J. W. Nienhuys

de Bruijn's COMBINATORICS

Classroom notes

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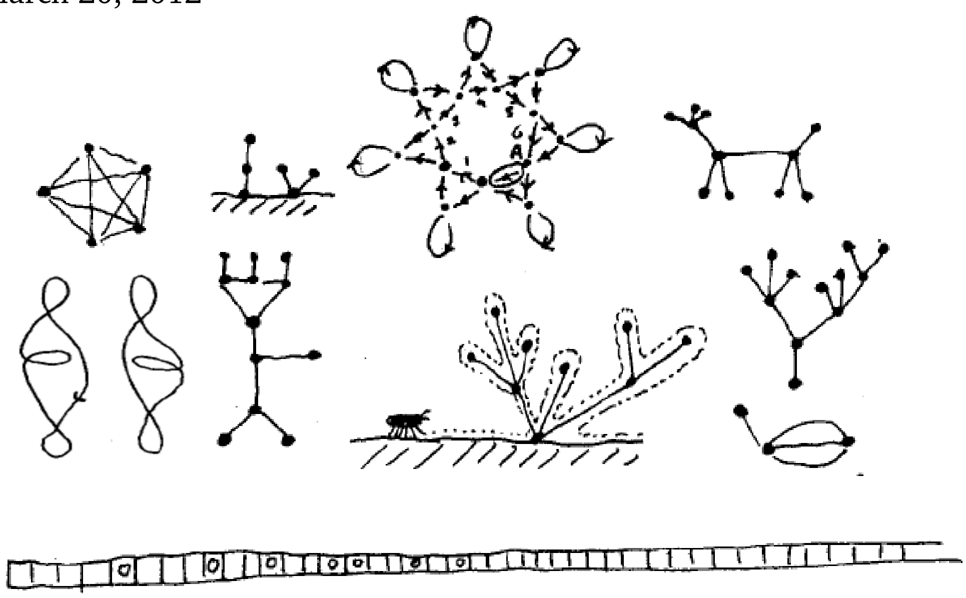

In honor of
N. G. de Bruijn and J. W. Nienhuys.

Preface

These are Nienhuys' lecture notes on a course given by de Bruijn starting in the late 1970s and all through the 1980s. We thought it was time to translate this classic text into English.

This text emphasizes the study of equivalences in enumeration methods. Nowadays, much research in graph theory and graph algorithms focuses on tree-decompositions of graphs. In order to obtain efficient algorithms using these tree-structures, a good understanding of equivalent solutions and equivalent tree-structures are of immense importance. More than other texts in basic combinatorics, this text emphasizes enumeration techniques and thereby trains a student to recognize equivalent combinatorial objects. The center of the book is Pólya's enumeration theory. This theory is without doubt the best developed general tool for the enumeration of, *e.g.*, geometrical and chemical objects that are equivalent under certain group operations. The chapters on generating functions and permutations slowly build up to the chapter on Pólya's theory. The chapter on graphs further emphasizes enumerations of trees under certain equivalences. The chapter on bipartite graphs turns the attention in the direction of algorithmics. It emphasizes covering and representative problems.

Games are widely studied in mathematics, economics, and computer science. The chapter on games is a playful introduction, by various nim games, into Grundy function theory.

Translating this text brought one of us back into the classroom, listening to de Bruijn explaining his combinatorics. These notes are true classroom notes; the teachings of de Bruijn were written down verbatim by Nienhuys and the blackboard material was copied in numerous figures. This gives the text a unique, almost magical, speed and clarity which one cannot find in other text books.

The original text lacked exercises. Besides 'ordinary' exercises, we have tried to incorperate various classic papers here, of *e.g.*, van der Waerden, Berge, Tutte and, of course, de Bruijn. The student is guided through these beautiful papers in a step-by-step manner.

We taught this course in the spring-semester 2010 to undergraduate students at the Department of Computer Science and Information Engineering of National Chung Cheng University, Ming-Shiun, Chia-Yi, Taiwan. The enthusiasm with which our students welcomed this course surpassed our wildest dreams. It proved to us that, without doubt, the preservation of this text will bring pleasure to many generations of combinatorics students to come.

We are deeply indebted to professor Dan Buehrer for proofreading and correcting our English translation.

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1 Introduction

N. G. de Bruijn gave a course in Combinatorics during the 1980s at Eindhoven University of Technology. During the course J. W. Nienhuys took notes. This is a translation of these notes.

What is combinatorics? You could say that combinatorics is that part of mathematics that concerns itself with finite systems. But this is not entirely true: the theory of finite groups, rings and fields does not belong to combinatorics.

It is with combinatorics as it is with asymptotics. You would like to define asymptotics as the theory of limits, but large parts of the differential– and integration-calculus do not belong to asymptotics.

For a long time combinatorics consisted only of puzzles and games. Before the 1960s there was probably no course in combinatorics.

Combinatorics is mainly about counting. The nice thing about counting is that you learn something about the thing that you are counting. While counting, you notice that your knowledge about the subject is not sufficiently precise, and so counting becomes an educational activity. When two sets have the same number of elements, you try to understand why that is the case by establishing some natural bijection between the two.

N. G. de Bruijn recalls that sometime around 1975, he counted the number of a certain type of logics with three variables. There were a lot of them; some number with 14 digits. A little while later he saw an article that was about something completely different, but it concluded that the number of those objects was exactly that same number with 14 digits. He got curious and read the article very carefully. Indeed, if you thought deeply about it you could see that you could interpret those objects also as logics.

Where is combinatorics applied? There is a lot of counting going on in statistics; and in computer science combinatorics plays a role in considerations about complexity.

Let’s just start, and do some combinatorics.

2

Generating Functions

One very old idea in combinatorics is the idea of the generating function. This idea can be traced back to Laplace.

Given an infinite sequence

$$a_0, a_1, a_2, \ldots$$

of numbers, then the generating function of that sequence is the following series:

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

The technique of the generating function is that you try to follow the next flowchart:

$$\begin{tikzcd}
\boxed{\text{knowledge about the sequence}} \arrow[r, "\text{translation}", "\text{forward}"'] & \boxed{\text{knowledge about } A(x)} \arrow[d, "\text{operations}"] \\
\boxed{\text{more knowledge about the sequence.}} & \boxed{\text{more knowledge about } A(x)} \arrow[l, "\text{translation}"', "\text{backwards}"]
\end{tikzcd}$$

This flowchart is a bit like that of the Laplace integral

$$F(x) = \int_0^\infty e^{xt} f(t)\,dt.$$

The analogous flowchart is:

$$\begin{tikzcd}
\boxed{\text{Differential equation for } f} \arrow[r, "\text{Laplace}", "\text{transformation}"'] & \boxed{\text{Algebraic equation for } F} \arrow[d, "\text{solve algebraic equation}"] \\
\boxed{\text{solution of differential equation for } f.} & \boxed{\text{Solution of algebraic equation for } F} \arrow[l, "\text{transformation}"', "\text{backwards}"]
\end{tikzcd}$$

Remark 2.1. Besides $a_0 + a_1 x + a_2 x^2 + \ldots$ people also consider

$$\frac{a_0}{0!} + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \ldots$$

but we won't do that.

Notation.

Due to the confusion caused by N. Bourbaki about the natural numbers, we feel obliged to define:

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\} \quad \text{and}$$
$$\mathbb{N}_1 = \{1, 2, 3, \ldots\}.$$

Example 2.2. The Fibonacci sequence is defined as follows:

$$a_0 = 1; \quad a_1 = 1; \quad \text{and}$$
$$a_{n+1} = a_n + a_{n-1} \quad \text{for} \quad n \in \mathbb{N}_1.$$

The generating function for the Fibonacci sequence is:

$$A(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots.$$

After a bit of puzzling we find, based on the recurrence relation of the coefficients:

$$x(x+1)A(x) = A(x) - 1$$

and so

$$A(x) = \frac{-1}{-1 + x + x^2}.$$

Splitting this fraction gives:

$$A(x) = \left(\frac{1}{x - \alpha_1} - \frac{1}{x - \alpha_2}\right)\frac{-1}{\alpha_1 - \alpha_2}$$

where α_1 and α_2 are the roots of $x^2 + x - 1$.

When we write these fractions again as a power series and collect terms, we find:

$$a_n = \frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}}.$$

We can defend this hocus-pocus in three ways:

1. As a heuristic. We get a result – it does not matter how we got that – and we check it afterward, in this case for example by mathematical induction.
2. Via the theory of convergent power series. We look at A as an analytic function, defined in some small neighborhood of zero in $\mathbb{C}$. This justifies our calculations with A as if it is a function.
   The disadvantage of this method is that we need to examine the convergence. We have to show that $A(x)$ has some positive radius of convergence. In this case this is not a big problem:
   $$\forall_{n \in \mathbb{N}_0} \, |a_n| \leqslant 2^n$$
   is easy to prove by induction, from which it then follows that the radius of convergence is $\geqslant \frac{1}{2}$.
3. By developing the theory of formal power series. This can be done formally and exactly, but it is rather boring to read.[1] Of course, nothing new or surprising turns up in a proof for something like that.

Overview of the theory of formal power series.

We will give the theory for power series with complex coefficients, but with some minor adjustments it also works in rings.

You have to look at a power series as a polynomial with possible infinite degree. If you really want to do it nicely, then the sequence that corresponds with the power series is the power series itself; when you think about it in this way, then the power series is just a mapping

$$\mathbb{N}_0 \to \mathbb{C}, \quad n \to a_n.$$

Addition is defined as usual. Let

$$A(x) = a_0 + a_1 x + a_2 x^2 + \dots$$
$$B(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

Then

$$(A + B)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$$

Multiplication is defined by the so-called Cauchy product

[1] W. Tutte, On elementary calculus and the good formula, Journal of Combinatorial Theory, Series (B) 18, (1975), pp. 97–137.

$$(AB)(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots$$

The coefficient of x^n is $\sum_{j=0}^{n} a_j b_{n-j}$. If $b_0 \neq 0$, then $\frac{A(x)}{B(x)}$ is also defined. If $b_0 = 0$ then $A(B(x))$ is also defined and we can even determine a $C(x)$ such that $C(B(x)) = x$.

We can differentiate power series in a formal way.

We can define, under certain conditions, infinite series and products of power series.

Remark 2.3. The idea is to allow some (possibly infinite) operation if the computation of every coefficient takes only finitely many non-trivial steps. A trivial step is the addition of zero, multiplication by 0, or multiplication by 1.

Consider for example,

$$A(B(x)) = a_0 + a_1B(x) + a_2(B(x))^2 + a_3(B(x))^3 + \dots$$

When $b_0 \neq 0$, then every term $a_n(B(x))^n$ has a nonzero constant term, and collecting those terms gives a constant term:

$$a_0 + a_1b_0 + a_2b_0^2 + \dots$$

This is the kind of infinite operation that we do not allow.

If, for example, for $k \in \mathbb{N}_0$, $A_k(x)$ is a power series starting with $a_kx^k+\dots$, then we can define $\sum_{k\geqslant 0} A_k(x)$: for the k^{th} term we only have to sum up $k+1$ terms.

Let's look at another example.

2.1 Small change

Suppose we have to pay Mrs. Geerts of the university's cafeteria some enormous sum of money, say 67 cts. (A cozy old-fashioned example with those cents.) We can only pay with coins of 1, 5, 10, and 25 cents. Of each of these coins we have plenty (say infinitely many).

We ask ourselves in how many ways we can pay. Rather quickly we figure out that we need to define which ways to pay are really counted as 'different.' Does the order count? Do we put the coins in a circle – or in a square? Pretty quickly we concede that examples of ways to pay are:

(**1**)	(**5**)	(**10**)	(**25**)
2	2	3	1
7	2	0	2
7	0	1	2

in other words, a way to pay is completely characterized by its frequency function:

$$f : \{\mathbf{1}, \mathbf{5}, \mathbf{10}, \mathbf{25}\} \to \mathbb{N}_0.$$

The question is how many frequency functions there are with a total value of 67 cts. The total value of a frequency function f is

$$1 \times f(\mathbf{1}) + 5 \times f(\mathbf{5}) + 10 \times f(\mathbf{10}) + 25 \times f(\mathbf{25}).$$

Solution.

We are not doing this systematically, yet. The answer is the coefficient of x^{67} in

$$\begin{array}{l}(1 + \ x \ + \underline{x^2} + \ x^3 + \ldots)\times \\ (1 + x^5 + \underline{x^{10}} + x^{15} + \ldots)\times \\ (1 + x^{10} + x^{20} + \underline{x^{30}} + \ldots)\times \\ (1 + \underline{x^{25}} + x^{50} + \ x^{75} + \ldots)\end{array}$$

which is 87.

If we write C_E for "the coefficient of E," then our answer could be written as:

$$C_{x^{67}} \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}.$$

In this way we can also find other coefficients, or determine, if we like, the asymptotics of C_{x^n} for $n \to \infty$.

You can see that the solution is correct because we can find the coefficient of x^{67} in the product by working out the multiplication and collecting the terms. Before collecting the terms you have a lot of terms, all with coefficient 1. Every term corresponds to choosing a term from the first line, a term in the second line, one from the third line, and one from the fourth line. For example, one of the terms is $x^2x^{10}x^{30}x^{25}$. This corresponds to the first line that we put in the little table above. Every term in the product corresponds to a way to pay.

2.2 A summation formula

We guess that you are familiar with the following notation:

$$|S| \quad \text{is the number of elements of } S.$$

If R and D are sets, and $D \neq \varnothing$, then we denote the collection of mappings $D \to R$ by R^D. The formula $|R^D| = |R|^{|D|}$ is a nice mnemonic to remember the structure of the formula R^D.

Theorem 2.4. *Let* K *be a commutative ring, and let* D *and* R *be sets, and let* $\phi : D \times R \to K$ *be a mapping. Then*

$$\sum_{f \in R^D} \prod_{d \in D} \phi(d, f(d)) = \prod_{d \in D} \sum_{r \in R} \phi(d, r).$$

Proof. It's pretty obvious when you look at it for a second, but let's just pretend that we 'prove' it. We write $n = |D|$. Number the elements of D, say $d_1, \ldots, d_n$. The product $\prod_{d \in D} \phi(d, f(d))$ is

$$\phi(d_1, f(d_1))\phi(d_2, f(d_2)) \ldots \phi(d_n, f(d_n)). \tag{2.1}$$

For every n-tuple $(r_1, \ldots, r_n) \in R^n$ there is exactly *one* $f \in R^D$ with $f(d_1) = r_1, f(d_2) = r_2, \ldots, f(d_n) = r_n$. For this f, equation 2.1 becomes:

$$\phi(d_1, r_1) \ldots \phi(d_n, r_n). \tag{2.2}$$

Instead of summing over R^D, we sum over R^n and we do this in a multiple summation:

$$\begin{aligned}
&\sum_{f \in R^D} \prod_{d \in D} \phi(d, f(d)) = \\
&= \sum_{r_1 \in R} \cdots \sum_{r_n \in R} \prod_{i=1}^{n} \phi(d_i, r_i) = \\
&= \sum_{r_1 \in R} \phi(d_1, r_1) \sum_{r_2 \in R} \cdots \sum_{r_n \in R} \prod_{i=2}^{n} \phi(d_i, r_i) = \\
&= \sum_{r_1 \in R} \phi(d_1, r_1) \sum_{r_2 \in R} \phi(d_2, r_2) \ldots \sum_{r_n \in R} \phi(d_n, r_n).
\end{aligned}$$

The $r_1, \ldots, r_n$ in the formulas above are bound variables (dummies), and we can write as well:

$$\begin{aligned}
&= \sum_{r \in R} \phi(d_1, r) \sum_{r \in R} \phi(d_2, r) \ldots \sum_{r \in R} \phi(d_n, r) = \\
&= \prod_{i=1}^{n} \sum_{r \in R} \phi(d_i, r) = \prod_{d \in D} \sum_{r \in R} \phi(d, r).
\end{aligned}$$

□

As an application we can let K be the ring of formal power series over $\mathbb{Z}$, $D = \{\mathbf{1}, \mathbf{5}, \mathbf{10}, \mathbf{25}\}$, and $R = \mathbb{N}_0$, $\phi(d, r) = x^{dr}$. Then

$$\prod_{d \in D} \phi(d, f(d)) = x^{\text{(total value of } f)}.$$

Let us now consider a special case: assume that the cafeteria only accepts payments with **5**-coins in even numbers, for some reason. Or more generally, with every type of coin there is a set S_d of "allowed" numbers. Define

$$\phi(d, r) = x^{c_d r} \times \nu(r \in S_d)$$

where c_d is the "weight" of d and

$$\nu(\text{statement}) = \begin{cases} 0 & \text{if the statement is false} \\ 1 & \text{if the statement is true.} \end{cases}$$

Furthermore, if we now define

$$C = \{ f \in R^D \mid \forall_{d \in D} \ \ f(d) \in S_d \}$$

then we have the following theorem.

Theorem 2.5.

$$\sum_{f \in C} x^{\sum_{d \in D} c_d f(d)} = \prod_{d \in D} \sum_{r \in S_d} x^{c_d r}.$$

Proof. Just notice that

$$\begin{aligned}
&\sum_{f \in C} x^{\sum_{d \in D} c_d f(d)} = \\
&= \sum_{f \in C} \prod_{d \in D} x^{c_d f(d)} = \\
&= \sum_{f \in R^D} \prod_{d \in D} x^{c_d f(d)} \nu(f(d) \in S_d) \\
&= \sum_{f \in R^D} \prod_{d \in D} \phi(d, f(d)) = \\
&= \prod_{d \in D} \sum_{r \in R} \phi(d, r) = \\
&= \prod_{d \in D} \sum_{r \in R} x^{c_d r} \nu(r \in S_d) = \prod_{d \in D} \sum_{r \in S_d} x^{c_d r}.
\end{aligned}$$

□

Thus the number of ways we can pay the cafeteria is

$$\begin{aligned}
C_{x^{67}} \quad & (1 + x + x^2 + \ldots) \quad \times \\
& (1 + x^{10} + x^{20} + \ldots) \times \\
& (1 + x^{10} + x^{20} + \ldots) \times \\
& (1 + x^{25} + x^{50} + \ldots) = 47.
\end{aligned}$$

Example 2.6. Assume we have coins of **1**, **2**, **4**, **8**, and **16** cents. For payments we are allowed to use only *one* coin of each type. Thus we have $D = \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{16}\}$, $c_{\mathbf{1}} = 1$, $c_{\mathbf{2}} = 2$, etc., and $S_d = \{0, 1\}$ for all d. Then the question is, what is

$$C_{x^k}\ (1+x) \times (1+x^2) \times \ldots \times (1+x^{16}).$$

If we work this out, we see that it is

$$C_{x^k}\ (1+x+x^2+\ldots+x^{31}).$$

In other words, there is exactly *one* way to pay every sum of value 31 cents or less, and more cannot be paid at all. By the way, note that

$$\begin{aligned} &(1+x) \times (1+x^2) \times \ldots \times (1+x^{16}) = \\ &= \frac{1-x^2}{1-x} \times \frac{1-x^4}{1-x^2} \times \ldots \times \frac{1-x^{32}}{1-x^{16}} = \frac{1-x^{32}}{1-x}. \end{aligned}$$

If we have exactly one coin of every power of two, we get

$$(1+x) \times (1+x^2) \times \ldots = \frac{1-x^2}{1-x} \times \frac{1-x^4}{1-x^2} \times \frac{1-x^8}{1-x^4} \times \ldots = \frac{1}{1-x}$$

in other words, every amount can be paid in exactly one way. The connection with the binary number system is clear.

Analogously, we can work with coins of **1**, **10**, **100**, **1000**, etc. Let

$$S_d = \{0, 1, \ldots, 9\}.$$

In that case we get

$$(1+x+\ldots+x^9) \times (1+x^{10}+\ldots+x^{90}) \times (1+x^{100}+\ldots+x^{900}) \times \ldots = \frac{1}{1-x}.$$

So we see that every amount can be paid in exactly one way; there is exactly one way to write any number in the decimal number system.

Example 2.7. We ask ourselves in how many ways we can write a number n like

$$n = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k, \tag{2.3}$$

where k is arbitrary and $\epsilon_i \in \{1, 2\}$ for all i. First we need to figure out, again, exactly what we mean by a split of n like that. In this case, the order counts.

