mass of one is by far larger than that of the other, in which case the acceleration suffered by the larger body in the gravitational field of the smaller body is negligible. This is the situation between the Sun and a planet for instance.
3. For the motion of a planet round the Sun, the following post-Newtonian approximations to $\vec{g}_{21\,\text{eff}}$ and $\vec{g}_{12\,\text{eff}}$, obtained with $2GM_{0a1}/r'c_g^2 \approx 0$ and $2GM_{0a2}/r'c_g^2 \approx 0$, are adequate:

$$\vec{g}_{21\,\text{eff}} = \left(-\frac{GM_{0a1}}{r'^2} + \frac{6G^2M_{0a1}^2}{r'^3c_g^2} + \frac{2G^2M_{0a1}M_{0a2}}{R'_2r'^2c_g^2}\right)\frac{\vec{r}}{r} \quad (70)$$

and

$$\vec{g}_{12\,\text{eff}} = \left(-\frac{GM_{0a2}}{r'^2} + \frac{6G^2M_{0a2}^2}{r'^3c_g^2} + \frac{2G^2M_{0a2}M_{0a1}}{R'_1r'^2c_g^2}\right)(-\frac{\vec{r}}{r}) \quad (71)$$

Further more, given that the body M_1 is the Sun, that is, $M_1 = M_S$ ($\equiv M_\odot$) and the body M_2 is the planet, that is $M_2 = M_P$ of radius R_p, then the effective gravitational acceleration $\vec{g}_{12\,\mathrm{eff}}$ suffered by the Sun in the gravitational field of the planet can be neglected, while the effective gravitational acceleration suffered by the planet in the gravitational field of the Sun $\vec{g}_{21\,\mathrm{eff}}$ shall be re-written as follows

$$\vec{g}_{\mathrm{p\,s}} = \left(-\frac{GM_{0\mathrm{as}}}{r'^2_{\mathrm{p\,s}}} + \frac{6G^2M^2_{0\mathrm{as}}}{r'^3_{\mathrm{p\,s}}c^2_g} + \frac{2G^2M_{0\mathrm{as}}M_{0\mathrm{ap}}}{R'_{\mathrm{p}}r'^2_{\mathrm{p\,s}}c^2_g}\right)\frac{\vec{r}_{\mathrm{p\,s}}}{r_{\mathrm{p\,s}}} \tag{72}$$

where $r_{\mathrm{p\,s}}$ is the radial distance from the center of the inertial mass M_{p} of the planet to the center of the inertial mass M_{S} of the Sun in the relativistic Euclidean 3-space Σ, which corresponds to the radial distance $r'_{\mathrm{p\,s}}$ from the center of the rest mass $M_{0\mathrm{p}}$ of the planet to the center of the rest mass $M_{0\mathrm{S}}$ of the Sun in the proper Euclidean 3-space Σ', and $\vec{r}_{\mathrm{p\,s}}/r_{\mathrm{p\,s}}$ is the unit vector along the radial direction from the center of the planet to the center of the Sun in Σ.

The approximate post-Newtonian gravitational acceleration (72) suffered by the planet towards the center of the Sun, when both the Sun and the planet are considered as two interacting gravitational field sources (or as two-body system), is a slightly modified form of Eq. (97) of [4], when the planet is considered as a test particle of negligible gravitational field (or when the Sun – planet system is considered as a one-body system). Since the extra third term inside the parentheses in (72) is an inverse-square-law acceleration term like the Newtonian first term, it does not give rise to further perihelion precession of the planetary orbit than caused by the second term calculated in section two of [8].

4. A moderate star may be bound in orbit round a neutron star. Then both the star and neutron star will be in motion and the exact equations of motion (67) for the moderate star of mass M_2 and Eq. (69) for the neutron star of mass M_1, must be employed.

5. If we replace the neutron star by a black hole in item 4, so that a moderate star is bound in orbit round a black hole, then the effective gravitational acceleration $\vec{g}_{12\,\mathrm{eff}}$ of Eq. (68), which the black hole suffers in the gravitational field of the moderate star is zero; and this is true even if we replace the moderate star by a star more massive than the black hole, since $(1 - 2GM_{0\mathrm{a}1}/R'_1c^2_g) = 0$ for a black hole of rest mass M_{01} and radius R'_1 (of M_{01})—this is the radius of the gravitational event horizon of the black hole. Thus the black hole will remain stationary, while the star will be in motion relative to it. A black hole is absolutely stationary (relative to all observers) always, since a black hole possesses it maximum velocity of zero always relative to all observers

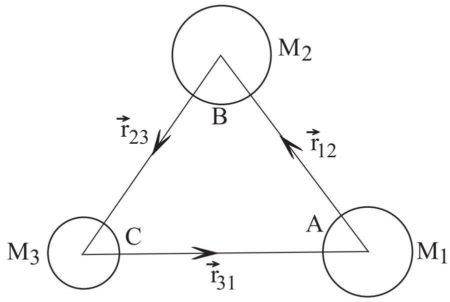

Fig. 6: Three interacting gravitational field sources in the Euclidean 3-space of the theory of gravitational relativity.

as established in [9]. Further support for this shall also be derived in the next article.

Let us now consider the case of interacting three isolated gravitational field sources (or bodies), where the inertial masses M_1, M_2 and M_3 of the bodies are scattered in the relativistic Euclidean 3-space Σ of TGR, such as illustrated in Fig. 6 at a given instant as the bodies are in motion. This is the three-body problem.

The bodies M_1 and M_3 establish gravitational potential at the location B of M_2. Hence the net gravitational potential at B in the context of TGR is,

$$\begin{aligned}\Phi_{\mathrm{B\,net}} &= \Phi_{21}(r'_{12}) + \Phi_{23}(r'_{23}) \\ &= -\frac{GM_{0a1}}{r'_{12}}(1 - \frac{2GM_{0a1}}{r'_{12}c_g^2})^{1/2} - \frac{GM_{0a3}}{r'_{23}}(1 - \frac{2GM_{0a3}}{r'_{23}c_g^2})^{1/2} \qquad (73)\end{aligned}$$

The net gravitational potential at C due to M_1 and M_2 is,

$$\begin{aligned}\Phi_{\mathrm{C\,net}} &= \Phi_{32}(r'_{23}) + \Phi_{31}(r'_{31}) \\ &= -\frac{GM_{0a2}}{r'_{23}}(1 - \frac{2GM_{0a2}}{r'_{23}c_g^2})^{1/2} - \frac{GM_{0a1}}{r'_{31}}(1 - \frac{2GM_{0a1}}{r'_{31}c_g^2})^{1/2} \qquad (74)\end{aligned}$$

And the net gravitational potential at A due to M_2 and M_3 is,

$$\begin{aligned}\Phi_{\mathrm{A\,net}} &= \Phi_{12}(r'_{12}) + \Phi_{13}(r'_{31}) \\ &= -\frac{GM_{0a2}}{r'_{12}}(1 - \frac{2GM_{0a2}}{r'_{12}c_g^2})^{1/2} - \frac{GM_{0a3}}{r'_{31}}(1 - \frac{2GM_{0a3}}{r'_{31}c_g^2})^{1/2} \qquad (75)\end{aligned}$$