For example $\quad 3 = \begin{cases} 1+1+1 \\ 1+2 \\ 2+1 \end{cases} \qquad 4 = \begin{cases} 1+1+1+1 \\ 2+1+1 \\ 1+2+1 \\ 1+1+2 \\ 2+2. \end{cases}$

We cannot do this at once, but we need to analyze first how many ways there are to split n in exactly k summands. For k summands you find the following generating function

$$(x+x^2)\times(x+x^2)\times\ldots\times(x+x^2)$$

with k factors, and so we find the following generating function for the original problem:

$$\sum_{k=0}^{\infty}(x+x^2)^k=\frac{1}{1-x-x^2}.$$

So the answer is

$$C_{x^n}\frac{1}{1-x-x^2}.$$

But that's a surprise! That is our old friend the n^{th} Fibonacci number! We should try to find a recursive relation for M_n, the number of ways to split n, as in Equation 2.3.

Obviously, $M_1=1$, since there is only one way to split 1. And $M_2=2$, since we can split 2 either as $1+1$ or as 2. Now, a split of n either starts with a 1 or with a 2. Thus for n there are M_{n-1} splits that start with 1, and that is followed by a split of $n-1$, and there are M_{n-2} splits that start with a 2 and that is followed by a split of $n-2$. So we get that $M_n=M_{n-1}+M_{n-2}$ for $n\geqslant 3$, and this is exactly the recursive definition of the Fibonacci sequence.

Remark 2.8. With some restrictions Theorem 2.5 also holds for infinite D. Consider $D=\mathbb{N}_0$. The sum

$$\sum_{d\in D}c_d f(d)$$

in the exponent of x on the lefthand side has to be essentially finite. So we need to require that $0\in S_d$ if d is large enough.

If we write

$$\sum_{r\in S_d}x^{c_d r}=1+x^{k_d}\cdot g_{r,d}(x)$$

for some function $g_{r,d}$, then the infinite product on the righthand side is well-defined as long as

$$\lim_{d\to\infty}k_d=\infty.$$

If these two requirements are satisfied, Theorem 2.5 also holds for $D=\mathbb{N}_0$.

2.3 Multisets

Recapitulation.

We have seen the following interesting theorem. Let

$$C = \{\, f \in \mathbb{N}_0^D \mid \forall_{d \in D} \ \ f(d) \in S_d \,\}$$

and let $c : D \to \mathbb{Z}$ (or some other ring). Then

$$\sum_{f \in C} x^{\sum_{d \in D} c_d f(d)} = \prod_{d \in D} \sum_{r \in S_d} x^{c_d r}.$$

We proved that by writing down the factors of the righthand side below each other. Every way to get a term of the product corresponds with a summand of the lefthand side. In this section we look at some more examples.

Example 2.9 (Multisets). Let $D = \{a, b, c, d, e\}$. (This is a bit confusing, because a, b, c, etc. are usually names of variables, and not the elements of sets itself.)

A map $D \to \mathbb{N}_0$ can be seen as a *multiset*: a multiset is, intuitively, a set in which every element has a multiplicity. For $\delta \in D$, and a multiset $f : D \to \mathbb{N}_0$, we call $f(\delta)$ the *multiplicity*, or the *frequency* of δ in f. The sum of all frequencies is called the *weight* of f.[2]

For example, we can write (aaa, bb, d) for a multiset with frequencies 3, 2, 0, 1, and 0 respectively. The total weight is 6.

Question:

What is the number of multisets with weight n?

We use the formula. We have $D = \{a, b, c, d, e\}$, $S_\delta = \mathbb{N}_0$, $c_\delta = 1$ for all $\delta \in D$. Then the weight of a multiset f is:

$$\sum_{\delta \in D} c_\delta f(\delta).$$

We chose to write δ instead of d in order to avoid confusion with the set-element. And the answer to our question is, according to the theorem:

$$C_{x^n} \ (1 + x + x^2 + \dots)^{|D|}.$$

In case the original set D has k elements, the answer is

$$\begin{aligned} C_{x^n} \ \ (1 + x + x^2 + \dots)^k &= \\ C_{x^n} \qquad (1 - x)^{-k} \qquad &= \binom{-k}{n}(-1)^n = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}, \end{aligned}$$

and this is a result that can be traced back to Euler.

Could we have detected that in an easier way? Well, we could encode the multiset (aaa, bb, d) in the example above like

[2] By the way, the word "multiset" was coined by N. G. de Bruijn back in the 1970s; at least that's what D. Knuth says about it.

$$a \mid \mid \mid b \mid \mid c \; d \mid e$$

and since there is a fixed order of letters, we can just as well write this as

$$\cdot \mid \mid \mid \cdot \mid \mid \cdot \; \cdot \mid \cdot$$

Apart from the first dot, there are 10 positions (actually $n+k-1$ positions), and for a multiset with 6 elements (n elements) we can choose 6 bars out of those 10 positions in an arbitrary way.

Let's look at another example.

2.4 The weighing problem

Suppose we have scales with two pans.

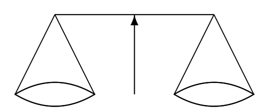

Fig. 2.1. Scales with two pans

What is the best way to choose a box with weights? A classical way to do that is to choose weights of $1, 2, 2, 5, 10, 20, 20, 50$ etc. units. Swiss money is still organized in that way! But according to Bachet the best way to do it is to choose weights that are powers of 3.

Take for example a box with weights of $1, 3, 9$, and 27 units. Put all the weights on the right pan. A weight of G units can be taken off the scales or moved from the right pan to the left pan. The difference in weight on the left pan minus the weight on the right pan then increases by either G or $2G$. The total increase is the sum of all these individual increases.

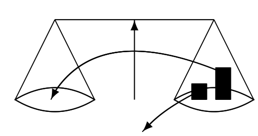

Fig. 2.2. Moving weights

We now have $D = \{0, 1, 2, 3\}$, $S_d = \{0, 3^d, 2 \cdot 3^d\}$, for $d \in D$, and $c_d = 1$ for all $d \in D$. For an admissible f the sum $\sum_{d \in D} c_d f(d)$ is exactly the total increase. Our formula gives the number of ways to have an increase of n:

$$\begin{aligned}
& C_{x^n}\ (1+x+x^2)\times(1+x^3+x^6)\times(1+x^9+x^{18})\times(1+x^{27}+x^{54}) = \\
&= C_{x^n}\ \frac{1-x^3}{1-x}\times\frac{1-x^9}{1-x^3}\times\frac{1-x^{27}}{1-x^9}\times\frac{1-x^{81}}{1-x^{27}} = \\
&= C_{x^n}\ \frac{1-x^{81}}{1-x} = C_{x^n}\ (1+x+x^2\cdots+x^{80}).
\end{aligned}$$

In other words, every increase from 0 to 80 can be realized in exactly *one* way.

Remark 2.10. We could have obtained the same answer by $S_d = \{0,1,2\}$ and $c_d = 3^d$ for all d. In that case the c_d's are "real weights."

Remark 2.11. One other way to get the answer is to put the weights in either the left pan or in the right pan, starting with both pans empty. Then you get

$$\begin{aligned}
&(x^{-1}+1+x)\times(x^{-3}+1+x^3)\times(x^{-9}+1+x^9)\times(x^{-27}+1+x^{27}) = \\
&x^{-40}+x^{-39}+\ldots+1+x+\ldots+x^{39}+x^{40}.
\end{aligned}$$

The disadvantage is that we then have to deal with formal power series with finitely many negative powers, *i.e.*, with formal Laurent series, and we haven't defined those properly. But if you get the idea of the formal power series in this kind of counting problems, then that is not such a big problem.

2.5 Partitions of numbers

Remark 2.12. Elsewhere we will consider "partitions of a set." A partition of a set S is an element $a \in P(P(S))$ such that

1) $\forall_{x\in a}\ \ x \neq \varnothing$,
2) $\forall_{x\in a}\forall_{y\in a}\ \ x = y\ \vee\ x\cap y = \varnothing$, and
3) $S = \cup_{x\in a} x$.

Here we discuss a somewhat similar idea, namely the partition of a *number*.

A partition of a number $n > 0$ is a way to write n as the sum of numbers that are $\neq 0$. For example, the partitions of 5 are:

$$5 = \begin{cases} 5 \\ 1+4 \\ 2+3 \\ 1+1+3 \\ 1+2+2 \\ 1+1+1+2 \\ 1+1+1+1+1 \end{cases}$$

ordered according to nondecreasing summands. The ordering does not matter.

It looks a bit like what we did with multisets but with a different "weight." In our formula we now have

1. $D = \mathbb{N}_1$,
2. $S_d = R = \mathbb{N}_0$ for all $d \in D$, and
3. $c_d = d$ for all $d \in D$.

A partition of n is a function $f : \mathbb{N}_1 \to \mathbb{N}_0$ such that

$$n = \sum_{d=1}^{\infty} d f(d).$$

We find now that $p(n)$, the number of partitions of n, is

$$\begin{aligned} p(n) &= C_{x^n}\ (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots \\ &= C_{x^n}\ \frac{1}{(1-x)(1-x^2)(1-x^3)\dots} \end{aligned}$$

For example, the partition $5 = 1 + 1 + 3$ corresponds to a term x^5 that you get by multiplying x^2 from the first factor, 1 from the second factor, x^3 from the third factor, and 1's from all other factors.

Some variations on the same theme.

We are interested in partitions into *odd* parts. For example, 7 can be partitioned into

$$7 = \begin{cases} 7 \\ 1 + 1 + 5 \\ 1 + 1 + 1 + 1 + 3 \\ 1 + 3 + 3 \\ 1 + 1 + 1 + 1 + 1 + 1 + 1 \end{cases}$$

if only odd summands can be used.

In that case we find:

$$\sum_{n=0}^{\infty} q(n) x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \dots$$

Or, we could be interested in $r(n)$, the number of partitions of n in *different* parts. Then we have that $S_d = \{0, 1\}$ for all $d \in D$, and the rest the same as before, and then we find

$$\sum_{n=0}^{\infty} r(n) x^n = (1 + x)(1 + x^2)(1 + x^3)\dots =$$
$$\frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \dots$$

All factors in the numerator and denominator of the type $1 - x^{2k}$ cancel, and surprisingly, we find that

$$r(n) = q(n).$$

For example, 7 has also exactly 5 partitions without repetitions:

$$7; \quad 6+1; \quad 5+2; \quad 3+4; \quad \text{and} \quad 1+2+4.$$

A combinatorial explanation of this phenomenon is the following. Each summand $2^m k$ with odd k in a partition without repetitions corresponds to 2^m summands with odd k. In the case of 7 we get:

$$7; \quad (3+3)+1; \quad 5+(1+1); \quad 3+(1+1+1+1); \quad 1+(1+1)+(1+1+1+1).$$

This correspondence is evidently a bijection; we leave it as an exercise to check the other direction.

For more examples we refer to Pólya and Szegö.[3]

Ordered partitions.

$$3 = \begin{cases} 3 \\ 2+1 \\ 1+2 \\ 1+1+1 \end{cases} \qquad\qquad 4 = \begin{cases} 4 \\ 3+1 \\ 1+3 \\ 2+2 \\ 2+1+1 \\ 1+2+1 \\ 1+1+2 \\ 1+1+1+1. \end{cases}$$

To deal with these we first look at the generating function of the number of ordered partitions with k summands. This is of course

$$(x + x^2 + x^3 + \ldots)^k = \left(\frac{x}{1-x}\right)^k.$$

Summation over k gives

$$\frac{x}{1-2x} = x + 2x^2 + 4x^3 + 8x^4 + \ldots.$$

In other words, the number of ordered partitions of n is 2^{n-1}. This follows easily by induction, and a combinatorial argument goes as follows. Here are the numbers 1 up to n:

$$1, 2, 3 \mid 4, 5, 6, 7 \mid \ \ldots \mid n-2, n-1 \mid n$$

An ordered partition corresponds to a way to put bars between numbers. For example, the way illustrated above corresponds to the partition

[3] G. Pólya und G. Szëgo, [Erster Abschnitt, 1. Kapitel, §1, Aufgaben 1–31.] *Aufgaben und Lehrsätze aus der Analysis I*, Springer-Verlag, Heidelberger Taschenbücher, 1925.

$$n = 3 + 4 + \ldots + 2 + 1.$$

Every ordered partition corresponds to the sequence of the number of elements between consecutive bars. There are $n - 1$ possible positions for the bars.

In this way we can interpret many combinatorial identities. There are lots of them. See for example Riordan's book [38].

One way to derive combinatorial identities is by means of substitution. Let f be a generating function which is a polynomial. Obviously, we have

$$\sum_{k=0}^{\infty} C_{x^k}\ f(x) = f(1).$$

For example, $f(x) = (1 + x)^n$ is the generating function for the number of ways to choose k elements from a set of n elements. This gives:

$$\sum_{k=0}^{\infty} \binom{n}{k} = 2^n.$$

Another way is by means of combinatorial interpretation. The following formula

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

can be understood as follows. From a set of n red and m white marbles, we choose k. The number of different possibilities is to choose j white marbles and $k - j$ red marbles, summed over j.

2.6 Notes

1. The article of de Bruijn that counts the number of different logics is:
   N.G. de Bruijn, Exact finite models for minimal propositional calculus over a finite alphabet. Technical report Eindhoven University T.H. 75, Wsk 02, 1975.
   The number of equivalence classes of formulas with letters a, b, and c, $\Rightarrow$ and $\wedge$ is 623.662.965.552.330. The title of the other article is
   Piotr S. Krystek, On the free relatively pseudo complemented semi-lattice with three generators, Report on Mathematical Logic 9 (1977), pp. 31–38.
2. Pierre Simon Laplace (1749–1827), author of Traité de mécanique céleste (1796), and Théorie analytique des probabilités (1812), and many other publications. The generating functions are in the Théorie.
   Well-known sayings of Laplace's were: “Il est aisé à voir” when he left out a lengthy proof; his last words “What we know is not so much; what we do

not know is immense;" and his remark to Napoleon "Sire, I had no need of that hypothesis," in answer to Napoleon's remark: "M. Laplace, they tell me that you have written this large book on the system of the universe, and have never even mentioned its Creator."

3. Fibonacci (Leonardo of Pisa, ca 1180–1250), describes in his "Liber Abaci" (1202) the problem of the total rabbit population. When you start with one pair of rabbits, that every other month produces two young rabbits, but that starts producing only after 4 months, then the number of pairs after every other month forms exactly the Fibonacci sequence. By the way, François Edouard Anatole Lucas (1842–1891) first called it the Fibonacci sequence.
4. Claude Gaspard Bachet de Méziriac (1581–1638). Problèmes plaisans et délectables qui se font par les nombres, 1612 (enlarged edition in 1624). Bachet observed that apparently every positive number can be expressed as a sum of at most four squares; he said that he checked it for more than 300 numbers but couldn't prove it. Lagrange proved it much later.

2.7 Problems

2.1. We want to dissect a convex $(n+2)$-sided polygon into triangles by connecting points with non-intersecting diagonals. Let D_n be the number of possible dissections. Let $D_0 = 1$. The following table gives the first few values:

n	0	1	2	3	4	5	6	7	8	9	10
D_n	1	1	2	5	14	42	132	429	1430	4862	16796

(a) Prove that

$$D_{n+1} = D_0 D_n + D_1 D_{n-1} + \ldots + D_k D_{n-k} + \ldots + D_n D_0.$$

(b) Find a closed form for the generating function

$$D(x) = D_0 + D_1 x + D_2 x^2 + D_3 x^3 + \ldots$$

(c) Prove that

$$D_n = \frac{1}{n+1}\binom{2n}{n}.$$

2.2. Assume that

$$\sum_{n=0}^{\infty} a_n x^n = \exp\left(\sum_{k=1}^{\infty} b_k x^k\right).$$

Then $a_0 = 1$ and

$$a_n = \frac{1}{n}\sum_{k=1}^{n} k b_k a_{n-k} \quad \text{for } n \geqslant 1.$$

2.3. Consider the Fibonacci sequence

$$a_0 = 1; \quad a_1 = 1; \quad \text{and} \quad a_{n+1} = a_n + a_{n-1} \quad \text{for } n \in \mathbb{N}_1.$$

(a) Show that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} a_2 & a_1 \\ a_1 & a_0 \end{pmatrix},$$

(b) and prove by induction that:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}.$$

(c) Compare the determinants to show that

$$a_{n+1}a_{n-1} - a_n^2 = (-1)^{n+1} \quad \text{for } n \in \mathbb{N}_1.$$

2.4. Let u_n be the number of ways you can climb a staircase with n steps if at each point you can either take one or two steps. Let $u_0 = 1$. Show that u_n is the n^{th} Fibonacci number. Compare this with the surprise we got in Example 2.7.

2.5 (Riordan[4]). Let $a_0 = 1$, $b_0 = 0$, and let

$$a_n = \sum_{k=0}^{n} \binom{n+k}{2k} \quad \text{and} \quad b_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1}, \quad \text{for } n = 1, 2, \ldots.$$

(a) Show that

$$a_{n+1} = a_n + b_{n+1} \quad \text{and} \quad b_{n+1} = a_n + b_n \quad \text{for } n = 0, 1, \ldots.$$

(b) The first few values are:

n	a_n	b_n
0	1	0
1	2	1
2	5	3
3	13	8
4	34	21
5	89	55

(c) Note that

$$a_n = F_{2n+1} \quad \text{and} \quad b_n = F_{2n}, \quad \text{for } n = 0, 1, \ldots$$

where F_n is the n^{th} Fibonacci number with $F_0 = 0$, $F_1 = 1$, etc.

[4] J. Riordan, Generating functions. In (E. F. Beckenbach ed.) *Applied Combinatorial Mathematics*, John Wiley & Sons 1964.

(d) Let

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

and let

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad b(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then

$$a(x^2) = \frac{1}{2x}\left(F(x) - F(-x)\right) = \frac{1-x^2}{1-3x^2+x^4}$$
$$b(x^2) = \frac{1}{2}\left(F(x) + F(-x)\right) = \frac{x^2}{1-3x^2+x^4}.$$

(e) Deduce also

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \quad \text{for } n > 0.$$

2.6. The Catalan numbers can be defined by the following recurrence:

$$c_0 = 1 \quad \text{and} \quad c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}.$$

Consider the generating function $C(x) = \sum_{n=0}^{\infty} c_n x^n$.

(a) Prove that

$$C(x) = 1 + xC^2(x) \quad \Rightarrow C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

(b) Show that this implies

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

(c) Compare this with the result of Exercise 2.1.

2.7. Let $p(n,k)$ be the number of partitions of n into exactly k parts. Since

$$5 = \begin{cases} 5 \\ 1+4 \\ 2+3 \\ 1+1+3 \\ 1+2+2 \\ 1+1+1+2 \\ 1+1+1+1+1 \end{cases}$$

we have that

$$p(5,1) = 1;\ p(5,2) = 2;\ p(5,3) = 2;\ p(5,4) = 1;\ \text{and } p(5,5) = 1.$$

(a) Show that

$$p(n, 1) = p(n, n) = 1 \quad \text{and} \quad \sum_{k=1}^{m} p(n, k) = p(n + m, m).$$

(b) Let $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition with $\lambda_1 \geqslant \ldots \geqslant \lambda_p$. A Young diagram (or Ferrers graph, or Ferrers diagram) of such a partition is an arrangement of dots, in p left-justified rows with λ_i dots in the i^{th} row. By considering the conjugate of such a diagram, prove that $p(n, m)$ is the number of partitions of n into parts of which the largest is m.

2.8 (de Bruijn[5]). We want to distribute m counters over three persons P_1, P_2, and P_3, with the condition that P_1 and P_2 obtain the same number. So we need functions $f : \{P_1, P_2, P_3\} \to \mathbb{N}_0$ such that $f(P_1) = f(P_2)$ and such that $f(P_1) + f(P_2) + f(P_3) = m$.

Show that the number that we are seeking is

$$\begin{aligned}
C_{x^m} & \quad (1 + x^2 + x^4 + \ldots) \cdot (1 + x + x^2 + \ldots) = \\
C_{x^m} & \quad \frac{1}{(1 - x^2)(1 - x)} = \\
C_{x^m} & \quad \frac{1}{4} \cdot \frac{1}{1 + x} + \frac{1}{2} \cdot \frac{1}{(1 - x)^2} + \frac{1}{4} \cdot \frac{1}{1 - x} = \\
& = \frac{1}{2}(m + 1) + \frac{1}{4}\left(1 + (-1)^m\right).
\end{aligned}$$

[5] N. G. de Bruijn, Pólya's theory of counting. In (E. F. Beckenbach ed.) *Applied Combinatorial Mathematics*, Wiley 1964.

3

Permutations

To count sheep, you count the legs and then you divide by 4.[1]

3.1 The shepherd's principle

Theorem 3.1. *Let W and V be two sets and let $k \in \mathbb{N}_1$. Let f be a surjection $W \to V$ with the following property*

$$\forall_{v \in V} \;\; |f^{\leftarrow}(v)| = k.$$

Then $W = k|V|$.

The interpretation is that W is the set of legs and V is the set of sheep.

Theorem 3.2. *Let W and V be two sets and $f : W \to V$, Then*

$$|f(W)| = \sum_{w \in W} \frac{1}{|f^{\leftarrow}(f(w))|}$$

Proof.

$$\sum_{w \in W} \frac{1}{|f^{\leftarrow}(f(w))|} = \sum_{v \in f(W)} \sum_{w \in f^{\leftarrow}(v)} \frac{1}{|f^{\leftarrow}(f(w))|} =$$
$$\sum_{v \in f(W)} \sum_{w \in f^{\leftarrow}(v)} \frac{1}{|f^{\leftarrow}(v)|} = \sum_{v \in f(W)} 1 = |f(W)|.$$

□

Example 3.3. Consider a function $f : W \to V$ as in this figure. The upper three points get a weight of $\frac{1}{3}$ and the lower two points get a weight of $\frac{1}{2}$. The sum of all weights in W is now 2, namely the number of elements of $f(W)$.

[1] We assume that there are no sheep with 5 legs, nor other irregularities.

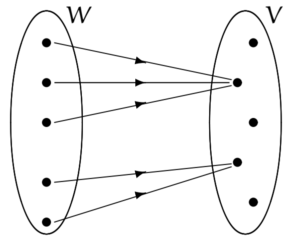

Fig. 3.1. Example of a function $f : W \to V$

Remark 3.4. The ordinary way to count the number of subsets of k elements in a set of n elements is also based on the shepherd's principle: first you count ordered subsets and then divide by $k!$.

3.2 Permutations

Example 3.5. Let $S = \{1, 2, \ldots, 7\}$. We call a bijective mapping $\pi : S \to S$ a permutation of S. We denote a permutation by writing the image of each argument under that argument. For example

$$\pi : \quad \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 6\,1\,4\,3\,5\,2\,7 \end{pmatrix}, \quad \text{thus} \quad \pi(1) = 6.$$

Remark 3.6. In the old days people used to call the image $(6, 1, 4, 3, 5, 2, 7)$ the permutation, but nowadays a permutation is a mapping.

We could depict the permutation above also as a diagram.

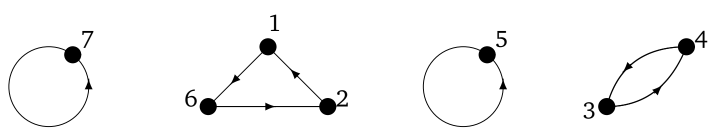

Fig. 3.2. Diagram of a permutation

A cycle-notation is the following:

$$(1, 6, 2)(3, 4)(5)(7).$$

The ordering of the cycles does not matter. Also the starting points of the cycles does not matter: $(1, 6, 2) = (6, 2, 1) = (2, 1, 6)$.

For any $k \in \mathbb{N}_1$ we denote the number of cycles of a permutation π by $b_k(\pi)$. The *type* of π is the sequence $(b_k(\pi))_{k \in \mathbb{N}_1}$. For example, the type of the permutation π above is $(2, 1, 1, 0, 0, \ldots)$.

The degree of π is the number of permuted objects, which is $\sum k b_k(\pi)$.

We introduce an equivalence relation between permutations as follows:

$$\sigma_1 \sim \sigma_2 \quad \Leftrightarrow \quad \exists_\pi \;\; \pi\sigma_1\pi^{-1} = \sigma_2.$$

Equivalent permutations have the same type (the same b-vector). The best way to see this is by looking at an example. Let

$$\sigma = (1,3)(2,4,5)(6)(7)$$

be a permutation and let ρ be an arbitrary permutation of $\{1,2,\ldots,7\}$. Then σ is equivalent to

$$\tau = (\rho_1, \rho_3)(\rho_2, \rho_4, \rho_5)\rho_6)(\rho_7)$$

Why? Well, just look; $\tau(\rho_4) = \rho_5$ and

$$\rho\sigma\rho^{-1}(\rho_4) = \rho\sigma(4) = \rho(5)$$

etc. So indeed

$$\rho\sigma\rho^{-1} = \tau.$$

We can also see that σ is equivalent to the original π; just take

$$\rho : \quad \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 4\,1\,3\,6\,2\,5\,7 \end{pmatrix}$$

In the same way you can show that with any pair of permutations with the same cycle-pattern, you can find a ρ that makes them equivalent.

Theorem 3.7.

$$\sigma_1 \sim \sigma_2 \quad \Leftrightarrow \quad \textit{the type of } \sigma_1 \quad = \quad \textit{the type of } \sigma_2.$$

Question.

What is the number of permutations of degree n and of a type $(b_1, b_2, \ldots)$?

We make use of the shepherd's principle to answer this question.

The idea is that you think of the type as a pattern of brackets and dots; that is the cycle notation without the numbers. For the type π above the pattern looks like

$$(\cdot,\cdot,\cdot)(\cdot,\cdot)(\cdot)(\cdot)$$

A permutation of type $(b_1, b_2, \ldots)$ is one sheep. The legs of the sheep are the ways to write that permutation in the pattern of brackets and dots.

The set of all permutations consist of $n!$ elements. That is the number of ways to fill in the dots. That fixes the sheep.

Every sheep has

$$b_1!1^{b_1} \cdot b_2!2^{b_2} \cdot b_3!3^{b_3} \cdot \ldots$$

legs. Namely, the b_k k-cycles occur in an arbitrary order, and there are k possibilities to start any of the b_k k-cycles.

Conclusion: there are

$$\frac{n!}{\prod_{k=1}^{\infty} b_k!k^{b_k}} \quad \text{permutations of type } (b_1, b_2, \ldots). \tag{3.1}$$

Let's see if it checks.

$$\sum_{(b_1,b_2,\ldots) \text{ is a type of degree } n} \frac{1}{b_1!1^{b_1} b_2!2^{b_2} \ldots} = 1 \quad \text{and} \quad x^n = x^{b_1+2b_2+\ldots}$$

$$\begin{aligned}
\text{and so} \quad \sum_{n=0}^{\infty} x^n &= \sum_{(b_1,b_2,\ldots)} \frac{1}{b_1!}\left(\frac{x}{1}\right)^{b_1} \cdot \frac{1}{b_2!}\left(\frac{x^2}{2}\right)^{b_2} \cdot \ldots \\
\frac{1}{1-x} &= \sum_{b_1=0}^{\infty} \frac{1}{b_1!}\left(\frac{x}{1}\right)^{b_1} \sum_{b_2=0}^{\infty} \frac{1}{b_2!}\left(\frac{x^2}{2}\right)^{b_2} \sum_{b_3=0}^{\infty} \ldots \\
&= e^x \cdot e^{\frac{1}{2}x^2} \cdot e^{\frac{1}{3}x^3} \cdot \ldots = e^{-\ln(1-x)}.
\end{aligned}$$

Of course we knew that already, but now we have proven it also for formal power series. It may seem silly but in the end it is the easiest way to do it.

Remark 3.8. And, anyway, how do you show that you get the series $1 + x + x^2 + \ldots$ by substituting $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ in $1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$? That is not so easy!

A permutation is even, or odd, whenever $\sum_{k=1}^{\infty} b_{2k}$ is even or odd.

We count the number of even permutations minus the number of odd permutations of degree n. Let $d(n)$ be this number. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{d(n)}{n!} x^n &= \sum_{(b_1,b_2,\ldots)} (-1)^{b_2+b_4+b_6+\ldots} \frac{x^{b_1+2b_2+\ldots}}{b_1!1^{b_1} b_2!2^{b_2} \ldots} = \\
&= \quad \text{similar calculations} \quad = \\
&= e^{\ln(1+x)} = 1 + x.
\end{aligned}$$

In other words

$$d(0) = 1; \quad d(1) = 1; \quad \text{and} \quad d(n) = 0 \quad \text{for all } n > 1.$$

Remark 3.9. About $d(0)$ etc. We have taken the type $(0, 0, \ldots)$ into consideration and assumed implicitly that there is exactly one permutation of zero objects. Whether this assumption is correct depends on the definition of a mapping (is $\varnothing \to \varnothing$ also a mapping?) that we used. Let's just be satisfied with the fact that it agrees with the convention that $0! = 1$.

3.3 Stirling numbers

Let us continue with $Q_k(n)$, the number of permutations of $\{1, \ldots, n\}$ with k cycles. The calculations are as follows.

$$\begin{aligned}
\sum_{k,n\in\mathbb{N}_0} \frac{Q_k(n)}{n!} x^n y^k &= \sum_{(b_1,b_2,\ldots)\text{ is a type}} \frac{x^{b_1+2b_2+3b_3+\ldots} y^{b_1+b_2+\ldots}}{b_1!1^{b_1}\cdot b_2!2^{b_2}\cdot b_3!3^{b_3}\cdot\ldots} = \\
&= \sum_{(b_1,\ldots)\text{ a type}} \prod_j \frac{1}{b_j!}\left(\frac{x^j y}{j}\right)^{b_j} = \\
&= \sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\sum_{b_3=0}^{\infty}\cdots\prod_j \frac{1}{b_j!}\left(\frac{x^j y}{j}\right)^{b_j} = \\
&= \prod_{j\geqslant 1}\sum_{b_j=0}^{\infty}\frac{1}{b_j!}\left(\frac{x^j y}{j}\right)^{b_j} = \prod_{j\geqslant 1}\exp\left(\frac{x^j y}{j}\right) = \\
&= \exp\left(xy + \frac{x^2 y}{2} + \ldots\right) = e^{-y\ln(1-x)} = \\
&= (1-x)^{-y} = \sum_{n=0}^{\infty}\binom{-y}{n}(-x)^n = \\
&= \sum_{n=0}^{\infty}\frac{y(y+1)(y+2)\ldots(y+n-1)}{n!}x^n.
\end{aligned}$$

We now define

$$\mathfrak{S}(n,k) = (-1)^{n+k}Q_k(n).$$

The numbers $\mathfrak{S}(n,k)$ are the Stirling numbers of the first kind. Thus

$$\mathfrak{S}(n,k) = n!\ C_{y^k}\binom{y}{n}.$$

See also the book of Comtet (Chapter 5 and 6) and the book of Riordan.

Now it is easy to find the *average* number of cycles, given n. We get that by calculating

$$\sum_{k,n\in\mathbb{N}_0}\frac{kQ_k(n)}{n!}x^n y^k$$

for $y = 1$.

This is the same as

$$\left[\sum_{k,n\in\mathbb{N}_0,k\geqslant 1}\frac{1}{n!}kQ_k(n)x^n y^{k-1}\right]_{y=1} =$$

$$\left[\frac{d}{dy}(1-x)^{-y}\right]_{y=1} =$$

$$\left[-\ln(1-x)\cdot(1-x)^{-y}\right]_{y=1} = -\frac{\ln(1-x)}{1-x} =$$

$$(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots)(1+x+x^2+\ldots).$$

The coefficient of x^n in this series is $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

Intermezzo

You will later appreciate what we do next, although right now, maybe it doesn't seem to make much sense.

Consider a permutation of type $(b_1, b_2, \ldots)$. We define a weight, which is the monomial

$$x_1^{b_1}x_2^{b_2}x_3^{b_3}\ldots$$

with variables $x_1, x_2, x_3, \ldots$. (NB $x_i^0 = 1$ for all i.)

We sum all weights of all permutations, first for fixed n:

$$\sum_{\substack{b_1,b_2,\cdots\geqslant 0\\ b_1+2b_2+\ldots=n}}\frac{n!}{b_1!1^{b_1}b_2!2^{b_2}\ldots}x_1^{b_1}x_2^{b_2}x_3^{b_3}\ldots$$

and we can also write this as

$$C_{x^n}\ n!\sum_{b_1=0}^{\infty}\sum_{b_2=0}^{\infty}\sum_{b_3=0}^{\infty}\ldots\frac{x_1^{b_1}x_2^{b_2}\ldots}{b_1!1^{b_1}\cdot b_2!2^{b_2}\cdot b_3!3^{b_3}\ldots}x^{b_1+2b_2+\ldots}$$

which gets rid of the extra condition.

We can calculate this jumble, and we get:

$$C_{x^n}\ n!\prod_{j=1}^{\infty}\sum_{b_j=0}^{\infty}\frac{x_j^{b_j}x^{jb_j}}{b_j!j^{b_j}} = \tag{3.2}$$

$$C_{x^n}\ n!\prod_{j=1}^{\infty}\sum_{b_j=0}^{\infty}\frac{1}{b_j!}\left(\frac{x_jx^j}{j}\right)^{b_j} = \tag{3.3}$$

$$C_{x^n}\ n!\exp(\frac{x_1x}{1}+\frac{x_2x^2}{2}+\frac{x_3x^3}{3}+\ldots). \tag{3.4}$$

We come back to this later.

3.4 Partitions of sets

We are going to discuss partitions of sets. Notice that these are *not* the partitions of numbers that we discussed earlier. In a while we will also talk about numbered partitions, and these are once more something completely different. A set S is cut into pieces. If S is finite, then we consider the type of that

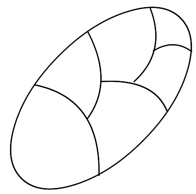

Fig. 3.3. Partition of a set

partition, which is $(b_1, b_2, b_3, \ldots)$, where b_i is the number of pieces of size i. For example, the type of the partition illustrated in Figure 3.4 is

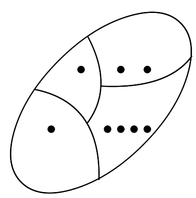

Fig. 3.4. Partition of a set into pieces

$$(2, 1, 0, 1, 0, 0, \ldots).$$

We want to count partitions, and we want to count them per type. Assume that we want to know how many partitions there are of type $(2, 3, 1, 0, 0, 0, \ldots)$ of a set with 11 elements. Let's use the idea of the pattern with brackets and dots:

$$(\cdot)(\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot\cdot)$$

Distribute the 11 elements among the dots. You can do that in 11! ways. This fixes the sheep. The number of ways you can represent the partition in this pattern is

$$\underbrace{2!}_{\substack{\text{\# of ways to}\\ \text{arrange boxes}\\ \text{with \textit{one} dot.}}} \times \quad 3!\underbrace{(2!)^3}_{\substack{\text{\# of ways to}\\ \text{arrange two}\\ \text{elements in}\\ \text{each of the 3}\\ \text{boxes with 2}\\ \text{dots.}}} \times \quad 1!(3!)^1 \quad (\text{\# of legs per sheep}).$$

Thus the total number of partitions of 11 elements with type $(2, 3, 1, 0, 0, \dots)$ is

$$\frac{11!}{2!(1!)^3 \cdot 3!(2!)^3 \cdot 1!(3!)^1} = 69300.$$

In general, with n elements and type $(b_1, b_2, \dots)$ the number becomes

$$\frac{n!}{b_1!(1!)^{b_1} \cdot b_2!(2!)^{b_2} \cdot b_3!(3!)^{b_3} \dots}$$

Note the difference with the formula that we found for the number of permutations of that type.

Let $P(n)$ be the number of partitions of a set with n elements. When we calculate $\sum_{n=0}^{\infty} \frac{P(n)}{n!} x^n$ as in Section 3.2 we find for the generating function:

$$\sum_{n=0}^{\infty} \frac{P(n)}{n!} x^n = \exp\left(\frac{x}{1!} + \frac{x^2}{2!} \dots\right) = \exp(e^x - 1).$$

Of course there is the usual dilemma with the case $n = 0$: how many partitions does a set with 0 elements have? Let's not worry about it.

We can obtain the same result also in another way, namely with recurrence relations. Let $n > 0$. Choose a point $a \in S$. That is in some part with j other

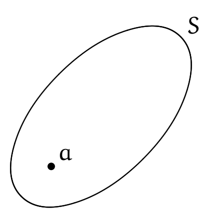

Fig. 3.5. Set S with fixed element a

elements, for some $j \geqslant 0$. We can choose that piece in $\binom{n-1}{j}$ ways, and then it turns out that

$$P(n) = \sum_{j=0}^{n-1} \binom{n-1}{j} P(n-1-j)$$

and in case $j = n-1$ we should apparently choose $P(0) = 1$. So

$$\frac{P(n+1)}{n!} x^n = \sum_{j=0}^{n} \frac{P(n-j)}{(n-j)!} x^{n-j} \frac{x^j}{j!}$$
$$\Rightarrow \sum_{n \geqslant 0} \frac{P(n+1)}{n!} x^n = e^x \sum_{n \geqslant 0} P(n) \frac{x^n}{n!}.$$

If we now let

$$F(x) = \sum_{n \geqslant 0} P(n) \frac{x^n}{n!},$$

then we obtain

$$F'(x) = e^x F(x) \quad \text{which implies} \quad F(x) = C \exp(e^x).$$

We can determine C by substituting $x = 0$. We find that $P(0) = C \exp(e^0) = Ce$, and since we chose $P(0) = 1$ this implies that

$$F(x) = \exp(e^x - 1)$$

as before.

3.5 Arrangements of multisets

We already know what a multiset is: Let S be a set, then a mapping $f : S \to \mathbb{N}_0$ is a multiset, and we also call f the frequency-function of that multiset. We now want to formulate in some way or other that the *order* matters.

Example 3.10. Let $S = \{a, b, c\}$. (NB letters are a nuisance as elements of a set. To get around that you can assume that $a \neq b$, $b \neq c$, and $a \neq c$, and then a, b, and c can represent either variables or elements, as you like.)

A frequency function is, for example

$$\begin{array}{ccc} a & b & c \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{array}$$

A word that contains the letters with that frequency is for example

$$a\ b\ a\ c\ b\ b$$

How many words are there that fit the frequency function f?

You can use the shepherd's theorem for counting that. There are 6 letters (say a_1, a_2, b_1, b_2, b_3, c_1) and these can be ordered in 6! ways into a word. If we rub away the indices then we get 2!3!1! times the same word. Thus there are

$$\frac{6!}{1! \cdot 2! \cdot 3!} = 60$$

words that fit the frequency-function f.

This easily generalizes. Let D be an alphabet and let $n \in \mathbb{N}_1$. A word over D is a mapping $\phi : \{1, \ldots, n\} \to D$. The type of a word, or the frequency-function, is $f : D \to \mathbb{N}_0$ defined by

$$\forall_{d \in D} \quad f(d) = |\phi^{\leftarrow}(d)|$$

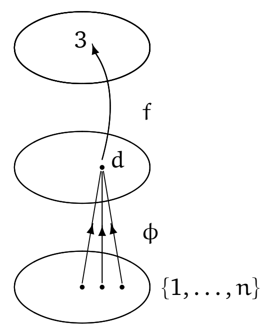

Fig. 3.6. Mappings f and ϕ

We now have the following theorem which, by itself is not of major importance, but which is funny because it looks the same as Theorem 2.5. Let

$$C = \{ f \in \mathbb{N}_0^D \mid \forall_{d \in D} \quad f(d) \in S_d \}$$

and let $A(n)$ be the number of n-letter words with $f \in C$. Then

$$\sum_{n=0}^{\infty} \frac{A(n)}{n!} x^n = \prod_{d \in D} \sum_{r \in S_d} \frac{x^r}{r!}.$$

To see that this is correct, simply observe that when d_1, d_2, and d_3 have terms $\frac{x^{r_1}}{r_1!}$, $\frac{x^{r_2}}{r_2!}$, and $\frac{x^{r_3}}{r_3!}$ in the sum, then the term $\frac{x^{r_1}}{r_1!} \cdot \frac{x^{r_2}}{r_2!} \cdot \frac{x^{r_3}}{r_3!}$ in the product corresponds with a word of $r_1 + r_2 + r_3$ letters, namely r_1 d_1's, r_2 d_2's, and r_3 d_3's.

Let's do one more example.

3.6 Numbered partitions

The example is a bit fabricated; we don't look at ordinary partitions, but at partitions in which the pieces are numbered.

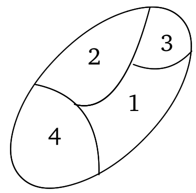

Fig. 3.7. Set S with a numbered partition

We will look at partitions into k pieces, numbered 1 up to k. Furthermore, we assume that the set has n elements.