The inertial mass M_2 of the body at B is related to its rest mass M_{02} in its own gravitational field and the gravitational fields of the bodies M_1 and M_3 at B as

$$M_2 = M_{02}(1 - \frac{2GM_{0a2}}{R'_2 c_g^2})(1 - \frac{2GM_{0a1}}{r'_{12} c_g^2})(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2}) \tag{76}$$

The inertial mass M_3 of the body at C is related to its rest mass M_{03} in its own gravitational field and the gravitational fields of the bodies M_1 and M_2 at B as

$$M_3 = M_{03}(1 - \frac{2GM_{0a3}}{R'_3 c_g^2})(1 - \frac{2GM_{0a1}}{r'_{31} c_g^2})(1 - \frac{2GM_{0a2}}{r'_{23} c_g^2}) \tag{77}$$

And the inertial mass M_1 of the body at A is related to its rest mass M_{01} in its own gravitational field and the gravitational fields of the bodies M_2 and M_3 at A as

$$M_1 = M_{01}(1 - \frac{2GM_{0a1}}{R'_1 c_g^2})(1 - \frac{2GM_{0a2}}{r'_{12} c_g^2})(1 - \frac{2GM_{0a3}}{r'_{31} c_g^2}) \tag{78}$$

where R'_1, R'_2 and R'_3 are the radii of the rest masses M_{01}, M_{02} and M_{03} respectively.

The net gravitational potential energy possessed by M_2 at B is,

$$\begin{aligned}
U_{\mathrm{B\,net}} &= M_2 \Phi_{\mathrm{B\,net}}(r'_{12}, r'_{23}) \\
&= [-\frac{GM_{0a1}}{r'_{12}}(1 - \frac{2GM_{0a1}}{r'_{12} c_g^2})^{1/2} - \frac{GM_{0a3}}{r'_{23}}(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2})^{1/2}] \\
&\quad \times [M_{02}(1 - \frac{2GM_{0a2}}{R'_2 c_g^2})(1 - \frac{2GM_{0a1}}{r'_{12} c_g^2})(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2})] \\
&= -\frac{GM_{0a1} M_{02}}{r'_{12}}(1 - \frac{2GM_{0a2}}{R'_2 c_g^2})(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2})(1 - \frac{2GM_{0a1}}{r'_{12} c_g^2})^{3/2} \\
&\quad -\frac{GM_{0a3} M_{02}}{r'_{23}}(1 - \frac{2GM_{0a2}}{R'_2 c_g^2})(1 - \frac{2GM_{0a1}}{r'_{12} c_g^2})(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2})^{3/2}
\end{aligned} \tag{79}$$

The net gravitational potential energy possessed by M_3 at C is,

$$\begin{aligned}
U_{\mathrm{C\,net}} &= M_3 \Phi_{\mathrm{C\,net}}(r'_{31}, r'_{23}) \\
&= [-\frac{GM_{0a1}}{r'_{31}}(1 - \frac{2GM_{0a1}}{r'_{31} c_g^2})^{1/2} - \frac{GM_{0a2}}{r'_{23}}(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2})^{1/2}] \\
&\quad \times [M_{03}(1 - \frac{2GM_{0a2}}{R'_3 c_g^2})(1 - \frac{2GM_{0a1}}{r'_{31} c_g^2})(1 - \frac{2GM_{0a3}}{r'_{23} c_g^2})] \\
&= -\frac{GM_{0a1} M_{03}}{r'_{31}}(1 - \frac{2GM_{0a3}}{R'_3 c_g^2})(1 - \frac{2GM_{0a2}}{r'_{23} c_g^2})(1 - \frac{2GM_{0a1}}{r'_{31} c_g^2})^{3/2} \\
&\quad -\frac{GM_{0a2} M_{03}}{r'_{23}}(1 - \frac{2GM_{0a3}}{R'_3 c_g^2})(1 - \frac{2GM_{0a1}}{r'_{31} c_g^2})(1 - \frac{2GM_{0a2}}{r'_{23} c_g^2})^{3/2}
\end{aligned} \tag{80}$$

And the net gravitational potential energy possessed by M_1 at A is,

$$\begin{aligned} U_{\mathrm{A\,net}} &= M_1\Phi_{\mathrm{A\,net}}(r'_{12}, r'_{31}) \\ &= [-\frac{GM_{0a2}}{r'_{12}}(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})^{1/2} - \frac{GM_{0a3}}{r'_{31}}(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})^{1/2}] \\ &\times [M_{01}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})] \\ &= -\frac{GM_{0a2}M_{01}}{r'_{12}}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})^{3/2} \\ &-\frac{GM_{0a3}M_{01}}{r'_{31}}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})^{3/2} \end{aligned} \quad (81)$$

The effective gravitational potential at the position B of M_2 is,

$$\begin{aligned} \Phi_{\mathrm{B\,eff}}(r'_{12}, r'_{23}) &= U_{\mathrm{B\,net}}/M_{02} \\ &= -\frac{GM_{0a1}}{r'_{12}}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a3}}{r'_{23}c_g^2})(1-\frac{2GM_{0a1}}{r'_{12}c_g^2})^{3/2} \\ &-\frac{GM_{0a3}}{r'_{23}}(1-\frac{2GM_{0a2}}{R'_2c_g^2})(1-\frac{2GM_{0a1}}{r'_{12}c_g^2})(1-\frac{2GM_{0a3}}{r'_{23}c_g^2})^{3/2} \end{aligned} \quad (82)$$

The effective gravitational potential at the position C of M_3 is,

$$\begin{aligned} \Phi_{\mathrm{C\,eff}}(r'_{12}, r'_{23}) &= U_{\mathrm{C\,net}}/M_{03} \\ &= -\frac{GM_{0a1}}{r'_{31}}(1-\frac{2GM_{0a3}}{R'_3c_g^2})(1-\frac{2GM_{0a2}}{r'_{23}c_g^2})(1-\frac{2GM_{0a1}}{r'_{31}c_g^2})^{3/2} \\ &-\frac{GM_{0a2}}{r'_{23}}(1-\frac{2GM_{0a3}}{R'_3c_g^2})(1-\frac{2GM_{0a1}}{r'_{31}c_g^2})(1-\frac{2GM_{0a2}}{r'_{23}c_g^2})^{3/2} \end{aligned} \quad (83)$$

The effective gravitational potential at the position A of M_1 is,

$$\begin{aligned} \Phi_{\mathrm{A\,eff}}(r'_{12}, r'_{31}) &= U_{\mathrm{A\,net}}/M_{01} \\ &= -\frac{GM_{0a2}}{r'_{12}}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})^{3/2} \\ &-\frac{GM_{0a3}}{r'_{31}}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})^{3/2} \end{aligned} \quad (84)$$