We look at the n elements as the positions in a word, and we fill the word with letters $1, \ldots, k$. For example

$$\begin{array}{cccccccccc} 2 & 1 & 2 & 1 & 3 & 2 & 4 & 3 & 4 & 4 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

The "letter" indicates the piece in which the element is contained. We let

$$D = \{1, \ldots, k\} \quad \text{and} \quad S_d = \mathbb{N}_1$$

and we find (we leave it as an exercise to figure out R, f, and C in the formula of Theorem 2.5):

$$\sum_{n=0}^{\infty} \frac{A_k(n)}{n!} x^n = \left(\sum_{r \in \mathbb{N}_1} \frac{x^r}{r!} \right)^k = (e^x - 1)^k .$$

Thus the total number of numbered partitions of $\{1, \ldots, n\}$ is just what we get when we sum this on k, namely:

$$\sum_{n=0}^{\infty} \frac{A(n)}{n!} x^n = \sum_{k=1}^{\infty} (e^x - 1)^k = \frac{1}{2 - e^x} - 1$$

that is, if we take $A(0) = 0$; if $A(0) = 1$, then the term -1 vanishes.

We can also get this with a completely different approach.

Let the piece with the highest number have j elements, $j \geqslant 1$. There are $\binom{n}{j}$ ways to choose this piece, and there are $A(n-j)$ ways to partition the rest of the set in numbered pieces. So (summing over j using the convention that $A(0) = 1$):

$$A(n) = \sum_{j=1}^{n} A(n-j)\binom{n}{j} \quad \Rightarrow$$

$$2\frac{A(n)}{n!} = \sum_{j=0}^{n} \frac{A(n-j)}{(n-j)!j!}, \quad n \geqslant 1 \quad \Rightarrow$$

$$2\sum_{n=0}^{\infty} \frac{A(n)}{n!}x^n - 1 = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{A(n-j)}{(n-j)!}x^{n-j}\frac{x^j}{j!} \quad \left\{\text{The term with } n = 0 \text{ is now also OK: } 2A(0)x^0 - 1 = \tfrac{A(0)}{0!}x^0\tfrac{x^0}{0!}.\right.$$

If we now let $F(x) = \sum_{n=0}^{\infty} \frac{A(n)}{n!}x^n$, then we get

$$F(x)e^x = 2F(x) - 1 \quad \text{and so} \quad F(x) = \frac{1}{2 - e^x}$$

which is the same formula as above.

We close this topic.

3.7 Problems

3.1. Consider an $n \times n$-grid of points labeled (i, j) with $0 \leqslant i, j \leqslant n$. Pairs (i, j) and (k, ℓ) with $\{i, j\} \neq \{k, \ell\}$ are connected by a line if and only if either

1. $i = k$ and $|j - \ell| = 1$, or
2. $j = \ell$ and $|i - k| = 1$.

Suppose you want to walk in this grid from point $(0, 0)$ to point (n, n) along the lines, where you only follow lines that either go upwards or to the right. How many different routes are there?

3.2. A derangement is a permutation with $b_1 = 0$; thus it has no 1-cycles. Let $d(n)$ be the number of derangements of $\{1, \ldots, n\}$.

(a) Show that

$$n! = \sum_{k=0}^{n} \binom{n}{k} d(n-k) = \sum_{k=0}^{n} \binom{n}{k} d(k).$$

(b) Let

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{and} \quad G(x) = \sum_{n=0}^{\infty} \frac{d(n)}{n!}x^n.$$

Prove that $F(x) = e^x G(x)$, thus

$$G(x) = \frac{e^{-x}}{1-x}.$$

(c) By the way, you can derive the same result also directly, from Formula 3.1. It goes like this:

$$\begin{aligned}
G(x) &= \sum_{b_2,\ldots} \frac{x^{2b_2} + x^{3b_3} + \ldots}{b_2!2^{b_2} b_3!3^{b_3}\ldots} \\
&= \sum_{b_2=0}^{\infty} \sum_{b_3=0}^{\infty} \cdots \prod_{j>1} \frac{1}{b_j!}\left(\frac{x^j}{j}\right)^{b_j} \\
&= \prod_{j>1} \sum_{b_j=0}^{\infty} \frac{1}{b_j!}\left(\frac{x^j}{j}\right)^{b_j} \\
&= \prod_{j>1} \exp\left(\frac{x^j}{j}\right) \\
&= \exp\left(\frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) \\
&= e^{-x}\exp(-\ln(1-x)) \\
&= \frac{e^{-x}}{1-x}.
\end{aligned}$$

(d) Show that this implies

$$\frac{d(n)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

3.3. The Stirling numbers of the second kind, $\mathfrak{S}_2(n,k)$ are the number of partitions of an n-set with k nonempty subsets.

(a) Prove that

$$\mathfrak{S}_2(n,2) = 2^{n-1} - 1 \quad \text{and} \quad \mathfrak{S}_2(n,n-1) = \binom{n}{2}.$$

(b) Prove that

$$\mathfrak{S}_2(n,k) = k\mathfrak{S}_2(n-1,k) + \mathfrak{S}_2(n-1,k-1).$$

3.4. Show that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \quad \Rightarrow \quad b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k.$$

(a) The Stirling number of the second kind, $\mathfrak{S}_2(z,k)$ is the number of partition of a z-set into k nonempty subsets. Show that

$$n^z = \sum_{k=0}^{n} \binom{n}{k} k!\mathfrak{S}_2(z,k) \quad \Rightarrow \quad n!\mathfrak{S}_2(z,n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^z.$$

(b) Deduce

$$\begin{aligned}\sum_{z=0}^{\infty} \mathfrak{S}_2(z,n)\frac{t^z}{z!} &= \sum_{z=0}^{\infty} \frac{t^z}{z!} \cdot \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^z = \\ &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e^{tk} = \\ &= \frac{1}{n!} \left(e^t - 1\right)^n .\end{aligned}$$

3.5. Let p be prime.

(a) Show that $\binom{p}{k}$ is divisible by p for every $0 < k < p$.
(b) Show that $2^p - 2$ is a multiple of p.

3.6. Let $P(n)$ be the number of permutations π of $\{1, \dots, n\}$ with the property that $\pi^2 = \mathrm{id}$.

(a) Show that

$$P(0) = P(1) = 1 \quad \text{and} \quad P(n) = P(n-1) + (n-1)P(n-2), \quad \text{for } n \geqslant 2.$$

(b) Use Formula 3.1 on page 26 to show that

$$P(n) = n! \sum_{k+2\ell=n} \frac{1}{k!\ell!2^\ell}, \quad \text{for } n \geqslant 0.$$

(c) Deduce that the exponential generating function satisfies

$$\sum_{n=0}^{\infty} \frac{P(n)}{n!} x^n = \exp\left(x + \frac{x^2}{2}\right).$$

3.7. Prove that

$$\mathfrak{S}(n,k) = \mathfrak{S}(n-1,k-1) - (n-1)\mathfrak{S}(n-1,k).$$

3.8. Suppose we want to paint n houses with k colors such that each house is painted by a single color and such that each color is used. Find a formula that expresses in how many ways you can do that.

3.9. Let f_n be the number of functions $F : \{1, \dots, n\} \to \{1, \dots, n\}$ with the property that if F takes on the value k then F takes on all values ℓ for which $1 \leqslant \ell \leqslant k$. Let $f_0 = 1$.

(a) If $|F^{\leftarrow}(1)| = k$ then there are $\binom{n}{k}$ ways to choose $F^{\leftarrow}(1)$ and the rest of the function can be defined in f_{n-k} ways. This shows that

$$f_n = \sum_{k=1}^{n} \binom{n}{k} f_{n-k}. \tag{3.5}$$

(b) Recall the number of numbered partitions $A(n)$ of a set with n elements of Section 3.6, and note that $A(n)$ satisfies the same recurrence 3.5. Find a combinatorial explanation.

(c) Work out that the exponential generation function

$$A(x) = \sum_{n=0}^{\infty} \frac{f_n}{n!} x^n$$

satisfies

$$\begin{aligned} A(x) &= \frac{1}{2 - e^x} = \frac{1}{2} \frac{1}{1 - e^x/2} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{e^x}{2} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \frac{n^k}{k!} x^k. \end{aligned}$$

(d) Thus

$$f_k = k! C_{x^k} A(x) = \sum_{n=0}^{\infty} \frac{n^k}{2^{n+1}}.$$

4

Pólya theory

> How I need a drink, alcoholic of course, after the heavy chapters involving quantum mechanics. (This is George Pólya's mnemonic for the first fifteen digits of π; the lengths of the words are the digits.)

The Pólya enumeration theorem is about counting the ways in which a combinatorial structure can be colored. Pólya's original motivation was the counting of molecules of chemical structures.

A simple example is the number of ways in which you can color a cube. (Nowadays, when you ask a child what a cube is, it will tell you that a cube consists of 27 little, colored blocks, but we keep things simple.)

4.1 Black and white cubes

Of course, when the cube is fixed in space, then there are 2^6 ways to color the 6 faces of the cube with two colors. But a cube can be rotated; if we drop it then we don't know anymore how it stood before. We want to call colorings equivalent if they can be transformed into one another by rotations. What are the equivalence classes for example, of a cube with two colors?

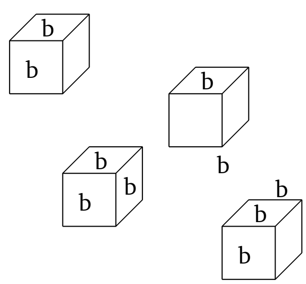

Fig. 4.1. Some black and white colored cubes

1) completely white
2) one black face
3) two black faces, meeting in an edge
4) two black faces, opposite each other
5) three black faces, meeting in a vertex
6) three black faces, in U-shape
7) two white faces, opposite
8) two white faces, meeting
9) one white face
10) completely black.

There are 10 equivalence classes.

Before we go on, here's some advice on notations of symmetries of geometrical objects. In *mechanics* material points usually get names, and when the object moves the names travel with them. In *geometry* this is not a clever idea: it's better to give the points names, and to describe a movement by giving for each point the name of its image. So the names do not travel.

Let's number the faces of the cube as in Figure 4.2.

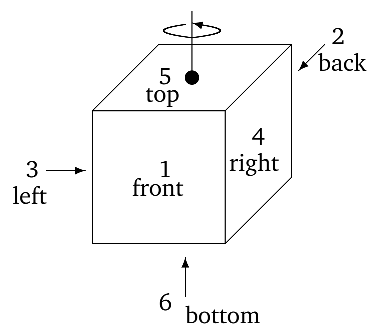

Fig. 4.2. Cube with numbered faces

A rotation around the vertical axis is described, for example, as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 2 & 5 & 6 \end{pmatrix}$$

In the general theory we assume that we have a group G, for example the group of the cube; *i.e.*, all rotations of the cube that map the cube onto itself. The group is not abstract, but *works* on a set D (for example the set of vertices, or the set of edges, or the set of faces of the cube), which means that we have a homomorphism:

$$\pi : G \to S_D \quad \text{such that} \quad \forall_{g_1,g_2\in G}\ \pi(g_1 g_2^{-1}) = \pi g_1 (\pi g_2)^{-1}.$$

You could also say that $\pi : G \to D^D$ such that

$$\forall_{g_1,g_2\in G}\ \pi(g_1 g_2) = \pi g_1 \pi g_2 \quad \text{and} \quad \pi(e) = \mathrm{Id}_D.$$

That amounts to the same thing. Usually we take π to be injective.

4.2 Cycle index

The *cycle index* of G and π is the polynomial

$$P_{G,\pi}(x_1, x_2, \ldots) = \frac{1}{|G|} \sum_{g\in G} x_1^{b_1(\pi(g))} x_2^{b_2(\pi(g))} \ldots$$

where $(b_1(\pi(g)), b_2(\pi(g)), \ldots)$ is, as usual, the type of $\pi(g)$.

> We will show that the number of ways to color the faces of the cube black and white is exactly $P_{G,\pi}(2, 2, \ldots)$.

There are of course a few trivialities, such as

$$P_{G,\pi}(x, x^2, \ldots) = x^{|D|}$$

(which implies $P_{G,\pi}(1, 1, \ldots) = 1$).

Let's do an example first. Let us calculate the cycle index of the cube-group working on the collection of faces:

rotation	type	number
all rotations over 90° are alike	$(2, 0, 0, 1, 0, \ldots)$	6
rotations 180°, axis through center of a face	$(2, 2, 0, \ldots)$	3
rotations 120°, axis through main diagonal	$(0, 0, 2, \ldots)$	8
rotations 180°, axis through middle of edges	$(0, 3, 0, \ldots)$	6
identity map	$(6, 0, 0, \ldots)$	1
These are all 24 elements of the cube-group.		24

Thus the cycle index polynomial is

$$\frac{1}{24}(6x_1^2x_4^1 + 3x_1^2x_2^2 + 8x_3^2 + 6x_2^3 + x_1^6).$$

Example 4.1. Check that the cycle index of the group of rotations working on the *edges* of the cube is

$$\frac{1}{24}(x_1^{12} + 3x_2^6 + 6x_4^3 + 6x_1^2x_2^5 + 8x_3^4).$$

The number of equivalence classes under black-and-white coloring of the edges of the cube is

$$\frac{1}{24}(4096 + 192 + 48 + 768 + 128) = 218$$

which we obtain by substituting $x_1 = x_2 = \ldots = 2$ into the cycle index.

Check that the cycle index of the group of rotations of the cube working on the *points* is

$$\frac{1}{24}(x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2).$$

Analogously, we can calculate the cycle index polynomial $P_{S_3,\mathtt{id}}$:

$$P_{S_3,\mathtt{id}} = \tfrac{1}{6}(\underset{\text{1 identity}}{x_1^3} + \underset{\text{2 3-cycles}}{2x_3^1} + \underset{\text{3 2-cycles}}{3x_1x_2}).$$

Thus there are 4 ways to color a set with three elements with two colors when all elements are similar; either all elements are black, or all elements are white, or one is black, or one is white.

4.3 Cauchy-Frobenius lemma

We first derive a lemma. This lemma used to be named after Burnside who wrote it down in a book in 1910.[1] Later, people[2] discovered that the lemma was already known to Cauchy[3] and Frobenius.[4]

The lemma is about a group that works on something. We make a distinction between the group G and the effect that its elements have on a set D, namely by defining:

[1] W. Burnside, *Theory of groups of finite order*, Cambridge University Press, 1897 (first edition; second edition 1911).

[2] N. G. de Bruijn, A note on the Cauchy-Frobenius lemma, *Indagationes Mathematicae* **41** (1979), pp. 225–228.

[3] A. L. Cauchy, Mémoire sur diverses propriétés remarquables des substitutiones régulières ou irrégulières, et des systèmes des substitutiones conjugées, *Comptes Rendus Acad. Sci. Paris* **21** (1845), pp. 972–987.

[4] F. G. Frobenius, Über die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul, *J. Reine Angew. Math.* **101** (1887), pp. 273–299.

$$\pi : G \to S_D, \quad \text{such that} \quad \forall_{g_1,g_2 \in G} \;\; \pi(g_1 g_2^{-1}) = (\pi g_1)(\pi g_2)^{-1}. \tag{4.1}$$

The mapping π is the "action" of G, and $\pi(g)$ is the "action" of $g \in G$ on D.

We define an equivalence relation $\sim$ on D as follows:

$$d_1 \sim d_2 \quad \Leftrightarrow \quad \exists_{g \in G} \;\; \pi(g)d_1 = d_2.$$

We call the equivalence class of $d \in D$ by K_d. These classes, or orbits, make up a partition of D. The lemma of Cauchy–Frobenius tells us how many equivalence classes there are.

Lemma 4.2 (Cauchy–Frobenius). *The number of equivalence classes* $|D/G|$*, under the equivalence relation* $\sim$*, is*

$$|D/G| = \frac{1}{|G|} \sum_{g \in G} \psi(g), \quad \textit{where}$$

$$\psi(g) = |\{\, d \in D \mid \pi(g)d = d \,\}| = \textit{ the number of cycles of } \pi(g) \textit{ of length 1 } =$$

$$= b_1(\pi(g)) = \frac{d}{dx} \;\; [P_{G,\pi}(x, 1, \ldots)]_{x=1} \,.$$

Proof. For $d \in D$ we define the stabilizer subgroup H_d of G as follows

$$\{\, g \in G \mid \pi(g)d = d \,\}.$$

There is a 1-1 correspondence between the elements of K_d and the left cosets gH_d of H_d, namely:

$$\begin{aligned} \text{Let} \quad & d' \in K_d \quad \text{(the equivalence class of d).} \\ \text{Then} \quad & d' = \pi(g_0)d \quad \text{for some } g_0 \in G. \\ \text{Let } g_1 \text{ be such that} \quad & d' = \pi(g_1)d. \\ \text{Then} \quad \pi(g_0)d &= \pi(g_1)d \\ \pi(g_0^{-1}g_1)d &= d \\ \therefore \quad g_0^{-1}g_1 &\in H_d \\ \therefore \quad g_1 &\in g_0 H_d. \end{aligned}$$

So $\{\, g \in G \mid d' = \pi(g)d \,\} \subseteq g_0 H_d$.

Vice versa, when you go backwards you can easily check that for $g_1 \in g_0 H_d$, $\pi(g_1)d = d'$, and so

$$\{\, g \in G \mid d' = \pi(g)d \,\} = g_0 H_d \quad \text{or} \quad \{\, g \in G \mid \pi(g)d = \pi(g_0)d \,\} = g_0 H_d.$$

In this way we associate with every element of the class K_d exactly *one* coset of H_d.

It is easy to see that the number of elements in each left coset is equal to $|H_d|$ and the consequence of this is that

$$|K_d| = \frac{|G|}{|H_d|} \quad \text{or} \quad |K_d| \cdot |H_d| = |G|.$$

Let us now apply the shepherd's counting principle to the mapping

$$\begin{aligned} f: \ D &\to D/G \quad \text{the set of equivalence classes,} \\ \text{defined by} \quad d &\to K_d \end{aligned}$$

then we find (ignore for the time being the asterisk):

$$\text{the number of classes } = |D/G| = \sum_{d \in D} \frac{1^*}{|K_d|} = \frac{1}{|G|} \sum_{d \in D} |H_d| \cdot 1^*$$

and when we write $|H_d| = \sum_{g \in G} \nu(\pi(g)d = d)$ we get:

$$|D/G| = \frac{1}{|G|} \sum_{d \in D} \sum_{g \in G} \nu(\pi(g)d = d) \cdot 1^*$$

A well-known principle in combinatorics says that you should always exchange the order of summation, unless of course when you have just done that. We obey and we get:

$$\begin{aligned} |D/G| &= \frac{1}{|G|} \sum_{g \in G} \sum_{d \in D} \nu(\pi(g)d = d) \cdot 1^* \\ &= \frac{1}{|G|} \sum_{g \in G} \psi(g). \end{aligned}$$

□

Intermezzo on a weighted version of Cauchy-Frobenius

To deduce a Cauchy–Frobenius formula with weights, we use a commutative algebra K over $\mathbb{Q}$. That is a ring in which we can multiply by scalars from $\mathbb{Q}$ such that

i. K with the ring-addition and scalar-multiplication over $\mathbb{Q}$ forms a linear space over $\mathbb{Q}$, and
ii. for $q \in \mathbb{Q}$ and $k, m \in K$, $q(km) = (qk)m$.

Examples of such algebras are rings of polynomials with rational coefficients, or ideals of such rings.

A weight function is a mapping $\omega : D \to K$. We demand that the weight function is constant on the equivalence classes K_d $(d \in D)$, so we can also interpret it as a mapping from the set of classes D/G into K. Then we have the following variation of Cauchy–Frobenius:

Lemma 4.3 (weighted version of Cauchy–Frobenius).

$$\textit{the sum of the weights of the classes of } D/G = \frac{1}{|G|} \sum_{g \in G} \Psi(g),$$

where $\Psi(g) = \sum_{d \in D} \omega(d) \cdot \nu(\pi(g)d = d)$.

In other words, we count the weights of the d's that remain fixed under $\pi(g)$ (instead of counting them all as 1).

The proof is similar; just replace the 1^* in the proof above by $\omega(d)$.

4.4 Colorings and color patterns

We are going to use this lemma to count different colorings of objects that have certain symmetries, for example the cube.