The effective gravitational acceleration suffered by M_2 at position B is,

$$\begin{aligned}\vec{g}_{2\,\text{eff}} &= -\frac{\partial \Phi_{B\,\text{eff}}(r'_{12}, r'_{23})}{\partial r'_{12}}(-\frac{\vec{r}_{12}}{r_{12}}) - \frac{\partial \Phi_{B\,\text{eff}}(r'_{12}, r'_{23})}{\partial r'_{23}}\frac{\vec{r}_{23}}{r_{23}} \\ &= -\frac{GM_{0\text{a}1}}{r'^2_{12}}(1-\frac{2GM_{0\text{a}2}}{R'_2c_g^2})(1-\frac{2GM_{0\text{a}3}}{r'_{23}c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{12}c_g^2})^{3/2}(-\frac{\vec{r}_{12}}{r_{12}}) \\ &\quad +\frac{3G^2M^2_{0\text{a}1}}{r'^3_{12}c_g^2}(1-\frac{2GM_{0\text{a}2}}{R'_2c_g^2})(1-\frac{2GM_{0\text{a}3}}{r'_{23}c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{12}c_g^2})^{1/2}(-\frac{\vec{r}_{12}}{r_{12}}) \\ &\quad -\frac{GM_{0\text{a}3}}{r'^2_{23}}(1-\frac{2GM_{0\text{a}2}}{R'_2c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{12}c_g^2})(1-\frac{2GM_{0\text{a}3}}{r'_{23}c_g^2})^{3/2}\frac{\vec{r}_{23}}{r_{23}} \\ &\quad +\frac{3G^2M^2_{0\text{a}3}}{r'^3_{23}c_g^2}(1-\frac{2GM_{0\text{a}2}}{R'_2c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{12}c_g^2})(1-\frac{2GM_{0\text{a}3}}{r'_{23}c_g^2})^{1/2}\frac{\vec{r}_{23}}{r_{23}} \qquad (85)\end{aligned}$$

The effective gravitational acceleration suffered by M_3 at position C is,

$$\begin{aligned}\vec{g}_{3\,\text{eff}} &= -\frac{\partial \Phi_{C\,\text{eff}}(r'_{31}, r'_{23})}{\partial r'_{31}}\frac{\vec{r}_{31}}{r_{31}} - \frac{\partial \Phi_{C\,\text{eff}}(r'_{31}, r'_{23})}{\partial r'_{23}}(-\frac{\vec{r}_{23}}{r_{23}}) \\ &= -\frac{GM_{0\text{a}1}}{r'^2_{31}}(1-\frac{2GM_{0\text{a}3}}{R'_3c_g^2})(1-\frac{2GM_{0\text{a}2}}{r'_{23}c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{31}c_g^2})^{3/2}\frac{\vec{r}_{31}}{r_{31}}) \\ &\quad +\frac{3G^2M^2_{0\text{a}1}}{r'^3_{31}c_g^2}(1-\frac{2GM_{0\text{a}3}}{R'_3c_g^2})(1-\frac{2GM_{0\text{a}2}}{r'_{23}c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{31}c_g^2})^{1/2}\frac{\vec{r}_{31}}{r_{31}} \\ &\quad -\frac{GM_{0\text{a}2}}{r'^2_{23}}(1-\frac{2GM_{0\text{a}3}}{R'_3c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{31}c_g^2})(1-\frac{2GM_{0\text{a}2}}{r'_{23}c_g^2})^{3/2}(-\frac{\vec{r}_{23}}{r_{23}}) \\ &\quad +\frac{3G^2M^2_{0\text{a}2}}{r'^3_{23}c_g^2}(1-\frac{2GM_{0\text{a}3}}{R'_3c_g^2})(1-\frac{2GM_{0\text{a}1}}{r'_{31}c_g^2})(1-\frac{2GM_{0\text{a}2}}{r'_{23}c_g^2})^{1/2}(-\frac{\vec{r}_{23}}{r_{23}}) \\ &\qquad (86)\end{aligned}$$

And the effective gravitational acceleration suffered by M_1 at position A is,

$$\begin{aligned}\vec{g}_{1\,\text{eff}} &= -\frac{\partial \Phi_{A\,\text{eff}}(r'_{12}, r'_{31})}{\partial r'_{12}}\frac{\vec{r}_{12}}{r_{12}} - \frac{\partial \Phi_{A\,\text{eff}}(r'_{12}, r'_{31})}{\partial r'_{31}}(-\frac{\vec{r}_{31}}{r_{31}}) \\ &= -\frac{GM_{0\text{a}2}}{r'^2_{12}}(1-\frac{2GM_{0\text{a}1}}{R'_1c_g^2})(1-\frac{2GM_{0\text{a}3}}{r'_{31}c_g^2})(1-\frac{2GM_{0\text{a}2}}{r'_{12}c_g^2})^{3/2}\frac{\vec{r}_{12}}{r_{12}}) \\ &\quad +\frac{3G^2M^2_{0\text{a}2}}{r'^3_{12}c_g^2}(1-\frac{2GM_{0\text{a}1}}{R'_1c_g^2})(1-\frac{2GM_{0\text{a}3}}{r'_{31}c_g^2})(1-\frac{2GM_{0\text{a}2}}{r'_{12}c_g^2})^{1/2}\frac{\vec{r}_{12}}{r_{12}}\end{aligned}$$

$$-\frac{GM_{0a3}}{r'^2_{31}}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})^{3/2}(-\frac{\vec{r}_{31}}{r_{31}})$$
$$+\frac{3G^2M^2_{0a3}}{r'^3_{31}c_g^2}(1-\frac{2GM_{0a1}}{R'_1c_g^2})(1-\frac{2GM_{0a2}}{r'_{12}c_g^2})(1-\frac{2GM_{0a3}}{r'_{31}c_g^2})^{1/2}(-\frac{\vec{r}_{31}}{r_{31}}) \quad (87)$$

The equations of motions of the bodies M_1, M_2 and M_3 must then be written respectively as follows

$$\frac{d^2\vec{x}_1}{dt^2}=\vec{g}_{1\,\mathrm{eff}}(r'_{12},r'_{31}) \quad (88)$$

$$\frac{d^2\vec{x}_2}{dt^2}=\vec{g}_{2\,\mathrm{eff}}(r'_{12},r'_{23}) \quad (89)$$

and

$$\frac{d^2\vec{x}_3}{dt^2}=\vec{g}_{3\,\mathrm{eff}}(r'_{31},r'_{23}) \quad (90)$$

Equations (88), (89) and (90) along with Eqs. (97), (85) and (86) respectively, must then be solved for the paths $\vec{x}_1(t)$, $\vec{x}_2(t)$ and $\vec{x}_3(t)$ of M_1, M_2 and M_3 in Σ. It is straight forward to extend this derivation to the 4-body problem, 5-body problem ... and N-body problem, except that it becomes increasingly cumbersome for N> 3.

4.3 The zero effect on the gravitational event horizon (or blackness) of a black hole of external gravitational field and motion of the observer relative to the black hole

Let us suppose that the kth gravitational field source of the N isolated gravitational field sources that give rise to the resultant factor $\overline{\gamma}_g^{\,t}$ of Eq. (41) at a point in space is a black hole. Let us suppose further that the point P is located at the gravitational event horizon surface of this black hole. The factor $\phi\gamma_{gk}$ at point P on the surface of the black hole, assuming all other gravitational field sources are absent, is given in the context of the intrinsic theory of gravitational relativity ϕTGR as follows

$$\phi\gamma_{gk}(\phi r'_k)^{-2}=(1-\frac{2G\phi M_{0ak}}{\phi r'_k\phi c_g^2})=0 \quad (91)$$

where ϕM_{0k} is the one-dimensional intrinsic rest mass of the black hole in the proper intrinsic space $\phi\rho'$ and $\phi r'_k$ is the length of ϕM_{0k}.

The intrinsic gravitational time dilation and intrinsic gravitational length contraction formulae at the surface of the black hole due to the gravitational field of the black hole solely are given respectively as follows

$$d\phi t = \phi\gamma_{gk}(\phi r'_k)d\phi t' = (1 - \frac{2G\phi M_{0ak}}{\phi r'_k \phi c_g^2})^{-1/2} d\phi t' = \infty \tag{92}$$

$$d\phi\rho = \phi\gamma_{gk}(\phi r'_k)^{-1} d\phi\rho' = (1 - \frac{2G\phi M_{0ak}}{\phi r'_k \phi c_g^2})^{-1/2} d\phi\rho' = 0 \tag{93}$$

When the presence nearby of the rest N-1 gravitational field sources is taken into consideration, then the resultant intrinsic gravitational time dilation and resultant intrinsic gravitational length contraction in the context of ϕTGR at the point P at the surface of the black hole, which is the kth gravitational field source, are the following respectively

$$\begin{aligned} d\phi t &= \phi\gamma_{g1}(\phi r'_1)\phi\gamma_{g2}(\phi r'_2)\phi\gamma_{g3}(\phi r'_3)\cdots\phi\gamma_{gk}(\phi r'_k)\cdots\phi\gamma_{gN}(\phi r'_N)d\phi t' \\ &= (1 - \frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2})^{-1/2}(1 - \frac{2G\phi M_{0a2}}{\phi r'_2 \phi c_g^2})^{-1/2}(1 - \frac{2G\phi M_{0a3}}{\phi r'_3 \phi c_g^2})^{-1/2}\cdots \\ &\quad \cdots(1 - \frac{2G\phi M_{0ak}}{\phi r'_k \phi c_g^2})^{-1/2}\cdots(1 - \frac{2G\phi M_{0aN}}{\phi r'_N \phi c_g^2})^{-1/2} d\phi t' = \infty \end{aligned} \tag{94}$$

$$\begin{aligned} d\phi\rho &= \phi\gamma_{g1}(\phi r'_1)^{-1}\phi\gamma_{g2}(\phi r'_2)^{-1}\cdots\phi\gamma_{gk}(\phi r'_k)^{-1}\cdots\phi\gamma_{gN}(\phi r'_N)^{-1}d\phi\rho' \\ &= (1 - \frac{2G\phi M_{0a1}}{\phi r'_1 \phi c_g^2})^{1/2}(1 - \frac{2G\phi M_{0a2}}{\phi r'_2 \phi c_g^2})^{1/2}(1 - \frac{2G\phi M_{0a3}}{\phi r'_3 \phi c_g^2})^{1/2}\cdots \\ &\quad \cdots(1 - \frac{2G\phi M_{0ak}}{\phi r'_k \phi c_g^2})^{1/2}\cdots(1 - \frac{2G\phi M_{0aN}}{\phi r'_N \phi c_g^2})^{1/2} d\phi\rho' = 0 \end{aligned} \tag{95}$$

where $\phi r'_i$ is the 'distance' along the the proper intrinsic space $\phi\rho'$ of the base of the intrinsic mass ϕM_{0i} of the ith gravitational field source from the surface of the black hole. Eqs. (94) and (95) are valid by virtues of Eqs. (92) and (93) respectively.

We find from equations (92) and (94) that a proper intrinsic time interval $d\phi t'$ is infinitely dilated at the surface of a black hole, relative to 3-observers in the relativistic Euclidean 3-space Σ, with or without the presence of other gravitational field sources at the neighborhood of the black hole. Equations (93) and (95) likewise show that an interval $d\phi\rho'$ of proper intrinsic space is contracted to zero interval at the surface of a black hole, relative to 3-observers in Σ, with or without the presence of other gravitational field sources at the neighborhood of the black hole.

Now, no matter how the inertial masses $M_1, M_2, M_3, \cdots, M_{N-1}$ of the other N-1 gravitational field sources are scattered in the Euclidean 3-space Σ of TGR about the black hole, their intrinsic inertial masses $\phi M_1, \phi M_2, \cdots, \phi M_{N-1}$ are all aligned with the intrinsic mass ϕM_k of the black hole along the isotropic relativistic intrinsic space $\phi\rho$ and their intrinsic rest masses $\phi M_{01}, \phi M_{02}, \phi M_{03} \cdots \phi M_{0N}$ lie along the curved proper intrinsic space $\phi\rho'$ in the context of TGR/ϕTGR. Consequently the intrinsic length contraction formula at the surface of the black hole (95) is valid no matter how the inertial masses of the other N-1 gravitational field sources are scattered in 3-space Σ about the black hole.

The outward manifestations on the flat four-dimensional relativistic spacetime (Σ, ct) of Eqs. (92) and (93) on flat two-dimensional relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, obtained by simply dropping the symbol ϕ in those equations, are the following respectively

$$dt = \gamma_{gk}(r'_k)dt' = (1 - \frac{2GM_{0ak}}{r'_k c_g^2})^{-1/2} dt' = \infty \tag{96}$$

$$d\rho = \gamma_{gk}(r'_k)^{-1} d\rho' = (1 - \frac{2GM_{0ak}}{r'_k c_g^2})^{-1/2} d\rho' = 0 \tag{97}$$

We shall for the present purpose replace $d\rho'$ by an elementary volume $d\Sigma'$ of the proper Euclidean 3-space Σ' at the surface of the rest mass M_{0k} in Σ' of the black hole and $d\rho$ by elementary volume $d\Sigma$ of the relativistic Euclidean 3-space Σ of TGR at the surface of the inertial mass M_k in Σ of the black hole, and re-write Eq. (97) as follows

$$d\Sigma = \gamma_{gk}(r'_k)^{-1} d\Sigma' = (1 - \frac{2GM_{0ak}}{r'_k c_g^2})^{-1/2} d\Sigma' = 0 \tag{98}$$

Equation (96) states that an interval dt' of proper time is infinitely dilated at the surface of a black hole relative to 3-observers in the 3-space Σ, with the assumption of the absence of every other gravitational field source, and Eq. (98) states that an elementary volume $d\Sigma'$ of proper Euclidean 3-space at the surface of a black hole is contracted to zero volume relative to 3-observers in Σ, with the assumption of the absence of every other gravitational field source.

The outward manifestations on the flat four-dimensional spacetime (Σ, ct) of Eqs. (94) and (95) on the flat two-dimensional intrinsic spacetime $(\phi\rho, \phi c\phi t)$ are likewise given respectively as follows

$$\begin{aligned} dt &= \gamma_{g1}(r'_1)\gamma_{g2}(r'_2)\gamma_{g3}(r'_3)\cdots\gamma_{gk}(r'_k)\cdots\gamma_{gN}(r'_N)dt' \\ &= (1 - \frac{2GM_{0a1}}{r'_1 c_g^2})^{-1/2}(1 - \frac{2GM_{0a2}}{r'_2 c_g^2})^{-1/2}(1 - \frac{2GM_{0a3}}{r'_3 c_g^2})^{-1/2}\cdots \\ &\quad \cdots(1 - \frac{2GM_{0ak}}{r'_k c_g^2})^{-1/2}\cdots(1 - \frac{2GM_{0aN}}{r'_N c_g^2})^{-1/2} dt' = \infty \end{aligned} \tag{99}$$

$$\begin{aligned} d\Sigma &= \gamma_{g1}(r'_1)^{-1}\gamma_{g2}(r'_2)^{-1}\cdots\gamma_{gk}(r'_k)^{-1}\cdots\gamma_{gN}(r'_N)^{-1}d\Sigma' \\ &= (1 - \frac{2GM_{0a1}}{r'_1 c_g^2})^{1/2}(1 - \frac{2GM_{0a2}}{r'_2 c_g^2})^{1/2}(1 - \frac{2GM_{0a3}}{r'_3 c_g^2})^{1/2}\cdots \end{aligned}$$

$$\cdots(1-\frac{2GM_{0\mathrm{a}k}}{r'_k c_g^2})^{1/2}\cdots(1-\frac{2GM_{0\mathrm{a}N}}{r'_N c_g^2})^{1/2}d\Sigma' = 0 \qquad (100)$$

Equations (99) and (100) are valid by virtues of Eqs. (96) and (98) respectively.