> Consider a *fixed* cube, with a set of faces D, and a set of colors R. For example
> $$R = \{\, r(\text{ed}), w(\text{hite}), b(\text{lue}) \,\}.$$

The fixed cube has $|R|^{|D|}$ colorings; namely every coloring corresponds with an elements of R^D.

If we detach the cube, and throw it on a table like a die, then the difference between “the top-face red and all other faces blue” and “the bottom-face red and all other faces blue” disappears. The cubes f_1 and f_2 in Figure 4.3 are

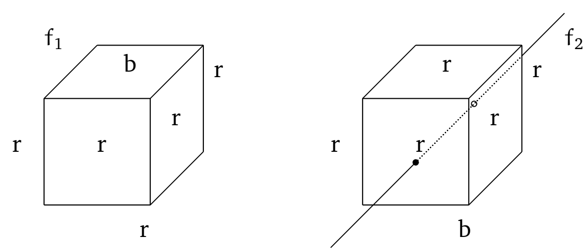

Fig. 4.3. The cubes f_1 and f_2 are equivalent

equivalent because there is a $g \in G$ (the symmetry group of rotations that leave the cube invariant) such that $f_1 \circ \pi(g) = f_2$, for example a half-turn around the depicted axis.

By considering a few examples you see how you can formalize geometric intuition: A *color pattern* is an equivalence class under the aforementioned equivalence relation. Thus a *coloring* is for example “top-face red and the rest blue” and a *color pattern* is for example “*one* face red and the rest blue.”

For $g \in G$ consider a function in S_{R^D} that maps a fixed, colored cube $f \in R^D$ to its rotated image $f \circ \pi(g)$. This map is almost, but not quite, a representation. The thing is that the order works out the wrong way. To get a map that *acts on the colorings* as in (4.1) on Page 43 we use a little trick (if the map doesn't work; then take the inverse):

$$\sigma : G \to S_{R^D} \quad \text{defined by} \quad \sigma(g)f = f \circ \pi(g^{-1}).$$

Let's check if (4.1) works now:

$$\begin{aligned}
\sigma(g_1 g_2)f &= f \circ \pi((g_1 g_2)^{-1}) = \\
&= f \circ \pi(g_2^{-1} g_1^{-1}) = \\
&= f \circ \pi(g_2^{-1}) \circ \pi(g_1^{-1}) = \\
&= \sigma(g_1)\left(f \circ \pi(g_2^{-1})\right) = \\
&= (\sigma(g_1)\sigma(g_2))\, f.
\end{aligned}$$

Note well that π permutes D and σ permutes R^D; just what we wanted; σ works on the colorings of D and the equivalence classes, or orbits, are the color patterns.

The partition into equivalence classes of D with G and π is rather dull since there is only one orbit. But the partition that we get with R^D and σ is a bit more complicated. The number of color patterns is equal to the number of equivalence classes, given by the Cauchy–Frobenius lemma.

We give away the answer right now: The number of color patterns is

$$P_{G,\pi}(|R|, |R|, |R|, \ldots).$$

How to prove that? We see that when we prove Pólya's theorem in the next, exciting section.

4.5 Pólya's theorem

We give the colors *weights*, namely r, w, and b. The weight of a coloring is

$$\Omega(f) = \prod_{d \in D} \omega(f(d)). \tag{4.2}$$

For example, the coloring of the cube in Figure 4.4 with three red faces and three blue faces has weight $\omega(f) = r^3 b^3$. Simply take the color of every

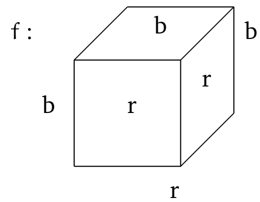

Fig. 4.4. A cube with three red faces and three blue faces

face and multiply them all together.

If you look at $f \circ \pi(g)$ instead of at f then only the factors are permuted, and therefore *the weight of every coloring in a color pattern* F *is the same*. So we can just as well write

$$\Omega(F) = \Omega(f) \quad \text{for every} \quad f \in F.$$

Bourbaki would call this "abus de langage."

Theorem 4.4 (Pólya's theorem). *The sum of the weights of the color patterns is*

$$P_{G,\pi}(\sum_{r\in R} \omega(r), \sum_{r\in R} \omega(r)^2, \sum_{r\in R} \omega(r)^3, \ldots)$$

Proof.

$$\sum_{F\in\{\text{ color patterns }\}} \Omega(F) = \frac{1}{|G|} \sum_{g\in G} \Psi(g)$$
$$\text{where} \quad \Psi(g) = \sum_{f\in R^D} \Omega(f) \cdot \nu(\sigma(g)f = f).$$

Let's try to understand this formula, as a mathematical finger exercise. We have that

$$\sum_{f\in R^D} \Omega(f) = \left(\sum_{r\in R} \omega(r)\right)^{|D|}$$

because if $f : D \to R$ is for example a function like this $f = \begin{pmatrix} d_1 & d_2 & d_3 \\ \downarrow & \downarrow & \downarrow \\ r_2 & r_1 & r_3 \end{pmatrix}$, then by (4.2), $\Omega(f)$ corresponds to the product of the underlined terms in

$$\begin{aligned} &(\omega(r_1) + \underline{\omega(r_2)} + \ldots) \\ \times\ &(\underline{\omega(r_1)} + \omega(r_2) + \ldots) \\ \times\ &(\omega(r_1) + \omega(r_2) + \underline{\omega(r_3)} + \ldots) \\ &\ldots \end{aligned}$$

Now let's look at the invariance-requirement $\sigma(g)f = f$. Let $g \in G$ be a group element. The action of $\pi(g)$ on D partitions D into cycles $D_1, D_2, D_3, \ldots$. Every D_i is permuted cyclically. The demand that $f \circ \pi(g^{-1}) = f$ means that $D_1, D_2, \ldots$ get colored monochromatically. In other words, all elements of D_i, the whole cycle, must get the same color. Thus

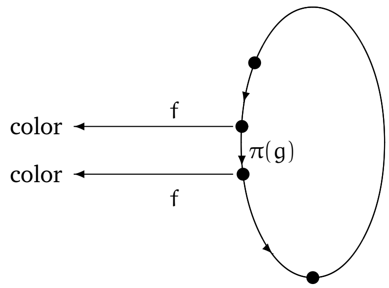

Fig. 4.5. Before or after $\pi(g)$ you get the same color

$$\Psi(g) = \sum_{\substack{f \in R^D \\ \sigma(g)f = f}} \Omega(f) = \sum_{r\in R} \underbrace{\omega(r)^{|D_1|}}_{\text{all terms corresponding to } d \in D_1 \text{ must give the same factor.}} \cdot \sum_{r \in R} \omega(r)^{|D_2|} \cdot \sum \ldots$$

Gathering cycles that have the same length gives

$$\Psi(g) = \left(\sum_{r \in R} \omega(r) \right)^{\#\text{ 1-cycles in } \pi(g)} \cdot \left(\sum_{r \in R} \omega(r)^2 \right)^{\#\text{ 2-cycles in } \pi(g)} \cdot \ldots$$

and this proves

$$\sum_{F \in \{\text{ color patterns }\}} \Omega(F) = P_{G,\pi}\Big(\sum_{r \in R} \omega(r), \sum_{r \in R} \omega(r)^2, \ldots\Big).$$

□

Example 4.5. Let's look at an example to clarify the difference between a *coloring* and a *color pattern*. Take

$$\begin{aligned} D &= \mathbb{Z} \quad \text{(the set of objects that we want to color is infinite)} \\ G &= (\mathbb{Z}, +) \quad \text{(the additive group)} \\ R &= \mathbb{R} \quad \text{(we take the reals as the set of colors)} \\ \pi(g)(z) &= z + g \quad \text{for } z \in \mathbb{Z}. \end{aligned}$$

A coloring is a function $\mathbb{Z} \to \mathbb{R}$. A color pattern is a class of colorings.

$\mathbb{Z}$	... -5	-4	-3	-2	-1	0	1	2	3	4	5	6	...
coloring ...	12	10	8	6	4	1	3	5	7	9	11	13	...
coloring ...	20	18	16	14	12	10	8	6	4	1	3	5	...

These are different colorings that belong to the same color pattern.

In general only constant colorings are invariant under *all* $\pi(g)$. Some colorings are invariant under certain permutations, for example

	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
...	...	3	4	2	1	3	4	2	1	3	4	2 ...

is invariant under $\pi(4)$.

No coloring is invariant under color permutations, but with color patterns that is different: the pattern above does not change if we permute the colors according to

$$1 \to 3 \to 4 \to 2 \to 1$$

Example 4.6 (The points of the cube). We call the representation group of the cube on the points π_3. The cycle index polynomial is:

$$\frac{1}{24}(x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2).$$

For colors we take "yes" and "no," thus a coloring indicates a subset of the corner points.

The color patterns are described by:

$$\frac{1}{24}((y+n)^8 + 9(y^2+n^2)^4 + 6(y^4+n^4)^2 + 8(y+n)^2(y^3+n^3)^2).$$

The coefficient of y^4n^4 gives the number of subset-equivalence classes. We find:

$$\begin{aligned} C_{y^4n^4} &= \frac{1}{24}\left(\binom{8}{4} + 9\binom{4}{2} + 6\binom{2}{1} + 8\binom{2}{1}\binom{2}{1}\right) \\ &= \frac{1}{24}(70 + 54 + 12 + 32) = \frac{168}{24} = 7. \end{aligned}$$

And indeed, Figure 4.6 on the following page gives the 7 equivalence classes.

Example 4.7. Consider once more the faces of the cube. We are now interested in patterns of subsets of faces with 3 elements. Many ways to choose 3 elements out of 6 are equivalent.

$$D = \text{ the set of faces, and } \quad K = \{\text{"yes"}, \text{"no"}\} \text{ is the set of colors.}$$

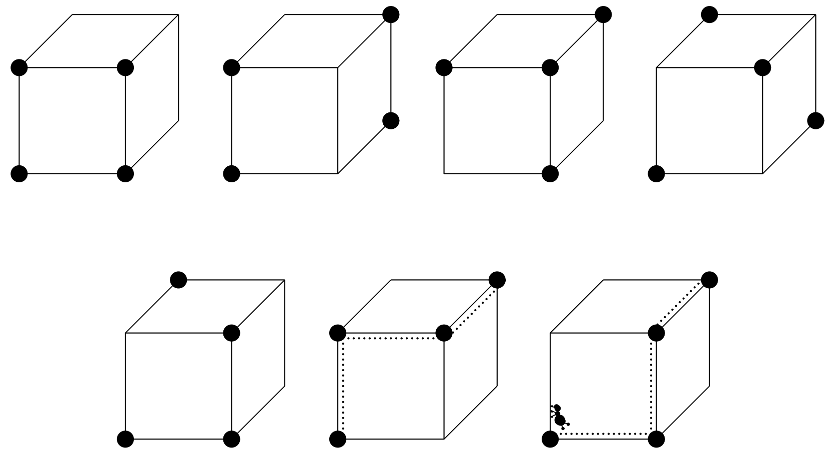

Fig. 4.6. Subset-equivalences; the last two are really different because a louse takes turns r and then ℓ in the first one of these two cubes, while he takes turns ℓ and then r in the second one of these two cubes

The weight of “yes” is $1 \cdot y \in \mathbb{Q}[y, n]$ and the weight of “no” is $1 \cdot n \in \mathbb{Q}[y, n]$.

$$\begin{aligned} G \;& \text{ is the cube group, and} \\ \pi : G \; \to S_D \;& \text{ is the representation as permutations of the faces.} \end{aligned}$$

The answer is

$$C_{y^3n^3}\ P_{G,\pi}(\sum \omega(r), \sum \omega^2(r), \sum \omega^3(r), \ldots).$$

The cycle index polynomial is the one that we calculated on Page 41.

$$\begin{aligned} \sum \omega(r) &= \text{ sum of the weights of the colors} \\ &= y + n, \\ \sum \omega^2(r) &= y^2 + n^2, \quad \text{etc.} \end{aligned}$$

So we get

$$\begin{aligned} C_{y^3n^3} \quad & \tfrac{1}{24}(\underbrace{(y+n)^6}_{A} + \underbrace{3(y+n)^2(y^2+n^2)^2}_{B} + \\ & + \underbrace{6(y+n)^2(y^4+n^4)}_{C} + \underbrace{6(y^2+n^2)^3}_{D} + \underbrace{8(y^3+n^3)^2}_{E}). \end{aligned}$$

The terms in C only give powers that are at least 4. The terms in D only give even powers. So we can forget about those. From the second factor in B only y^2n^2 contributes. So we are left with:

$$\frac{1}{24}\left(\binom{6}{3}+3\cdot 2\cdot 2+8\cdot 2\right)=2.$$

Of course there is another way to see that: the 3 faces either come together in one point, or they are in U-shape. See also the discussion on Page 40.

Let's make things a bit more complicated. For starters we have again a group G, a set D, a homomorphism $\pi : G \to S_D$, and an equivalence relation

$$d_1 \sim d_2 \quad \Leftrightarrow \quad \exists_{g\in G} \ \ \pi(g)d_1 = d_2.$$

We assume that we have a weight function $\omega : D \to K$, where K is a commutative ring as on Page 44, and now we add one other map $\rho : D \to D$ of which we assume that it is a permutation. (BTW, the fact that it is a permutation is not essential.)

We meet the demand of the weighted version of Cauchy-Frobenius that

$$d_1 \sim d_2 \quad \Rightarrow \quad \omega(d_1) = \omega(d_2).$$

Thus ω is constant on the orbits of G in D. Furthermore we assume that

$$d_1 \sim d_2 \quad \Rightarrow \quad \rho(d_1) \sim \rho(d_2),$$

thus we assume that ρ maps orbits into orbits.

Theorem 4.8. *The sum of the weights of the classes (orbits) that are invariant under* ρ *is*

$$\frac{1}{|G|}\sum_{g\in G}\sum_{d\in D}\omega(d)\cdot\nu(\rho(\pi(g)d) = d).$$

Note that when $\rho = \mathrm{Id}_D$ we get our old formula back.

Proof. There really is no very elegant way to prove this; it is just complicated. We just have to go through it the hard way. Here goes!

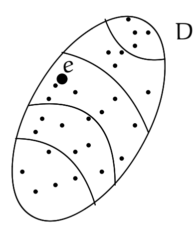

Fig. 4.7. Partitioned set with fixed element e

Step 1

We first calculate another sum, namely

$$\begin{aligned}\text{sum} &= \sum_{g\in G}\sum_{d\in D}\omega(d)\cdot\nu(\rho(\pi(g)d)=d)\nu(d\sim e)\\ &= \sum_{\substack{d\in D\\ d\sim e}}\sum_{g\in G}\omega(d)\cdot\nu(\rho(\pi(g)d)=d).\end{aligned}$$

Step 2

(It looks a bit like playing with the Hungarian cube; you turn a face, then do something else and then turn the face back; instead of turning the face we exchange the summation.) In the inner sum, d is fixed, and because $d\sim e$, $d=\pi(k)e$ for some suitable k:

$$\begin{aligned}\text{sum} &= \sum_{\substack{d\in D\\ d\sim e}}\sum_{g\in G}\omega(d)\cdot\nu(\rho(\pi(g)\pi(k)e)=d)\\ &= \sum_{\substack{d\in D\\ d\sim e}}\sum_{gk\in kG}\omega(d)\cdot\nu(\rho(\pi(gk)e)=d)\\ &\quad\text{(now we may replace } g \text{ by } gk \text{ and call it } h)\\ &= \sum_{\substack{d\in D\\ d\sim e}}\sum_{h\in G}\omega(d)\cdot\nu(\rho(\pi(h)e)=d)\\ &\quad\text{(because } kG=G)\\ &\quad\text{(observe that } d\sim e\Rightarrow\omega(d)=\omega(e))\\ &= \sum_{\substack{d\in D\\ d\sim e}}\sum_{h\in G}\omega(e)\cdot\nu(\rho(\pi(h)e)=d)\\ &\quad\text{(now change back)}\\ &= \sum_{h\in G}\sum_{\substack{d\in D\\ d\sim e}}\omega(e)\cdot\nu(\rho(\pi(h)e)=d).\end{aligned}$$

In the inner sum, there is exactly *one* nonzero term or otherwise all terms are zero. One term is nonzero when

$$\rho(\pi(h)e)\sim e$$

and otherwise all terms are zero. So we find that the sum is equal to

$$\text{sum} = \begin{cases}\omega(e)\cdot|G| & \text{if } \rho(\pi(h)e)\sim e, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

But the condition $\rho(\pi(h)e) \sim e$ means exactly that the equivalence class of e is invariant under ρ: $\pi(h)e$ runs through the complete class; if e is in some ρ-invariant class, then $\rho(\pi(h)e) \sim e$.

We now choose in every class K a representative e_K and sum over the classes. □

4.6 Invariant colorings

In this section we apply the previous theorem to colorings and color patterns. We have

i. a set R of colors,
ii. a set D,
iii. colorings $f : D \to R$,
iv. weights $\omega : R \to K$,
v. $\pi : G \to S_D$, and
vi. $\theta \in S_R$ (this one is new).

We have the set of colorings R^D instead of the set D in the previous theorem.

The weight of a coloring $f : D \to R$ is

$$\Omega(f) = \prod_{d \in D} \omega(f(d)).$$

We define an equivalence relation on colorings:

$$f_1 \sim f_2 \quad \Leftrightarrow \quad \exists_{g \in G} \; f_1 \circ \pi(g) = f_2.$$

Equivalent colorings have the same weight; the factors in $\Omega(f)$ and $\Omega(f \circ \pi(g))$ only differ by a permutation. We write

$$f \circ \pi(g^{-1}) = \sigma(g) f$$

like we did on page 46.

We introduce a mapping $\rho : R^D \to R^D$ defined by

$$\rho(f) = \theta \circ f$$

and we claim that

$$f_1 \sim f_2 \quad \Rightarrow \quad \rho f_1 \sim \rho f_2.$$

Indeed, let f_1 and f_2 be equivalent colorings. Then

$$\begin{aligned} f_1 &= f_2 \circ \pi(g) \;\; \text{for some } g. \text{ This implies that} \\ \theta \circ f_1 &= \theta \circ f_2 \circ \pi(g) \quad \Rightarrow \quad \rho(f_1) = \rho(f_2) \circ \pi(g), \\ \therefore \;\; & \rho f_1 \sim \rho f_2. \end{aligned}$$

A proof like this is really nothing else than a messy exercise with brackets. The reason for this is that our notation for functions is not brilliant. On the other hand, it is also not too bad; it works with a bit of effort, and if you'd do it some other way you get lots of complaints. It's hopeless.

We are going to determine the sum of the weights of θ-invariant colorings. But let's look at an example first so that we get some idea of what we are actually doing.

Example 4.9. Suppose we color the *edges* of a cube with two colors: white and black ("no" and "yes"). We are interested in the number of patterns that are invariant under a switch of the colors. Of course, this can only happen when there are 6 edges colored white and 6 edges colored black.

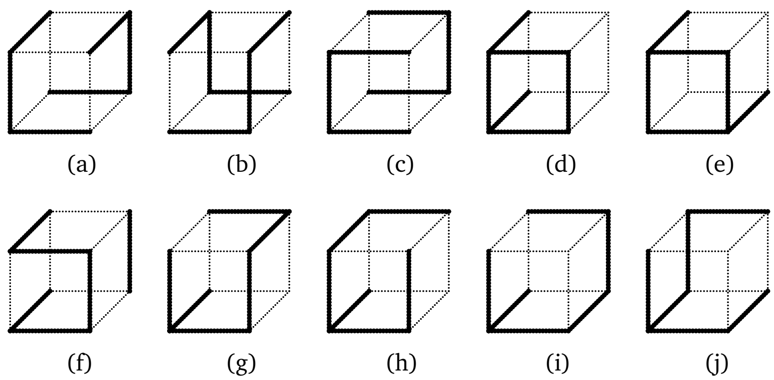

Fig. 4.8. Invariant edge-colorings of the cube; (a) and (b) are mirror images; (d) is a table with legs on the same side and (e) has the legs on opposite sides; (g) and (h) look the same but the branch is in a different place; (j) is a reflection of (h)

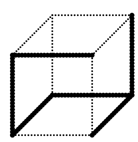

Fig. 4.9. Perhaps you think that there is one missing in Figure 4.8: the mirror-image of (g) (this one), but that is exactly (i).

Remark 4.10. The last 5 cases in Figure 4.8 are easiest characterized as the following coloring of the edges in Figure 4.10, with *one* of the dotted edges moved to one edge that is parallel to it.