Let us write the mass expression for the black hole due to its own gravitational field solely, that is, with the assumption of absence of the rest N-1 gravitational field sources. This is given as follows

$$M_k = M_{0\,k}(\gamma_{g\,k})^{-2} = M_{0\,k}(1-\frac{2GM_{0\mathrm{a}\,k}}{r'_k c_g^2}) = 0 \qquad (101)$$

A black hole possesses zero inertial mass in the relativistic Euclidean 3-space Σ according to Eq. (101), but we shall not deliberate further on this in this article. When the black hole is considered to be located at the neighborhood of the other N-1 gravitational field sources, then the mass relation for the black hole in the context of TGR that follows from Eq. (49) is

$$\begin{aligned} M_k &= \gamma_{g1}(r'_1)^{-2}\gamma_{g2}(r'_2)^{-2}\cdots\gamma_{gk}(r'_k)^{-2}\cdots\gamma_{gN}(r'_N)^{-2}M_{0\,k} \\ &= (1-\frac{2GM_{0\mathrm{a}1}}{r'_1 c_g^2})(1-\frac{2GM_{0\mathrm{a}2}}{r'_2 c_g^2})(1-\frac{2GM_{0\mathrm{a}3}}{r'_3 c_g^2})\cdots \\ &\quad \cdots(1-\frac{2GM_{0\mathrm{a}k}}{r'_k c_g^2})\cdots(1-\frac{2GM_{0\mathrm{a}N}}{r'_N c_g^2})M_{0\,k} = 0 \qquad (102) \end{aligned}$$

Equation (102) again shows that a black hole possesses zero inertial mass in Σ with the presence of any number $N-1$ of other gravitational field sources that are scattered about the black hole in Σ.

We find from Eqs. (96) and (98) and Eqs. (99) and(100) that any interval dt' of proper time is infinitely gravitationally dilated and any volume $d\Sigma'$ of the proper physical Euclidean 3-space is gravitationally contracted to zero volume at the surface of a black hole, while Eqs. (101) and (102) show that the inertial mass of the black hole in Σ is zero, both while the black hole is isolated from the gravitational field of any other source and while it is located within the gravitational field of any number of other sources. In other words, the properties of a black hole namely, the radius of the event horizon of a black hole; the zero inertial mass in Σ of a black hole; the infinite dilation of time at the surface of a black hole; the gravitational event horizon of the surface of a black hole and the blackness of a black hole, are unaltered by its location in the gravitational field of any number of other sources.

The derived facts about black hole in the context of the theory of gravitational relativity (TGR) in the foregoing paragraph shall be stated as a principle of black hole thus

> A black hole is the same with respect to all observers where ever it may be located in the universe.

This derived principle must be one of the principles of black hole physics in the context of TGR, to be developed in the next volume of this monograph series.

In order to show that the above principle is valid relative to all observers, as stated, let us modify Eq. (99), (100) and (102) by incorporating the effect of the velocity of the observer relative to the black hole into them (in the context of combined TGR and SR) to have as follows

$$\begin{aligned} dt &= \gamma_{g1}(r'_1)\gamma_{g2}(r'_2)\gamma_{g3}(r'_3)\cdots\gamma_{gk}(r'_k)\cdots\gamma_{gN}(r'_N)\,\gamma(v)\,dt' \\ &= (1-\frac{2GM_{0\mathrm{a}1}}{r'_1c_g^2})^{-1/2}(1-\frac{2GM_{0\mathrm{a}2}}{r'_2c_g^2})^{-1/2}(1-\frac{2GM_{0\mathrm{a}3}}{r'_3c_g^2})^{-1/2}\cdots \\ &\quad \cdots(1-\frac{2GM_{0\mathrm{a}k}}{r'_kc_g^2})^{-1/2}\cdots(1-\frac{2GM_{0\mathrm{a}N}}{r'_Nc_g^2})^{-1/2}(1-\frac{v^2}{c_\gamma^2})^{-1/2}dt' = \infty \end{aligned} \tag{103}$$

$$\begin{aligned} d\Sigma &= \gamma_{g1}(r'_1)^{-1}\gamma_{g2}(r'_2)^{-1}\cdots\gamma_{gk}(r'_k)^{-1}\cdots\gamma_{gN}(r'_N)^{-1}\gamma(v)^{-1}d\Sigma' \\ &= (1-\frac{2GM_{0\mathrm{a}1}}{r'_1c_g^2})^{1/2}(1-\frac{2GM_{0\mathrm{a}2}}{r'_2c_g^2})^{1/2}(1-\frac{2GM_{0\mathrm{a}3}}{r'_3c_g^2})^{1/2}\cdots \\ &\quad \cdots(1-\frac{2GM_{0\mathrm{a}k}}{r'_kc_g^2})^{1/2}\cdots(1-\frac{2GM_{0\mathrm{a}N}}{r'_Nc_g^2})^{1/2}(1-\frac{v^2}{c_\gamma^2})^{1/2}d\Sigma' = 0 \end{aligned} \tag{104}$$

$$\begin{aligned} M_k &= \gamma_{g1}(r'_1)^{-2}\gamma_{g2}(r'_2)^{-2}\cdots\gamma_{gk}(r'_k)^{-2}\cdots\gamma_{gN}(r'_N)^{-2}\gamma(v)M_{0k} \\ &= (1-\frac{2GM_{0\mathrm{a}1}}{r'_1c_g^2})(1-\frac{2GM_{0\mathrm{a}2}}{r'_2c_g^2})(1-\frac{2GM_{0\mathrm{a}3}}{r'_3c_g^2})\cdots \\ &\quad \cdots(1-\frac{2GM_{0\mathrm{a}k}}{r'_kc_g^2})\cdots(1-\frac{2GM_{0\mathrm{a}N}}{r'_Nc_g^2})(1-\frac{v^2}{c_\gamma^2})^{-1/2}M_{0k} = 0 \end{aligned} \tag{105}$$

Equations (103), (104) and (105) show that the above principle is not affected by the velocity of the observer relative to the black hole (or of the black hole relative to the observer). Hence the principle is indeed valid with respect to all observers in the universe as stated.

Now the gravitational speed at the surface of a black hole has a numerical value of 3×10^8 m/s. That is, $V'_g(r'_k)/c_g = (2GM_{0\mathrm{a}k}/r'_k)^{1/2} = 1$; hence $V'_g(r'_k) = c_g = 3\times10^8\,\mathrm{m\,s^{-1}}$. The principle derived above, which states that the event horizon of a black hole is unaltered by its location in an external gravitational field, where the gravitational speed due to the other sources is non-zero, implies that the maximum gravitational speed c_g, (at the event horizon of a black hole), is an invariant with both location in the universe and the observer.