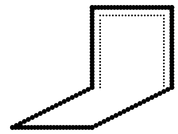

Fig. 4.10. U-shape edge coloring

To find the sum of the θ-invariant colorings we apply Theorem 4.8. The sum is

$$\frac{1}{|G|}\sum_{g\in G}\sum_{f\in R^D}\Omega(f)\cdot\nu(f=\rho\sigma(g)f)$$

and we have to live with that.

Note that

$$\rho\sigma(g)f=\theta\circ f\circ\pi(g^{-1}).$$

Now, this sum is pretty easy to understand. For fixed g we draw the cycles of $\pi(g)$ in Figure 4.11.

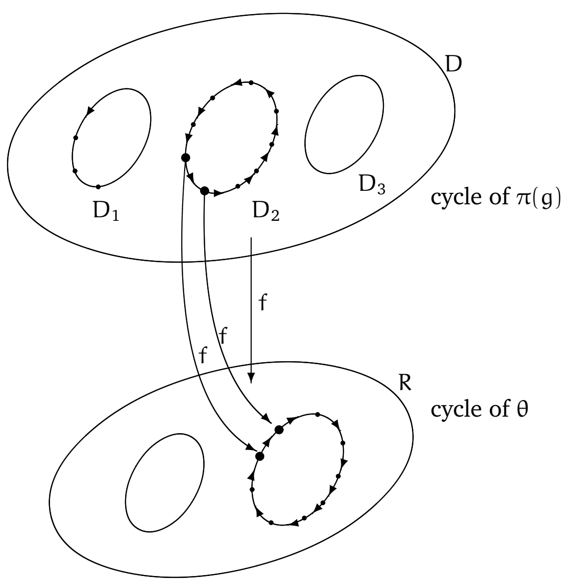

Fig. 4.11. $f\circ\pi(g)=\theta\circ f$

Now

$$f=\rho\sigma(g)f \quad\Rightarrow\quad f\circ\pi(g)=\theta\circ f.$$

In other words, if you "go down with" f and then "take one step with" θ then you end up at the same place as when you had first "taken a step with" $\pi(g)$ and then "gone down with" f.

Choose $d_1 \in D_1, d_2, \in D_2, \ldots$. Then f is determined by $fd_1, fd_2, \ldots$. But it only works out when f lands you in a cycle that is compatible with the cycle in which you started: when you have completed a cycle in D then you should have completed the cycle in R exactly a whole number of times. That is,

$$\theta^{|D_1|} f(d_1) = f(d_1).$$

Or, when we write $f(d_1) = r_1$:

$$\theta^{|D_1|} r_1 = r_1.$$

The sum of the weights blah blah is

$$\sum_{\substack{r_1,\ldots,r_h \\ \theta^{|D_i|} r_i = r_i \\ r_i = fd_i}} \omega\left(fd_1\right)\omega\left(\theta fd_1\right)\ldots\omega\left(\theta^{|D_1|-1} fd_1\right) \times \text{similar terms with } D_2 \text{ etc.}$$

Let's introduce a shorthand for

$$\lambda_{|D_i|} = \sum_{\substack{r \in R \\ \theta^{|D_i|} r = r}} \omega(r)\omega(\theta r)\ldots\omega\left(\theta^{|D_i|-1} r\right).$$

Then the sum becomes

$$\lambda_{|D_1|} \cdot \lambda_{|D_2|} \cdot \ldots \cdot \lambda_{|D_h|}.$$

So for one fixed g we get

$$\lambda_1^{b_1(\pi(g))} \cdot \lambda_2^{b_2(\pi(g))} \cdot \ldots$$

and our answer becomes

$$P_{G,\pi}(\lambda_1, \lambda_2, \ldots), \quad \text{where} \quad \lambda_k = \sum_{\substack{r \in R \\ \theta^k r = r}} \omega(r)\omega(\theta r)\ldots\omega\left(\theta^{k-1} r\right).$$

In the special case where all weights are 1 we have

$$\lambda_k = \text{ number of colors that are invariant under } \theta^k.$$

Consider a color x that appears in a cycle of length j. Then x is invariant under θ^k only when $j|k$, in other words

$$\lambda_k = \sum_{j|k} jc_j,$$

where $(c_1, c_2, \ldots)$ is the type of θ.

If there are two colors that are switched by θ, then this becomes

$$\lambda_k = \begin{cases} 2 & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd,} \end{cases}$$

and then the answer is

$$P_{G,\pi}(0, 2, 0, 2, \ldots),$$

which, for the case of the edges of a cube gives

$$\frac{1}{24}(0 + 3 \cdot 2^6 + 6 \cdot 2^3 + 0 + 0) = \frac{1}{24}(192 + 48) = 10.$$

This is exactly the number that we illustrated in Figure 4.8 on page 54.

4.7 Super patterns

We are going to discuss *super patterns*. We need a few lemmas for that.

As before, we have a set D and a set R, and *one* permutation η of D, and *one* permutation θ of R.

Question.

What is the number of mappings

$$f \in R^D \quad \text{such that} \quad \theta f = f\eta?$$

The condition is that the image of a cycle D_i is contained in a cycle whose length divides $|D_i|$. More precisely, if we choose one element in every D_i, say d_i, then fd_i must land in a cycle whose length divides $|D_i|$. This fixes the image of D_i. See Figure 4.12.

Let $(b_1, b_2, \ldots)$ be the type of η and let $(c_1, c_2, \ldots)$ be the type of θ. The number of possibilities to map a fixed representative d_i of D_i with length $k = |D_i|$, is

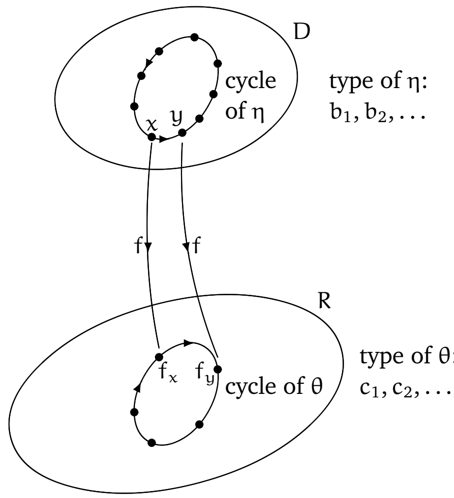

Fig. 4.12. Cycles of η in D mapped to cycles of θ in R

$$\sum_{d|k} dc_d \quad \text{(for every “good” } \theta\text{-cycle there are } d \text{ possibilities).}$$

Thus we find that $\displaystyle\prod_{k=1}^{\infty}\Bigg(\sum_{d|k} dc_d\Bigg)^{b_k} =$

$$\prod_{k=1}^{\infty}\left(\frac{\partial}{\partial x_k}\right)^{b_k} \overbrace{\exp\left(c_1(x_1+x_2+\dots)+2c_2(x_2+x_4+\dots)+3c_3(x_3+x_6+\dots)\right)}^{\boxed{=\prod_{k=1}^{\infty}\left(e^{x_k+x_{2k}+x_{3k}+\dots}\right)^{kc_k}}}$$

evaluated at $(x_1, x_2, \dots) = (0, 0, 0, \dots)$.

If we want f to be injective, then we don’t allow any proper divisors. Then the sum $\sum_{d|k} dc_d$ becomes kc_k (where $k = |D_i|$). *Furthermore*, f must map the D_i’s to different cycles. This can be done in two steps. Inject the cycles of length k in η into the cycles of length k in θ. You can do this in

$$\frac{c_k!}{(c_k - b_k)!}$$

ways. For each injection like that, there are k^{b_k} ways to choose the image of the representative.

This sums up to

$$\prod_{k=1}^{\infty} k^{b_k} \cdot \frac{c_k!}{(c_k - b_k)!}$$

ways to do it, that is, when

$$\forall_k \;\; c_k \geqslant b_k.$$

We can write

$$k^{b_k} \cdot \frac{c_k!}{(c_k - b_k)!} = \left[\left(\frac{\partial}{\partial x_k} \right)^{b_k} (1 + kx_k)^{c_k} \right]_{x_k = 0}$$

So we find that the number of *injections* f with $\theta f = f\eta$ is equal to

$$\left[\prod_{k=1}^{\infty} \left(\frac{\partial}{\partial x_k} \right)^{b_k} (1 + kx_k)^{c_k} \right]_{(x_1, x_2, \ldots) = (0,0,\ldots)}$$

where the type of θ is $(c_1, c_2, \ldots)$ and the type of η is $(b_1, b_2, \ldots)$.

Recapitulation.

Let's recall what a pattern is. We have a set of colors R, a set of "objects" D, and a set of "colorings" $f : D \to R$. Using a group G and a homomorphism (representation) $\pi : G \to S_D$, we define an equivalence relation on the colorings:

$$f_1 \sim f_2 \quad \Leftrightarrow \quad f_1 \circ \pi(g) = f_2.$$

We call the equivalence classes color *patterns*.

We also introduced a permutation in the set of colors and, obviously, colorings are *not* invariant under that permutation (unless the permutation does not change the colors that are used in the coloring), but color patterns sometimes are.

We now make things a bit more interesting. Instead of a single permutation of the colors, we consider a whole group H of permutations of colors. We have to redefine our equivalence relation slightly:

$$f_1 \sim f_2 \quad \Leftrightarrow \quad \exists_{h,g} \;\; hf_1 = f_2 \circ \pi(g).$$

In this way we get *super patterns* (patterns where different colors are not such a big issue anymore; more like color *designs*).

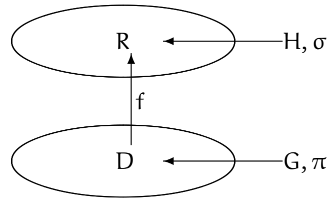

Fig. 4.13. H Permutes the colors R and G permutes the objects D

Example 4.11. Take the cube with its 6 faces. Let the set of colors be

$$R = \{\text{ purple, violet }\}.$$

Let H be the group of permutations of R. (The idea is that many people can distinguish the colors purple and violet, but when asked which is which, they find it hard to tell.)

We have the group of rotations of the cube G and this groups acts via π on the faces.

What are the super patterns? We show them in Figure 4.14.

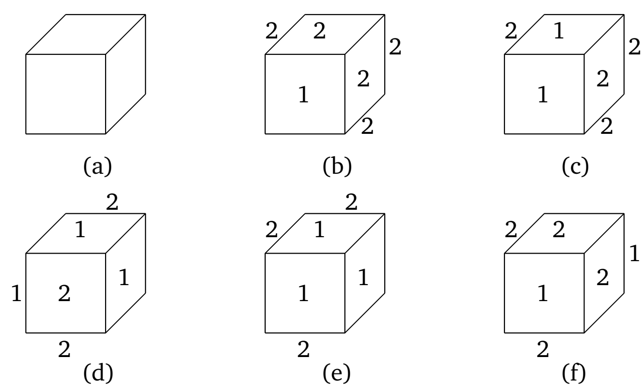

Fig. 4.14. Super patterns; (a) is monochromatic; (b) has one face of a different color; (c) has two faces next to each other of one color; (d) is a U-pattern; (e) has three faces of one color coming together in a corner; (f) has two opposite faces of the same color

We are going to do this now *très chique*. Let G and H be groups with representations π and σ:

$$\sigma : H \to S_R \quad \text{and} \quad \pi : G \to S_D.$$

We define

$$\tau : G \times H \to S_{R^D}$$

as follows:

$$(\tau(g,h))f = \sigma(h) \circ f \circ \pi(g^{-1}), \quad \text{for all } f \in R^D.$$

A super pattern is an equivalence class in R^D with the equivalence relation $\approx$ defined by:

$$f_1 \approx f_2 \quad \Leftrightarrow \quad \exists_{(g,h)\in G\times H}\ f_1 = \tau(g,h)f_2.$$

To determine the super patterns we now have two options:

1. either use Cauchy–Frobenius on R^D, $G \times H$, and τ, or
2. express super patterns as equivalence classes of patterns.

Let's explain that last item first.

R^D is partitioned by $\sim$ into subsets. If $f_1 \sim f_2$ then this is a refinement of $f_1 \approx f_2$ (to get $\sim$ as a special case of $\approx$ you can take h equal to the identity element of H). A collection of $\sim$-classes makes up one $\approx$-class; the $\approx$-partition is coarser. See Figure 4.15.

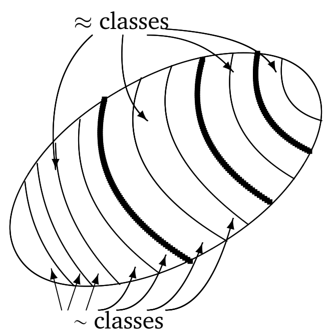

Fig. 4.15. Equivalence classes of equivalence classes

The collection of patterns

$R^D/(G,\pi)$ (which is R^D, where (G,π)-equivalent things are identified)

is permuted by a permutation $\rho(h)$, for each $h \in H$:

$$\rho(h)\underbrace{\{\, f \circ \pi(g^{-1}) \mid g \in G \,\}}_{\in R^D/(G,\pi)} = \{\, \sigma(h) \circ f \circ \pi(g^{-1}) \mid g \in G \,\}.$$

Now we must show formally that

i. the righthand side is again a class (independent of the choice of the representative h), and
ii. that ρ is a representation (action; see 4.1 on page 43).

Thereafter we can apply Cauchy–Frobenius with $R^D/(G,\pi)$, H, and ρ.

We go for the first option. We use Theorem 4.8 on page 51 but this time we do it without weights.

The number of super patterns =

$$\frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \text{the number of } f \in R^D \text{ with } \tau(g,h)f = f.$$

We write $\#_{f\in R^D}(\dots)$ instead of "the number of $f \in R^D$ with ..."
Note that

$$\#_{f\in R^D}(\tau(g,h)f = f) = \#_{f\in R^D}(\sigma(h)f = f\pi(g)).$$

Let's write

$$\theta = \sigma(h) \quad \text{and} \quad \eta = \pi(g),$$

and let

$$\begin{array}{ll} (c_1, c_2, \dots) & \text{be the type of } \theta \text{ and} \\ (b_1, b_2, \dots) & \text{be the type of } \eta. \end{array}$$

Then

$$\begin{aligned} &\#_{f\in R^D}\ (\theta f = f\eta) = \\ &= \left(\frac{\partial}{\partial x_1}\right)^{b_1} \left(\frac{\partial}{\partial x_2}\right)^{b_2} \cdots \underbrace{\left(e^{x_1+x_2+\dots}\right)^{c_1} \left(e^{2(x_2+x_4+\dots)}\right)^{c_2} \left(e^{3(x_3+x_6+\dots)}\right)^{c_3} \cdots}_{\substack{\text{if we sum this kettle of fish on } h \in H \text{ we get} \\ P_{H,\sigma}\left(e^{x_1+x_2+\dots},\, e^{2(x_2+x_4+\dots)},\, e^{3(x_3+x_6+\dots)}, \dots\right)}} \end{aligned}$$

If we now take the average over $g \in G$ we get

$$\left[P_{G,\pi}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\right) P_{H,\sigma}\left(e^{x_1+x_2+\dots}, e^{2(x_2+x_4+\dots)}, \dots\right)\right]_{(x_1,x_2,x_3,\dots)=(0,0,0,\dots)}$$

Another way to get this result is as follows. The number of super patterns is:

$$\tfrac{1}{|H|}\sum_{h\in H}\frac{1}{|G|}\sum_{g\in G}\#_{f\in R^D}\ (\sigma(h)f = f\circ\pi(g)) =$$

$$= \tfrac{1}{|H|}\sum_{h\in H}\frac{1}{|G|}\sum_{g\in G}\left(\sum_{d|1} dc_d\right)^{b_1}\left(\sum_{d|2} dc_d\right)^{b_2}\left(\sum_{d|3} dc_d\right)^{b_3}\ldots$$

where we write $c_d = c_d(\sigma(h))$ and $b_i = b_i(\pi(g))$

$$= \tfrac{1}{|H|}\sum_{h\in H} P_{G,\pi}\left(\sum_{d|1} db_d(\sigma(h)), \sum_{d|2} db_d(\sigma(h)),\ldots\right) =$$

$$= \tfrac{1}{|H|}\sum_{h\in H} P_{G,\pi}\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\ldots\right)\exp(c_1(\sigma(h))(x_1+x_2+\ldots)+$$

$$+2c_2(\sigma(h))(x_2+x_4+\ldots)+3c_3(\sigma(h))(x_3+x_6+\ldots)+\ldots)$$

and this evaluated at $(x_1,x_2,\ldots) = (0,0,\ldots)$. This gives the same result as before.

The nice thing is that we get the injective super patterns for free: "injective" means that things should be invariant under $\sigma(h)$ and under $\pi(g)$, thus under $\tau(g,h)$. When you compare the two formulas on Page 58 and 59 for the general and the injective case, then you see that we have to replace

$$e^{k(x_k+x_{2k}+x_{3k}+\ldots)} \quad \text{by} \quad 1+kx_k$$

and then we get:

$$\left[P_{G,\pi}\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots\right)P_{H,\sigma}(1+x_1,1+2x_2,1+3x_3,\ldots)\right]_{\underline{x}=\underline{0}}$$

As an application, let us take a group H with two elements, namely the permutations of $\{\text{white},\text{black}\}$. Then

$$P_{H,\sigma}(v_1,v_2,v_3,\ldots) = \frac{1}{2}(v_1^2+v_2),$$

and this gives

$$P_{H,\sigma}\left(e^{x_1+x_2+\ldots}, e^{2(x_2+x_4+\ldots)}+\ldots\right) = \frac{1}{2}\left(e^{2x_1+2x_2+\ldots}+e^{2x_2+2x_4+\ldots}\right).$$

If we now let loose

$$P_{G,\pi}\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots\right)\Big|_{x_1=0,x_2=0,\ldots}$$

on $P_{H,\sigma}$ then we obtain the following formula for the number of superpatterns with two colors.

$$\frac{1}{2}(\underbrace{P_{G,\pi}(2,2,2,\ldots)}_{A}+\underbrace{P_{G,\pi}(0,2,0,2,\ldots)}_{B}).$$

Can we also see this in another way?

i. B is the number of patterns that is invariant, and
ii. A is the total number of patterns.

Thus $A - B$ is the number of patterns that are *not* invariant. When we consider super patterns, the non-invariant patterns are pairwise identified, but the invariant patterns form super patterns by themselves. Thus the number of super patterns is indeed $\frac{1}{2}(A - B) + B = \frac{1}{2}(A + B)$.

4.8 Wreath product

We could do a whole lot more, but let's restrict ourselves to the wreath product (Pólya calls this "Kranz.")

To keep the discussion a bit simple, we consider $D_1 \times D_2$, where D_1 is a collection of m objects and $|D_2| = n$. We let G be a group of permutations of D_1 and we let H be a group of permutation of D_2. For simplicity we write P_G instead of $P_{G,\pi}$ and P_H instead of $P_{H,\sigma}$.

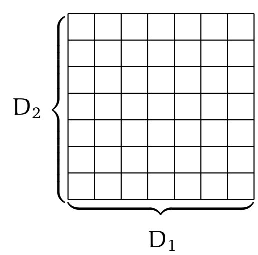

Fig. 4.16. $D_1 \times D_2$

Using G and H we can permute $D_1 \times D_2$:

> within each column we permute the elements according to some h (every column its own h), and next we permute the columns according to G. We denote the group of permutations that we get like that by $G[H]$.

More precisely, let H, σ, and D_2 and G, π, and D_1 be given. Then we equip $G \times H^{D_1}$ with the following operation:

$$(g, f)(g', f') = (gg', f \circ \pi(g') \bullet f')$$

where $\bullet$ stands for coordinate-wise multiplication.

Next, τ acts on $D_1 \times D_2$ as follows:

$$\tau(g, f)(d_1, d_2) = (\pi g(d_1), \sigma(f(d_1))d_2).$$

Notice that this defines an action:

$$\begin{aligned}
\tau(g, f) \circ \tau(g', f')(d_1, d_2) &= \\
\tau(g, f)\ (\pi g'(d_1), \sigma(f'(d_1))d_2) &= \\
((\pi g)(\pi g')d_1, \sigma(f(\pi g' d_1))\sigma(f'(d_1))d_2) &= \\
(\pi(gg')d_1, \sigma(f(\pi g' d_1))\sigma(f'(d_1))d_2) &= \\
(\pi(gg')d_1, \sigma((f\pi g' \bullet f')d_1)d_2) &= \\
\tau(gg', f \circ \pi(g') \bullet f')(d_1, d_2) &= \\
\tau((g, f)(g', f'))(d_1, d_2).&
\end{aligned}$$

If G and H are groups of permutations, then

$$(g, h_1, h_2, \ldots, h_m) : (s, t) \to (gs, h_s t).$$

One of the theorems of Pólya now says that

$$P_{G[H]}(x_1, x_2, \ldots) = P_G\left(P_H(x_1, x_2, \ldots), P_H(x_2, x_4, \ldots), P_H(x_3, x_6, \ldots) \ldots\right). \tag{4.3}$$

The idea to prove this is by using colorings with weights. A super color is the color pattern of a column (which is the color pattern of H acting on D_2). A color pattern of G[H] acting on $D_1 \times D_2$ is a super-color pattern of G acting on D_1.

The sum of the k^{th} powers of the weights of the super colors is, if $k = 1$:

$$P_H\left(\sum \omega_r, \sum \omega_r^2, \sum \omega_r^3, \ldots\right)$$

and for general k:

$$P_H\left(\sum \omega_r^k, \sum \omega_r^{2k}, \sum \omega_r^{3k}, \ldots\right).$$

This gives

$$\begin{aligned}
&P_{G[H]}\left(\sum \omega_r, \sum \omega_r^2, \ldots\right) = \\
&P_G\left(P_H\left(\sum \omega_r, \sum \omega_r^2, \ldots\right), P_H\left(\sum \omega_r^2, \sum \omega_r^4, \ldots\right), \ldots\right).
\end{aligned}$$

Finally, if you notice that $\sum \omega_r$, $\sum \omega_r^2$, etc. are algebraically independent, then you find Pólya's theorem.

Remark 4.12. The weight of a color pattern is, as usual, the product of the weights of the colors that are used (see Page 46).

Pólya's theory can be extended in numerous other ways, but we won't go into that. Actually, the proof above gets pretty complicated if you write it down in full detail. For a sketch of the proof see [8].

4.9 Problems

4.1 (de Bruijn[5]). Use Formula 3.1 on page 26 to check that the cycle index of the symmetric group S_n (the group of all permutations of $\{1, \ldots, n\}$) is:

$$P_{S_n,\pi}(x_1, x_2, \ldots) = \sum_{\substack{\lambda_1 \geqslant 0, \ldots, \lambda_n \geqslant 0 \\ \lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n}} \frac{x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}}{1^{\lambda_1}\lambda_1! 2^{\lambda_2}\lambda_2! \ldots n^{\lambda_n}\lambda_n!}.$$

Show that this is equal to

$$C_{z^n} \ \exp\left(zx_1 + \frac{z^2x_2}{2} + \frac{z^3x_3}{3} + \ldots\right).$$

Compare this with the jumble that we obtained in 3.4 on page 28.

4.2. Consider the rotations in $\mathbb{R}^3$ of the hexagon. Prove that the cycle index is

$$P_{D_6,\pi}(x_1, \ldots, x_6) = \frac{1}{12}(x_1^6 + 2x_6 + 2x_3^2 + x_2^3 + 3x_1^2x_2^2 + 3x_2^3).$$

4.3. In this exercise we derive the cycle index of the cyclic group C_n. This group acts on $1, \ldots, n$ by cyclic permutations. Let π be a generator of this group, *i.e.* the elements of C_n are $\pi, \pi^2, \ldots, \pi^n = \mathrm{Id}$. For example,

$$\pi = \begin{pmatrix} 1 \ 2 \ 3 \ldots & n \\ n \ 1 \ 2 \ldots & n-1 \end{pmatrix}.$$

(a) We need Euler's totient function $\phi(n)$, which is the number of positive integers less than or equal to n that are relatively prime to n. Recall that

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right), \quad \text{where } p_1, \ldots, p_k \text{ are the prime factors of } n.$$

Thus, for example, $\phi(1) = 1$, and $\phi(p) = p - 1$ if p is prime, and

$$\phi(36) = \phi\left(3^2 \cdot 2^2\right) = 36 \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{2}\right) = 36 \cdot \frac{2}{3} \cdot \frac{1}{2} = 12.$$

[5] N. G. de Bruijn, Pólya's theory of counting. In (E. F. Beckenbach ed.) *Applied Combinatorial Mathematics*, John Wiley & Sons 1964.

(b) Let $1 \leqslant k \leqslant n$. Let $d = \gcd(k, n)$. There exist k' and n' such that

$$k = k'd, \quad n = n'd, \quad \text{and} \quad \gcd(k', n') = 1.$$

(c) With Euclid's algorithm we can find integers a and b such that

$$ak' + bn' = \gcd(k', n') = 1.$$

(d) Then

$$\pi^d = \pi^{d(ak'+bn')} = \pi^{ak'd} \cdot \pi^{bn'd} = \pi^{ak} \cdot \pi^{bn} = \pi^{ak},$$

since $\pi^n = \mathrm{Id}$. Thus π^d is an element of the subgroup generated by π^k.

(e) Also π^k is an element of the subgroup generated by π^d since $k = k'd$. We conclude that π^d and π^k generate the same cyclic subgroup and thus they have the same type (see Page 24 if you need to recall the type of a permutation).

(f) Actually, the permutation π^d consists of d cycles of length $\frac{n}{d}$.

(g) For every k such that $\gcd(n, k) = d$, π^k has the same type as above. The number of these permutations is the number of k with $1 \leqslant k \leqslant n$ and $\gcd(k, n) = d$.
This is the number of integers $\leqslant \frac{n}{d}$ and relatively prime to $\frac{n}{d}$; thus $\phi\left(\frac{n}{d}\right)$.

(h) The conclusion is that

$$P_{C_n,\pi}(x_1, x_2, \ldots) = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) x_{n/d}^d.$$

(i) The dihedral group D_n is like the cyclic group, but it includes reflections. Thus this is the group of symmetries of a regular polygon.
Prove that

$$P_{D_n,\pi}(x_1, \ldots, x_n) = \frac{1}{2} P_{C_n,\pi'}(x_1, \ldots, x_n) + \\ + \begin{cases} \frac{1}{2} x_1 x_2^{(n-1)/2} & \text{if } n \text{ is odd} \\ \frac{1}{4}(x_1^2 x_2^{(n-2)/2} + x_2^{n/2}) & \text{if } n \text{ is even.} \end{cases}$$

(j) Check that the cycle index of the hexagon in $\mathbb{R}^2$ under rotations is

$$P_{C_6,\pi}(x_1, \ldots, x_6) = \frac{1}{6}(x_1^6 + x_2^3 + 2x_3^2 + 2x_6).$$

(k) Check if the cycle index for the dihedral group D_6 agrees with Exercise 4.2.

(l) Consider beads of two colors; black and white. How many different ways are there to make a circular necklace with those beads of length 6? Two necklaces are the same if one is obtained from the other by a cyclic permutation. How many are there with exactly 3 white beads and exactly 3 black beads? How many are there with 3 purple beads and 3 violet beads, if you also allow a permutation of colors?

4.4. Check that the cycle index of the rotations of the tetrahedron is

$$P_{G,\pi}(x_1, x_2, x_3) = \frac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2).$$

Note that the group acting on the faces is the same as the group acting on the corners.

4.5. Consider the diagonals of the cube. What is the cycle index? Does your answer agree with the following formula

$$\frac{1}{24}(x_1^4 + 6x_4 + 3x_2^2 + 6x_1^2x_2 + 8x_1x_3)?$$

It is the same as the cycle index of the symmetric group S_4.

4.6. Consider the triangular prism. Find the cycle index, where you consider the permutations of the points under rotations only (thus "mirror" reflections of the prism are not included). Does your answer agree with the following

$$\frac{1}{6}(x_1^6 + 2x_3^2 + 3x_2^3)?$$

Now also include reflections. Prove that the cycle index is

$$\frac{1}{12}(x_1^6 + 2x_3^2 + 3x_2^3 + x_2^3 + 3x_1^2x_2^2 + 2x_6).$$

It is the same as the cycle index of the rotations of the hexagon.

4.7. What is the cycle index of the lines of the triangular prism?

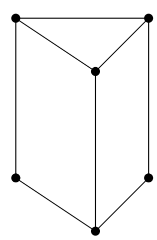

Fig. 4.17. A triangular prism

4.8. Consider the points of the triangular prism.

(a) How many ways are there to color the points of the triangular prism with 3 points white and 3 points black (both with and without reflections)?
(b) How many ways are there altogether to color the points either black or white?

(c) What about coloring the edges of the triangular prism black and white?
(d) Consider coloring the edges of the triangular prism with colors black and white, but allow also a permutation of colors. Thus two colorings are the same if one can be obtained from the other by a rotation and/or a switch in the colors. How many different colorings are there?

4.9 (de Bruijn[6]). Let G be a group of permutations of a set S and let H be a group of permutations of a set T. Assume that S and T are disjoint and let U be their union. For each choice $g \in G$ and $h \in H$ there corresponds a permutation u of U defined by

$$u \to \begin{cases} gu & \text{if} \quad u \in S \quad \text{and} \\ hu & \text{if} \quad u \in T. \end{cases}$$

Denote this permutation of U by $u = g \times h$.

(a) Show that these permutations form a group of order $|G| \cdot |H|$.
N. G. de Bruijn calls this group the direct product of G and H and denotes it by $G \times H$. Other authors[7,8] call this group the direct sum and denote it by, *e.g.*, $(G, S) \oplus (H, T)$.
(b) If $g \in G$ is of type $(b_1, b_2, \ldots)$ and $h \in H$ is of type $(c_1, c_2, \ldots)$ then $g \times h$ is of type $(b_1 + c_1, b_2 + c_2, \ldots)$, since each cycle in U lies either entirely in S or entirely in T.
(c) The term of the cycle index of $G \times H$ corresponding to the element $g \times h$ is equal to the product of the term in P_G corresponding to g and the term in P_H corresponding to the h. Thus

$$P_{G \times H} = P_G \cdot P_H.$$

4.10 (de Bruijn[9]). In this exercise we generalize Theorem 4.8 on page 51. Let G be a group and let π be a representation which assigns a permutation of a finite set X to each element of G. Thus we assume that

$$\forall_{g_1, g_2 \in G} \quad \pi(g_1)\pi(g_2)^{-1} = \pi(g_1 g_2^{-1}).$$

Define a relation $\sim$ on X by

$$x \sim y \quad \text{if} \quad \exists_{g \in G}\ x = \pi(g)y.$$

(a) Prove that $\sim$ is an equivalence relation, (the equivalence classes are called (G, π)-patterns).

[6] N. G. de Bruijn, Pólya's theory of counting. In (E. F. Beckenbach ed.) *Applied Combinatorial Mathematics*, John Wiley & Sons 1964.
[7] F. Harary, Exponentiation of permutation groups, *American Mathematical Monthly* **66** (1959), pp. 572–575.
[8] V. Krishnamurthy, *Combinatorics–theory and applications*, Ellis Horwood, 1986.
[9] N. G. de Bruijn, A note on the Cauchy-Frobenius lemma, *Indagationes Mathematicae* **41** (1979), pp. 225–228.

(b) Let $k \in G$. Prove that the map $\theta_k : G \to G$ defined by

$$\theta_k(g) = gk$$

is a bijection.

(c) Let Y be a complete set of representatives, *i.e.* Y consists of exactly one element of each equivalence class of X. We use the ν-indicator function that we introduced on Page 9. Prove

$$\forall_{x,y \in X} \quad x \sim y \;\Rightarrow\; \sum_{g \in G} \nu(\pi(g)x = x) = \sum_{g \in G} \nu(\pi(g)y = x).$$

(d) Now check the following derivation of the Cauchy-Frobenius lemma 4.2 on page 43.

$$\begin{aligned}
\sum_{g \in G} \sum_{x \in X} \nu(\pi(g)x = x) &= \sum_{g \in G} \sum_{y \in Y} \sum_{\substack{x \in X \\ x \sim y}} \nu(\pi(g)x = x) = \\
&= \sum_{y \in Y} \sum_{x \sim y} \sum_{g \in G} \nu(\pi(g)x = x) = \\
&\quad \text{(using (c))} \\
&= \sum_{y \in Y} \sum_{x \sim y} \sum_{g \in G} \nu(\pi(g)y = x) = \\
&= \sum_{y \in Y} \sum_{g \in G} \sum_{x \sim y} \nu(\pi(g)y = x) = \\
&\quad \text{(since there is exactly one } x\text{)} \\
&= \sum_{y \in Y} \sum_{g \in G} 1 = |Y| \cdot |G|.
\end{aligned}$$

(e) Let I be a finite set, and for each $i \in I$ let $\sigma_i : X \to X$ be a map. Let $\mathcal{A}$ be some additive, Abelian group and let $\omega : X \to \mathcal{A}$ be a weight function. Assume that

$$\forall_{x,y \in X} \quad x \sim y \;\Rightarrow\; \sum_{i \in I} \nu(\sigma_i x = x) = \sum_{i \in I} \nu(\sigma_i y = x). \tag{4.4}$$

Then, for any $x \in X$ (and Y a complete set of representatives),

$$\begin{aligned}
\sum_{i \in I} \nu(\sigma_i x = x) &= \sum_{\substack{y \in Y \\ y \sim x}} \sum_{i \in I} \nu(\sigma_i x = x) \\
&= \sum_{\substack{y \in Y \\ y \sim x}} \sum_{i \in I} \nu(\sigma_i y = x) \\
&= \sum_{y \in Y} \sum_{i \in I} \nu(x = \sigma_i y \sim y).
\end{aligned}$$

Sum over all $x \in X$, and multiply by weights:

$$\sum_{i\in I}\sum_{x\in X}\nu(\sigma_i x = x)\omega(x) = \sum_{y\in Y}\sum_{i\in I}\nu(\sigma_i y \sim y)\omega(\sigma_i y). \qquad (4.5)$$

(f) We specify as follows. Let $\rho : X \to X$ be *any* mapping (this is the generalization; in Theorem 4.8 we assumed that ρ is a permutation). Take $I = G$, and let $\sigma_k = \rho\pi(k)$, for $k \in G$.
Let's first check that (4.4) is satisfied; if $x \sim y$, then that means that there is some $h \in G$ such that $y = \pi(h)x$. Now clearly,

$$\sum_{k\in G}\nu(\rho\pi(k)x = x) = \sum_{k\in G}\nu(\rho(\pi(kh)x = x),$$

since if k runs through G then kh runs through G as well.

(g) Assume that ω is constant on each equivalence class; that is, we assume that

$$\forall_{x,y\in X} \quad x \sim y \;\Rightarrow\; \omega(x) = \omega(y).$$

Then (4.5) becomes:

$$\sum_{g\in G}\sum_{x\in X}\nu(\rho\pi(g)x = x)\omega(x) = \sum_{y\in Y}\sum_{g\in G}\nu(\rho\pi(g)y \sim y)\omega(y),$$

where Y is a complete set of representatives.
It follows that

$$\sum_{g\in G}\nu(\rho\pi(g)y \sim y)$$

does not change when y is replaced by another element in the same equivalence class, but this can easily be checked in other ways as well.

(h) Finally, we assume that ρ maps equivalence classes into equivalence classes:

$$\forall_{x,y\in X} \quad x \sim y \;\Rightarrow\; \rho x \sim \rho y.$$

Then

$$\rho y \sim y \quad\Rightarrow\quad \sum_{g\in G}\nu(\rho\pi(g)y \sim y) = |G|,$$

and thus (with Y a complete set of representatives)

$$\sum_{g\in G}\sum_{x\in X}\nu(\rho\pi(g)x = x)\omega(x) = |G| \cdot \sum_{y\in Y}\nu(\rho y \sim y)\omega(y).$$

The case where $\rho = \mathrm{Id}_X$ and $\omega(x) = 1$ for all $x \in X$ reduces to the Cauchy-Frobenius lemma that we proved in (d). The case where ρ is a permutation was used by de Bruijn[10] to obtain a Pólya-type theorem including a fixed

[10] N. G. de Bruijn, Color patterns that are invariant under a given permutation of the colors, *Journal of Combinatorial Theory* **2** (1967), pp. 418–421.

permutation h of the set R of colors. With the result above it follows that h needs not be a permutation; any mapping $h : R \to R$ gives exactly the same result. This is not a big surprise; if R' is a maximal subset of colors that is permuted by h, then no color outside R' is invariant under any positive power of h. Thus we can simply apply the theorem to R' instead.

4.11 (Balasubramanian[11]). Consider an array of objects

$$\begin{array}{c}(x_1, y_1)\ (x_1, y_2)\ \ldots\ (x_1, y_q)\\ \cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\\ (x_p, y_1)\ (x_p, y_2)\ \ldots\ (x_p, y_q)\end{array}$$

Consider permuting these objects within each row according to some $h \in H$ and then permuting the rows according to some $g \in G$. This set of permutations is denoted by $G[H]$ (the wreath product). If we denote an element of $G[H]$ as $(g; h_1, \ldots, h_p)$ then its effect on (x_i, y_j) is

$$(g; h_1, \ldots, h_p)(x_i, y_j) = (x_{g(i)}, y_{h_i(j)}) \quad \text{for } i = 1, \ldots, p \text{ and } j = 1, \ldots, q.$$

The group operation between elements of $G[H]$ is given by

$$(g; h_1, \ldots, h_p)(g'; h'_1, \ldots, h'_p) = (gg'; h_{g'(1)}h'_1, \ldots, h_{g'(p)}h'_p).$$

Consider polynomials A and B in indetertminates $s_1, s_2, \ldots$. Then the *plethysm* polynomial $A[B]$ is obtained as follows. For each r replace s_r in A by the polynomial obtained from B by multiplying by r the subscript of each indeterminate. For example, let

$$A = \frac{1}{2}(s_1^2 + s_2) \quad \text{and} \quad B = \frac{1}{6}(s_1^3 + 2s_3 + 3s_1s_2).$$

Then $A[B]$ is obtained by replacing s_1 in A by

$$\frac{1}{6}(s_1^3 + 2s_3 + 3s_1s_2)$$

and s_2 in A by

$$\frac{1}{6}(s_2^3 + 2s_6 + 3s_2s_4).$$

Thus

$$A[B] = \frac{1}{72}(s_1^6 + 4s_3^2 + 9s_1^2s_2^2 + 4s_1^3s_3 + 12s_1s_2s_3 + +6s_1^4s_2 + 6s_2^3 + 12s_6 + 18s_2s_4).$$

[11] K. Balasubramanian, Applications of combinatorics and graph theory to spectroscopy and quantum chemistry, *Chemical Reviews* **85** (1985), pp. 599–618. See the exposition in: V. Krishnamurthy, *Combinatorics—theory and applications*, Ellis Horwood, 1986.

Polya's theorem, Equation 4.3 on page 65 states that the cycle index of $G[H]$ is obtained by the plethysm of the cycle indices P_G and P_H:

$$P_{G[H]} = P_G[P_H].$$

(a) Show that the cycle index of $S_2[S_3]$ is the polynomial $A[B]$ described above.
(b) Consider the cycle indices of the cyclic groups C_2 and C_3 and show that the plethysm is

$$P_{C_2[C_3]} = \left(\frac{1}{2}(s_1^2 + s_2)\right)\left[\frac{1}{3}(s_1^3 + 2s_3)\right] = \frac{1}{18}(s_1^6 + 4s_3^2 + 4s_1^3 s_3 + 3s_2^3 + 6s_6).$$

(c) Consider the chemical molecule $C_2H_2Br_2Cl_2$ which consists of two carbon atoms, and two hydrogen, two bromine, and two chlorine atoms. Three of the hydrogen, bromine, and chlorine atoms are grouped around each carbon atom (see Figure 4.18 on the next page). Consider the following symmetries:
   (i) a rotation of the entire molecule around the midpoint of the CC-bond, and
   (ii) a local symmetry at each carbon atom, given by an action of the cyclic group C_3.

   Two equivalent molecules are called stereoisomers.
(d) Use weights B, C, and H, for the bromine, chlorine, and hydrogen atoms and compute the coefficient of $B^2C^2H^2$ in

$$P_{C_2[C_3]}(B + C + H, B^2 + C^2 + H^2, \ldots).$$

   Show that this is

$$\frac{1}{18}\left(\frac{6!}{2!2!2!} + 3\frac{3!}{1!1!1!}\right) = 6.$$

   Picture the 6 stereoisomers and check them with Figure 4.18.