As a matter of fact, the rule for composition of gravitational velocities in the context of TGR, which follows from gravitational local Lorentz transformation (GLLT) or its inverse, is the same as the rule for composition of dynamical velocities in SR. It is given as follows, as derived formally in [4]

$$v_g = \frac{V'_g(r'_{\text{es}}) + v'_g}{1 + V'_g(r'_{\text{es}})v'_g/c_g^2} \tag{106}$$

For instance, the earth, by virtue of its gravitational field, prescribes gravitational speed v'_g at its surface. The Sun also prescribes gravitational speed $V'_g(r'_{\text{es}})$ at the surface of the earth, where r'_{es} is the radial distance from the center of the Sun to the surface of the earth in the proper Euclidean 3-space Σ'. The resultant gravitational speed v_g of a point on the surface of the earth, which lies on the line joining the centers of the earth and the Sun, such that the gravitational speeds v'_g and $V'_g(r')$ are collinear, is given by Eq. (106).

If either $V'_g(r'_{\text{es}}) = c_g$ or $v'_g = c_g$, then $v_g = c_g$, and if $V'_g(r'_{\text{es}}) = c_g$ and $v'_g = c_g$, then $v_g = c_g$ again in Eq. (106). This shows that the gravitational speed c_g at the event horizon of a black hole is unaltered by the presence of other gravitational field sources. Thus the derived principle of black hole physics stated above can be stated equivalently as follows

> The gravitational speed c_g at the event horizon of a black hole (which is the speed of gravitational waves), is invariant with the observer (or frame of reference) and with location of the black hole in the universe.

Clearly this is the counterpart in the theory of gravitational relativity of the second principle of the special theory of relativity. This derived principle was stated without proof at the beginning of the analytical development of TGR in analogy with the analytical development of SR in [4].

5 The metric theory of absolute intrinsic gravity at the neighborhood of several isolated gravitational field sources

As has been well developed in [10] – [11] and [12], the theory of absolute intrinsic gravity (ϕAG) is composed of the metric theory of absolute intrinsic gravity (ϕMAG) with absolute intrinsic sub-Riemannian line element $d\phi\hat{s}^2 = \phi\hat{g}_{ik}dx^i dx^k$, on curved 'two-dimensional' absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$ with absolute intrinsic sub-Riemannian metric tensor $\phi\hat{g}_{ik}$, and the starred Newtonian theory of absolute intrinsic gravity (ϕNAG*) on the curved $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$, derived by the absolute intrinsic action principle with the aid of the absolute intrinsic line element of ϕMAG, with respect to 3-observers in the relativistic Euclidean 3-space Σ in [11] and [12], although ϕNAG* is non-detectable to these observers.

The ϕNAG* on the curved absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$ is invariantly projected as Newtonian theory of absolute gravity without star label (ϕNAG) into the underlying flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, which is made manifest in the Newtonian theory of absolute gravity (NAG) in the flat four-dimensional relativistic spacetime (Σ, ct), but which is non-observable and non-detectable to 3-observers in Σ. The non-detectable NAG co-exists with the observed theory of gravitational relativity (TGR) and relativistic Newton's law of gravity—relativistic in the context of TGR—on the flat four-dimensional relativistic spacetime (Σ, ct) in every gravitational field.

All the component theories of absolute intrinsic theory of gravity (ϕAG) namely, ϕMAG, ϕNAG*, ϕNAG and NAG, have been well developed at the exterior of a singular gravitational field source in [12]. Only the ϕMAG shall be extended to the neighborhood of several isolated gravitational field sources in this section, since such extensions of the other theories follow easily from the absolute intrinsic line element of ϕMAG.

The resultant absolute intrinsic line element and resultant absolute intrinsic metric tensor ϕMAG shall be derived, and this will be easy, since in the absolute intrinsic 2-geometry of ϕMAG, the 'one-dimensional' absolute intrinsic rest masses $\phi\hat{m}_0$ or $\phi\hat{M}_0$ of all particles and bodies in the universe lie along the 'one-dimensional' universal isotropic absolute intrinsic space $\phi\hat{\rho}$ and all their intrinsic inertial masses ϕm and ϕM lie along the one-dimensional isotropic universal relativistic intrinsic space $\phi\rho$, although their intrinsic masses m and M are scattered arbitrarily in the universal relativistic Euclidean 3-space Σ, with respect to all 3-observers in Σ. Hence all intrinsic gravitational field sources are collinear with any given point P in space in the context of ϕMAG. Moreover the components of the absolute intrinsic metric tensor are always related thus,

$$\phi\hat{g}_{00} = -\phi\hat{g}_{11}^{-1}; \phi\hat{g}_{12} = \phi\hat{g}_{21} = 0,$$

as has been established since [13].

Now let us revisit the theory of gravitational relativity (TGR) at the neighborhood of two gravitational field sources formulated in sub-section 2.1. By assuming that the mass M_1 is absent in Fig. 1a or 1b, the mass M_2 gives rise to TGR and establishes Lorentzian metric tensor at point P, as well as at every other point in spacetime in all its finite neighborhood. When the mass M_1 is then brought in place, operating upon the Lorentzian metric established everywhere by M_2, it also gives rise to TGR and establishes Lorentzian metric tensor at point P, as well as at every other point in spacetime again in all finite neighborhood of the field source. This can be continued until the resultant theory of gravitational relativity of as many isolated field sources as possible is obtained. Each new gravitational field source introduced operates upon the Lorentzian metric tensor established by the preceding field sources in the context of TGR. This superposition procedure in the context of TGR gives rise to the resultant factor $\overline{\gamma}_g^t$ of Eq. (43) at the neighborhood of two gravitational field sources, and Eq. (41) at the neighborhood of N gravitational field sources in the context of TGR.

The procedure for superposition of spacetime geometry in the context of the 'two-dimensional' metric theory of absolute intrinsic gravity (ϕMAG), is different from that of TGR described in the preceding paragraph. In ϕMAG, the absolute intrinsic rest mass $\phi\hat{M}_{01}$ of the gravitational field source of inertial mass M_1 in Σ, assuming mass M_2 is absent (in Figs. 1a and 1b), gives rise to the curved absolute intrinsic spacetime $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ relative to flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ and establishes an absolute intrinsic metric tensor $\hat{g}'_{ik}$ at every point on this curved $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$, with respect to 3-observers in the physical Euclidean 3-space Σ.

Thus when M_2 is brought in place, operating upon the curved absolute intrinsic spacetime $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ with absolute intrinsic sub-Riemannian metric tensor established by M_1, its absolute intrinsic rest mass $\phi\hat{M}_{02}$ establishes another curved absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$ relative to the curved $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ established by $\phi\hat{M}_{01}$, as illustrated in Fig. 7.

The two gravitational field sources thereby establish a resultant absolute intrinsic sub-Riemannian metric tensor on the upper curved absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$ relative to the flat relativistic intrinsic space $(\phi\rho, \phi c\phi t)$ at every point in spacetime in all their finite neighborhood, with respect to all 3-observers in the physical Euclidean 3-space Σ. Fig. 7 illustrates the curved absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$ established by absolute absolute intrinsic rest mass $\phi\hat{M}_{02}$ upon the curved absolute intrinsic spacetime $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ established by $\phi\hat{M}_{01}$.

Since the absolute intrinsic spacetine $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ in the absence of $\phi\hat{M}_{02}$ is curved relative to the flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$, the components of the absolute intrinsic metric tensor of $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ at P, due to the gravitational field source M_1, assuming M_2 is absent, in the context of the present metric theory of absolute intrinsic gravity is the following

$$\phi\hat{g}_{00} = -\phi\hat{g}_{11}^{-1} = 1 - \frac{2G\phi\hat{M}_{0a1}}{\phi\hat{r}_1\phi\hat{c}_g^2};\ \phi\hat{g}_{12} = \phi\hat{g}_{21} = 0 \qquad (107)$$

Now the components of the intrinsic Lorentzian metric tensor of the flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ relative to which $(\phi\hat{\rho}', \phi\hat{c}\phi\hat{t}')$ is curved in Fig. 7 are $\eta_{00} = -\eta_{11} = 1; \eta_{12} = \eta_{21} = 0$. Hence system (107) can be written alternatively as follows

$$\phi\hat{g}_{00} = -\phi\hat{g}_{11}^{-1} = \eta_{00} - \frac{2G\phi\hat{M}_{0a1}}{\phi\hat{r}_1\phi\hat{c}_g^2};\ \phi\hat{g}_{12} = \phi\hat{g}_{21} = 0 \qquad (108)$$

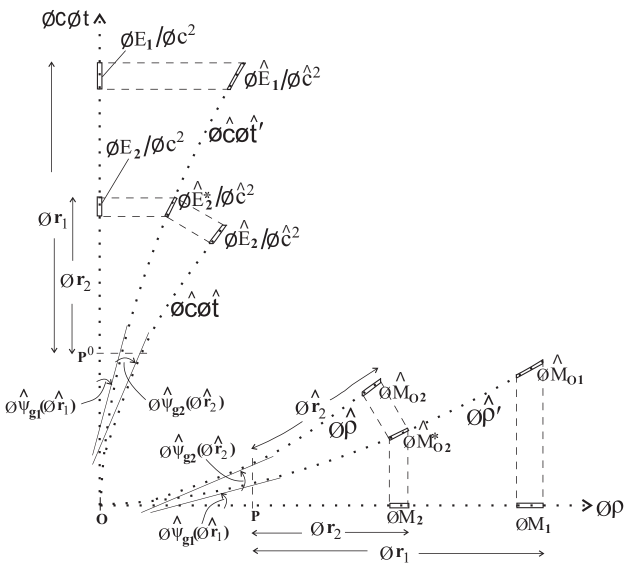

Fig. 7: The resultant curved absolute intrinsic spacetime at the neighborhood of two isolated gravitational field sources at the second stage of evolutions of space-time/intrinsic space-time and of parameters/intrinsic parameters.

System (108) shows that the field source M_1 establishes curved absolute intrinsic spacetime (or absolute intrinsic Riemannian spacetime geometry) relative to the flat relativistic intrinsic spacetime with intrinsic Lorentzian metric tensor (or relative to an intrinsic Lorentzian boundary condition). Now when the gravitational field source M_2 is brought in place, it will establish curved absolute intrinsic spacetime (or absolute intrinsic Riemnnian spacetime geometry) relative to the absolute intrinsic Riemannian spacetime geometry, (or absolute intrinsic sub-Riemannian metric tensor) that M_1 established. Hence the components of the resultant absolute intrinsic metric tensor at P due to M_1 and M_2 jointly, (in the context of ϕMG), are given as follows, as has been derived formally in [13],

$$\begin{aligned}\phi\hat{\hat{g}}_{00} = -\phi\hat{\hat{g}}_{11}^{-1} &= \phi\hat{g}_{00} - \frac{2G\phi\hat{M}_{0a2}}{\phi\hat{r}_2\phi\hat{c}_g^2};\ \phi\hat{\hat{g}}_{12} = \phi\hat{\hat{g}}_{21} = 0 \\ &= 1 - \frac{2G\phi\hat{M}_{0a1}}{\phi\hat{r}_1\phi\hat{c}_g^2} - \frac{2G\phi\hat{M}_{0a2}}{\phi\hat{r}_2\phi\hat{c}_g^2};\ \phi\hat{\hat{g}}_{12} = \phi\hat{\hat{g}}_{21} = 0 \qquad (109)\end{aligned}$$

And when another gravitational field source of inertial mass M_3 is brought in place at radial distance r_3 from point P in the physical Euclidean 3-space Σ, then the components of the resultant absolute intrinsic metric tensor at P are the following

$$\phi\hat{\bar{g}}_{00} = -\phi\hat{\bar{g}}_{11}^{-1} = 1 - \frac{2G\phi\hat{M}_{0a1}}{\phi\hat{r}_1\phi\hat{c}_g^2} - \frac{2G\phi\hat{M}_{0a2}}{\phi\hat{r}_2\phi\hat{c}_g^2} - \frac{2G\phi\hat{M}_{0a3}}{\phi\hat{r}_3\phi\hat{c}_g^2};\ \phi\hat{\bar{g}}_{12} = \phi\hat{\bar{g}}_{21} = 0 \quad (110)$$

In general, the components of the resultant absolute intrinsic metric tensor of the ‘two-dimensional’ metric theory of absolute intrinsic gravity (ϕMAG) at a point P in space, due to N gravitational field sources scattered in space about this point is the following

$$\phi\hat{\bar{g}}_{00} = -\phi\hat{\bar{g}}_{11}^{-1} = 1 - \sum_{i=1}^{N} \frac{2G\phi\hat{M}_{0a\,i}}{\phi\hat{r}_i\phi\hat{c}_g^2};\ \phi\hat{\bar{g}}_{12} = \phi\hat{\bar{g}}_{21} = 0 \quad (111)$$

System (109) can be written in terms of absolute intrinsic angles $\phi\hat{\psi}_{g1}(\phi\hat{r}_1)$ and $\phi\hat{\psi}_{g2}(\phi\hat{r}_2)$ and the resultant absolute intrinsic angle, $\phi\hat{\psi}_{g\mathrm{res}} = \phi\hat{\psi}_{g1}(\phi\hat{r}_1) + \phi\hat{\psi}_{g2}(\phi\hat{r}_2)$, in Fig. 7, knowing that

$$\sin^2\phi\hat{\psi}_{g1}(\phi\hat{r}_1) = 2G\phi\hat{M}_{0a1}/\phi\hat{r}_1\phi\hat{c}_g^2 \ \text{ and } \ \sin^2\phi\hat{\psi}_{g2}(\phi\hat{r}_2) = 2G\phi\hat{M}_{0a2}/\phi\hat{r}_2\phi\hat{c}_g^2,$$

as follows

$$\sin^2\phi\hat{\psi}_{g\mathrm{res}} = \sin^2[\phi\hat{\psi}_{g1}(\phi\hat{r}_1) + \phi\hat{\psi}_{g2}(\phi\hat{r}_2)] = \sin^2\phi\hat{\psi}_{g1}(\phi\hat{r}_1) + \sin^2\phi\hat{\psi}_{g2}(\phi\hat{r}_2) \quad (112)$$

Hence

$$\phi\hat{\bar{g}}_{00} = -\phi\hat{\bar{g}}_{11}^{-1} = \cos^2\phi\hat{\psi}_{g\mathrm{res}} = 1 - \sin^2\phi\hat{\psi}_{g1}(\phi\hat{r}_1) - \sin^2\phi\hat{\psi}_{g2}(\phi\hat{r}_2);\ \phi\hat{\bar{g}}_{12} = \phi\hat{\bar{g}}_{21} = 0 \quad (113)$$

The generalization of Eqs. (112) and (113) to the case of N isolated gravitational field sources are the following respectively

$$\sin^2\phi\hat{\psi}_{g\mathrm{res}} = \sin^2[\phi\hat{\psi}_{g1}(\phi\hat{r}_1) + \phi\hat{\psi}_{g2}(\phi\hat{r}_2) + \ldots + \phi\hat{\psi}_{gN}(\phi\hat{r}_N)] = \sum_{i=1}^{N}\sin^2\phi\hat{\psi}_{gi}(\phi\hat{r}_i) \quad (114)$$

Hence

$$\phi\hat{\bar{g}}_{00} = -\phi\hat{\bar{g}}_{11}^{-1} = \cos^2\phi\hat{\psi}_{g\mathrm{res}} = 1 - \sum_{i=1}^{N}\sin^2\phi\hat{\psi}_{gi}(\phi\hat{r}_i);\ \phi\hat{\bar{g}}_{12} = \phi\hat{\bar{g}}_{21} = 0 \quad (115)$$

Equations (112) and (113) give the rules for finding the sine and cosine of the sum of two absolute intrinsic angles of rotation on the vertical absolute intrinsic spacetime plane, in the context of the ‘two-dimensional’ metric theory of absolute intrinsic gravity, while Eqs. (114) and (115) give their generalization to the composition of N absolute intrinsic angles.

Finally the absolute intrinsic curvature parameters $\phi\hat{k}_{g1}(\phi\hat{r}_1)$ and $\phi\hat{k}_{g2}(\phi\hat{r}_2)$ at point P of the curved absolute intrinsic spaces $\phi\hat{\rho}'$ and $\phi\hat{\rho}$ in Fig. 7 and the resultant absolute intrinsic curvature parameter $\phi\hat{k}_{g\mathrm{res}}$ are related to the absolute intrinsic angles $\hat{\psi}_1(\phi\hat{r}_1)$ and $\hat{\psi}_2(\phi\hat{r}_2)$ as follows, as derived in [13] – [14]

$$\begin{aligned} \phi\hat{k}^2_{g1}(\phi\hat{r}_1) &= \sin^2\phi\hat{\psi}_{g1}(\phi\hat{r}_1) = 2G\phi\hat{M}_{0\mathrm{a}1}/\phi\hat{r}_1\phi\hat{c}^2_g \\ \phi\hat{k}^2_{g2}(\phi\hat{r}_2) &= \sin^2\phi\hat{\psi}_{g2}(\phi\hat{r}_2) = 2G\phi\hat{M}_{0\mathrm{a}2}/\phi\hat{r}_2\phi\hat{c}^2_g \end{aligned}$$

Hence

$$\phi\hat{k}^2_{g\mathrm{res}} = \phi\hat{k}^2_{g1}(\phi\hat{r}_1) + \phi\hat{k}^2_{g2}(\phi\hat{r}_2) \qquad (116)$$

and

$$\phi\hat{\hat{g}}_{00} = -\phi\hat{\hat{g}}^{-1}_{11} = 1 - \phi\hat{k}^2_{g\mathrm{res}} = 1 - \phi\hat{k}^2_{g1}(\phi\hat{r}_1) - \phi\hat{k}^2_{g2}(\phi\hat{r}_2);\ \phi\hat{\hat{g}}_{12} = \phi\hat{\hat{g}}_{21} = 0 \qquad (117)$$

The generalizations of Eqs. (116) and (117) to the situation of the neighborhood of N isolated gravitational field sources are given respectively as follows

$$\phi\hat{k}^2_{g\mathrm{res}} = \sum_{i=1}^{N} \phi\hat{k}^2_{gi}(\phi\hat{r}_i) \qquad (118)$$

and

$$\phi\hat{\hat{g}}_{00} = -\phi\hat{\hat{g}}^{-1}_{11} = 1 - \phi\hat{k}^2_{g\mathrm{res}} = 1 - \sum_{i=1}^{N} \phi\hat{k}^2_{gi}(\phi\hat{r}_i);\ \phi\hat{\hat{g}}_{12} = \phi\hat{\hat{g}}_{21} = 0 \qquad (119)$$

Again Eq. (116) and its generalization (118) give the rule for the composition of absolute intrinsic curvature parameters for the purpose of deriving resultant absolute intrinsic metric tensor in the context of the ‘two-dimensional’ metric theory of absolute intrinsic gravity.

The absolute intrinsic line element of ϕMAG, which is valid with respect to 3-observers in the physical Euclidean 3-space Σ at the neighborhood of isolated N gravitational field sources is then given as follows from Eqs. (111), (113) and (115) and (119)

$$d\phi\hat{s}^2 = \phi\hat{\hat{g}}_{00}\phi\hat{c}^2 d\phi\hat{t}^2 - \phi\hat{\hat{g}}_{11}\, d\phi\hat{\rho}^2$$

$$= \left(1 - \sum_{i=1}^{N} \sin^2 \phi\hat{\psi}_{gi}(\phi\hat{r}_i)\right) \phi\hat{c}^2 d\phi\hat{t}^2 - \left(1 - \sum_{i=1}^{N} \sin^2 \phi\hat{\psi}_{gi}(\phi\hat{r}_i)\right)^{-1} d\phi\hat{\rho}^2 \tag{120}$$

$$= \left(1 - \sum_{i=1}^{N} \phi\hat{k}_{gi}(\phi\hat{r}_i)^2\right) \phi\hat{c}^2 d\phi\hat{t}^2 - \left(1 - \sum_{i=1}^{N} \phi\hat{k}_{gi}(\phi\hat{r}_i)^2\right)^{-1} d\phi\hat{\rho}^2 \tag{121}$$

$$= \left(1 - \sum_{i=1}^{N} \frac{2G\phi\hat{M}_{0\mathrm{a}\,i}}{\phi\hat{r}_i\phi\hat{c}_g^2}\right) \phi\hat{c}^2 d\phi\hat{t}^2 - \left(1 - \sum_{i=1}^{N} \frac{2G\phi\hat{M}_{0\mathrm{a}\,i}}{\phi\hat{r}_i\phi\hat{c}_g^2}\right)^{-1} d\phi\hat{\rho}^2 \tag{122}$$

This first part of this article shall be ended at this point with a recap of its essential accomplishments. These are the extensions of TGR/ϕTGR on flat relativistic spacetime (Σ, ct) and its underlying flat relativistic intrinsic spacetime $(\phi\rho, \phi c\phi t)$ and ϕMAG on curved absolute intrinsic spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$, to the neighborhood of several isolated gravitational field sources; formulation of the relativistic Newton's law of gravity (RNG) in the context of TGR at the neighborhood of several gravitational field sources; validation of Einstein's principle of equivalence at the neighborhood of any number N of isolated gravitational field sources and consequently within the entire universe and the N-body problem for N = 2 and N = 3, which admits of straight forward extension to N = 4 and larger, in the context of TGR and any value of N in the context of ϕMAG. The corresponding theories of dynamics namely, SR/ϕSR and ϕMAM, shall be incorporated into the results of this article in the second part of it.

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