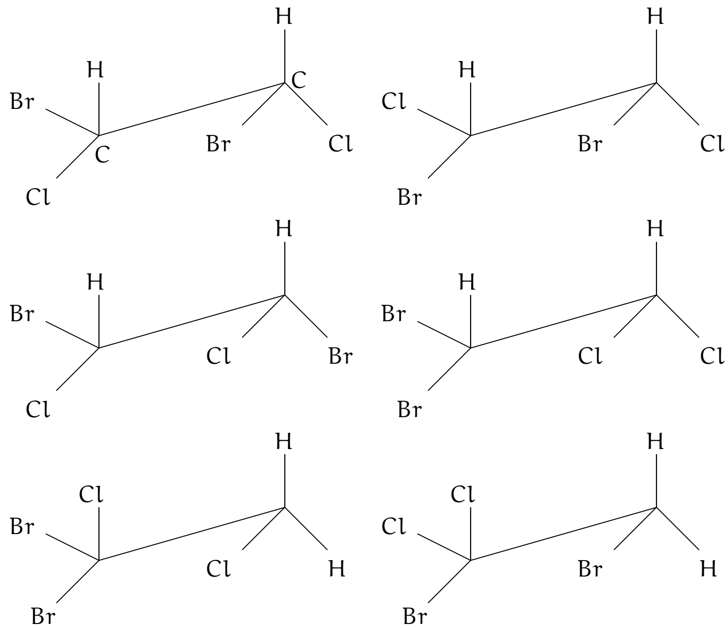

Fig. 4.18. The 6 stereoisomers of $C_2H_2Br_2Cl_2$

5

Graphs

We come to the second part of this course. Also this part is mainly about counting. You see things much more clearly when you count them. What we will do now is called graph theory.

5.1 Introduction

A graph is not somebody who can write, but the following examples do show graphs. The first graph is not called incomplete because it misses an arm; in a

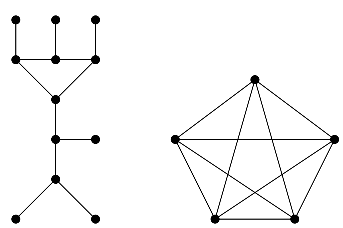

Fig. 5.1. Incomplete graph and complete 5-graph

complete graph all the possible connections are there.

Originally a graph was somebody who could write, some high official (what they call Doctors nowadays) who lived in some castle. The word graph in graph is etymologically the same as in photography, and there is no reason why we shouldn't use it also for some mathematical object. But joking apart; let's get serious.

Let G be a set and let $\mathcal{P}_2(G)$ be the collection of all subsets of G with two elements. Then a *graph on* G is a pair

$$(G, \Gamma) \quad \text{with} \quad G \neq \varnothing \quad \text{and} \quad \Gamma \subseteq \mathcal{P}_2(G).$$

The pairs of points that are connected in the picture above are given by Γ. Examples of graphs and non-graphs according to this definition are:

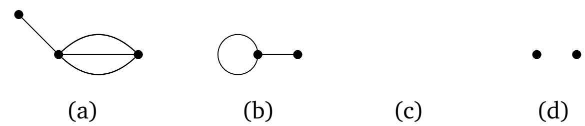

Fig. 5.2. Examples of graphs and non-graphs: (a)-(c) non-graphs; (d) a graph

The elements of G are called points, or nodes, or vertices, the elements of Γ are called connections, or lines, or edges (in analogy with geometric complexes). Before we go on, we mention one other important concept, namely equivalence, or better, isomorphism:

$$(G_1, \Gamma_1) \quad \text{and} \quad (G_2, \Gamma_2) \quad \text{are isomorphic if there is a bijection}$$

$$\phi : G_1 \to G_2 \quad \text{such that} \quad \phi\Gamma_1 = \Gamma_2,$$

that is

$$\{g, h\} \in \Gamma_1 \Leftrightarrow \{\phi g, \phi h\} \in \Gamma_2.$$

Of course this is "abus de langage" as Bourbaki puts it so eloquently in French. We mean that

$$\Gamma_2 = \{\, \{\phi p, \phi q\} \mid \{p, q\} \in \Gamma_1 \quad \text{and} \quad p, q \in G_1\},$$

or also

$$\Gamma_2 = \{A \in \mathcal{P}_2(G_2) \mid \exists_{p \in G_1} \exists_{q \in G_1} \; A = \{\phi p, \phi q\}\,\}.$$

Some people call an equivalence class in this equivalence relation a graph. You should always pay attention. Sometimes you don't want to identify equivalent graphs, and sometimes you do.

5.2 More introduction

Graph theory is a field with many names and agreements. Much of the terminology you can find in the book by C. Berge. Usually, when we use some terminology, we just say what we mean.

We mention the concept of an oriented graph (it is *not* a graph). An oriented graph is a pair

$$(G, \Delta) \quad \text{where } G \text{ is a set and } \Delta \subseteq G \times G.$$

Thus Δ consists of *pairs*, with a first element and a second element. For example the oriented graph

$$(\{1, 2, 3\}, \{(1, 2), (1, 3), (3, 1), (1, 1)\})$$

is depicted in Figure 5.3. With oriented graphs we speak about the head and

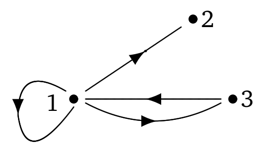

Fig. 5.3. Oriented graph

tail of a connection.

Planar graphs are graphs. They are graphs that can be embedded in the plane (with points and continuous curves). Below we show a picture of the complete graph with 4 points. This one is planar, but not every finite graph is planar. We will come back to this in a minute.

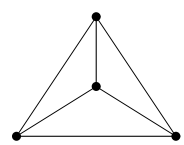

Fig. 5.4. Example of a planar graph

Every finite graph can be embedded in $\mathbb{R}^3$ with a little trick: Put $|G|$ points on a sphere, such that no 4 are in a plane. You can do that because in every step of the construction there are only finitely many planes "forbidden," and there are infinitely many points on the sphere left. We call this collection of points G. The complete graph on G has a realization: if two straight lines between points would cross each other, then the endpoints lie in a plane.

There are two famous isomorphism-classes of nonplanar graphs. The first is called K_5 and has $|G| = 5$ and $\Gamma = \mathcal{P}_2(G)$ and the second one is called $K_{3,3}$ and has $|G| = 6$, and can be written as $G = G_1 \cup G_2$, with $|G_1| = |G_2| = 3$ and $\Gamma = \{\{g_1, g_2\} \mid g_1 \in G_1, g_2 \in G_2\}$. An important theorem of Kuratowski says that these are more or less the only bad guys:

Every nonplanar graph has a subgraph that is a delayed K_5 or a delayed $K_{3,3}$.

We illustrate the notion of a delayed graph with an example.

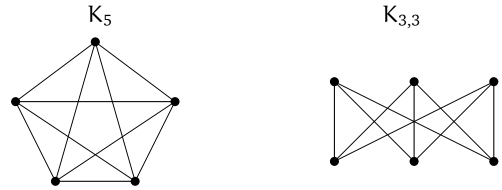

Fig. 5.5. K_5 and $K_{3,3}$

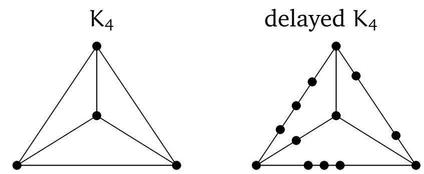

Fig. 5.6. Example of a delayed K_4

Some graphs can be embedded in the plane in two basically different ways. The two graphs below are isomorphic, but their embeddings are topologically not equivalent.

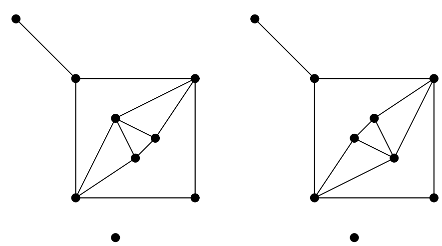

Fig. 5.7. Two topologically non-equivalent embeddings of isomorphic graphs

Degree.

Let (G, Γ) be a graph and let $P \in G$.

$$\text{The degree of } P \;=\; \text{the number } Q \in G \text{ with } \{Q, P\} \in \Gamma.$$

Let us draw the graph of the last figure again, this time adding the degrees of the points as shown in Figure 5.8.

A *tip* of G is a vertex of G with degree 1.

In an oriented graph we speak about indegree and outdegree. For example, the indegree of P in Figure 5.9 is 4 and the outdegree of P is 3. Note that a connection of P to P counts for indegree as well as for outdegree!

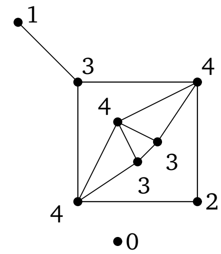

Fig. 5.8. Graph with degrees

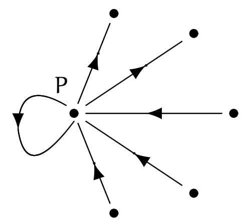

Fig. 5.9. Indegree of P is 4 and outdegree of P is 3

We now come to a few concepts that cause confusion as often as not:

Chains and circuits.

Let $n \in \mathbb{N}_0$. A chain of length n is a sequence of points $P_0, P_1, \ldots, P_n$ with

1. $\forall_{i \in \{0,\ldots,n\}}\ P_i \in G$, and
2. $\forall_{i \in \{1,\ldots,n\}}\ \{P_{i-1}, P_i\} \in \Gamma$.

Sometimes we put extra conditions on a chain.

(a) Chain without repeated points: $i \neq j \Rightarrow P_i \neq P_j$.[1]
(b) Chain without repeated lines:[2]

$$i > j > 0 \ \Rightarrow\ \{P_{i-1}, P_i\} \neq \{P_{j-1}, P_j\}.$$

Note that with repeated lines, the same line can be traversed in either the same "direction" or in the opposite direction; see Figure 5.10.

A closed chain is a chain $P_0, \ldots, P_n$ with $P_0 = P_n$.

A circuit is a closed chain without repeated lines. The length of a circuit is the number of lines in it.[3]

[1] Chains without repeated points are called paths.
[2] A chain without repeated lines is called a trail.
[3] A circuit without repeated points other than the start and endpoint is called a cycle. The length of a cycle is the number of lines in it.

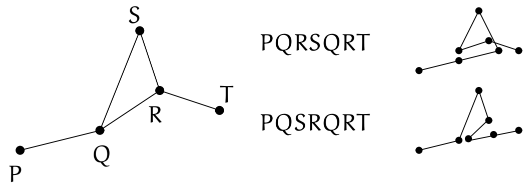

Fig. 5.10. With repeated lines: PQRSQRT and PQSRQRT

For circuits there are some identification possibilities. You can start at another point:

$$P_0, P_1, \dots, P_n \quad \text{becomes} \quad P_j, P_{j+1}, \dots, P_0, P_n, \dots, P_{j-1}$$

or you can reverse the order in which you traverse the circuit:

$$P_0, \dots, P_n \quad \text{becomes} \quad P_n, P_{n-1}, \dots, P_0.$$

A graph (G, Γ) is called connected if there is for every pair P and Q in G with $P \neq Q$ an $n \in \mathbb{N}_1$ and a chain $P_0, \dots, P_n$ with $P_0 = P$ and $P_n = Q$. "Chain-connectedness" is an equivalence relation, and the equivalence classes are called components (together with their lines).

5.3 Trees

There are a lot of ways to define trees. We do it like this:

A tree is a connected graph without circuits.

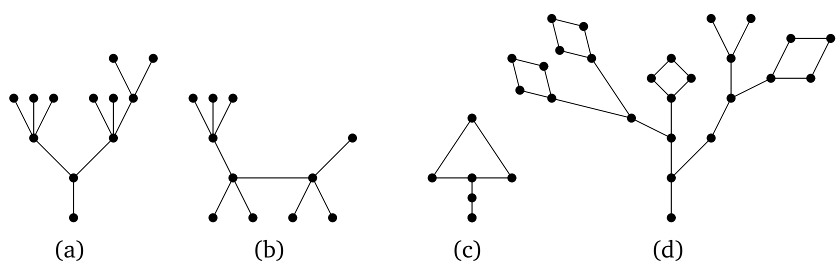

Fig. 5.11. Examples of trees and non-trees

Theorem 5.1. *A finite tree consists of either only* one *node,* i.e.*,* $|G| = 1$ *and* $\Gamma = \emptyset$*, or it has a least two tips,* i.e.*, two nodes of degree one.*

Proof. If (G, Γ) is connected, then we define for all $P, Q \in G$:

$$d(P, Q) = \begin{cases} \text{length } n \text{ of the shortest chain} \\ \text{that connects } P \text{ and } Q & \text{if} \quad P \neq Q \\ 0 & \text{if} \quad P = Q. \end{cases}$$

(Here $d(P, Q)$ counts the lines in a chain, not the points.)

Assume that $|G| > 1$. Then

$$\delta = \max\{\, d(P, Q) \mid (Q, P) \in G \times G\}$$

is attained in two points, say P and Q. We claim that P and Q are tips.

Assume that P is not a tip. Let $\{P, P_1\}$ be the first line in a shortest chain. Then there is another line $\{P, P'\}$ with $P' \neq P_1$. The point P' is connected to Q via some chain. And the distance from P' to Q is at most δ. Let

$$P' = P'_0, P'_1, \ldots, P'_k = Q$$

be a shortest chain. Let $m > 0$ be the first element in this sequence that also occurs in $P_0, \ldots, Q$. (NB that $P_0, \ldots, P_\delta = Q$ has no repeated points, since otherwise the chain would not be of minimal length. Likewise, there are no repeated points in $P' = P'_0, \ldots, P'_k = Q$.)

Now

$$P'_0, \ldots, P'_m (= P_\ell) P_{\ell-1}, \ldots, P_0, P'_0$$

is a closed chain. If $\ell > 0$ then this chain has no repeated lines, since there are no repeated lines in $P'_0, \ldots P'_m$, and none in $P_\ell, \ldots, P_0$, and a line in $P'_0, \ldots, P'_{m-1}$ does not occur in $P_\ell, \ldots, P'_0$. The line $\{P'_{m-1}, P'_m\}$ can only occur in $P_\ell, \ldots, P_0, P'_0$ if it is $\{P'_0, P_0\}$, but then $\ell = 0$.

Since a tree has no circuits, we have that $\ell = 0$. In other words, the shortest connection between P' and Q goes via P, and so

$$k = d(P', Q) = \delta + 1.$$

Thus $\delta + 1 \leqslant k \leqslant \delta$, which is a contradiction. Thus P is a tip. Likewise Q is a tip. □

Remark 5.2. The proof relies on the following observation: Let A and B be points in a graph and let $A \neq B$. Assume that A and B are connected by a chain. Then A and B are connected by a chain without repeating points. This observation is easily checked by considering a shortest chain.

Intermezzo on ordering multisets

Let S be a linearly ordered set. That is, for any two distinct elements of S, one is the largest and one is the smallest. For example, $S = \mathbb{N}_1$ or $S = \mathbb{N}_0$ are linearly ordered sets. We order the set of multisets over S lexicographically as follows. Let

$$\phi : S \to \mathbb{N}_0 \quad \text{and} \quad \psi : S \to \mathbb{N}_0$$

be two such multisets. Then

$$\phi < \psi \quad \text{means that} \quad \exists_{k \in S} \ \phi(k) < \psi(k) \wedge \forall_{\ell \in S} \ \ell < k \Rightarrow \phi(\ell) = \psi(\ell).$$

The set of multisets thereby becomes linearly ordered, and if S is finite, then every nonempty subset Φ of multisets has a smallest element.

Proof. Let $\Phi \neq \varnothing$. For simplicity assume that $S = \{1, \dots n\}$. Let $\Phi_0 = \Phi$ and define recursively for $i \geqslant 1$,

$$\begin{aligned} k_i &= \text{the smallest element of } \{\phi(i) \mid \phi \in \Phi_{i-1}\} \text{ and} \\ \Phi_i &= \{\phi \in \Phi_{i-1} \mid \phi(i) = k_i\} \end{aligned}$$

up to k_n and Φ_n. None of the Φ_i are empty and Φ_n contains exactly one element; the smallest element of Φ. □

If $S = \mathbb{N}_1$ then you get a greatest lowerbound $i \to k_i$ which does not have to be in the set.

We apply this to graphs. Let (G, Γ) be a finite graph. Number the elements of Γ with numbers $1, 2, \dots, |\Gamma|$, anyway you like. Color a line of Γ red if it is a "circuit champion," that is, if there is a circuit in which this line has the lowest number.

Example 5.3. Here is an example of a numbering of a graph and the set of red lines $\{1, 2, 3, 4, 5, 7, 9, 13, 14\}$.

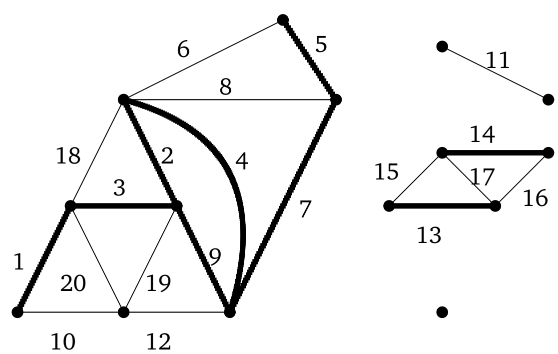

Fig. 5.12. Graph with numbered lines and subset of red lines

We color the rest of the lines blue.

Claim. We get blue trees and the red lines connect points in the trees. The set of points of each tree plus all the connections between them, form a component of the graph.

In other words, you get so-called "spanning trees," and together they form a spanning forest. We prove now that this is indeed the case.

Theorem 5.4. *If* P *and* Q *are connected by a chain, then also by a blue chain.*

Proof. Consider a chain from P to Q:

$$P = P_0, P_1, \ldots, P_n = Q.$$

Each chain like that corresponds to a multiset ϕ in Γ, namely

$$\phi : i \quad \to \quad \begin{cases} \text{the number of times that line } i \\ \text{occurs in the chain.} \end{cases}$$

Now take a chain with the *smallest* ϕ. By the intermezzo, that chain exists.

If it contains a red line, then that line was a circuit champion, and then you can replace that line with a chain that has only higher numbers. The resulting chain has a *smaller* ϕ. (If the number of the red line is a, then $\phi(a)$ is lowered, and $\phi(x)$ with $x < a$ is the same.) □

As a corollary we obtain:

Theorem 5.5. *If* (G, Γ) *is a graph and* Γ *is linearly ordered, then*

$$\#(\texttt{components}) + \#(\texttt{blue lines}) = \#(\texttt{points}).$$

Let's do one more:

Theorem 5.6.

$$\#(\texttt{components}) + \#(\texttt{lines}) - \#(\texttt{points}) =$$
$$= \begin{cases} 0 & \textit{if there are no circuits} \\ > 0\ (\#(\texttt{red lines})) & \textit{if there are circuits.} \end{cases}$$

The following theorem summarizes the result.

Theorem 5.7. *Let* (G, Γ) *be a finite graph. Then the following are equivalent.*

1) (G, Γ) *is a tree.*
2) $\#(\texttt{components}) = 1$ *and* $\#(\texttt{circuits}) = 0$.
3) $|G| \leqslant |\Gamma| + 1$ *and there are no circuits.*
4) $|G| \geqslant |\Gamma| + 1$ *and the graph is connected.*
5) *There is no circuit and every line-addition results in a circuit.*
6) *Connected and every line-removal breaks connectivity.*
7) *Every pair of points* P *and* Q, $P \neq Q$, *is connected by exactly* one *chain without repeating points.*

5.4 Prüfer code

In this section we discuss the Prüfer encoding of trees. A consequence of the Prüfer encoding is that there are n^{n-2} trees on a given set of n points (when $n \geqslant 2$.) If G is a set with n points then

$$\{\Gamma \subseteq \mathcal{P}_2(G) \mid (G, \Gamma) \text{ is a tree }\}$$

contains exactly n^{n-2} elements. Cayley proved this. For example, Figure 5.13 shows that there are 3 trees with 3 (numbered) points and 16 trees with 4 points.

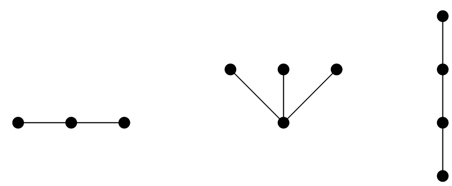

Fig. 5.13. Trees with 3 and 4 points

Example 5.8. By way of example we number the points of a tree arbitrarily as in Figure 5.14. Next, we make a sequence of $n - 2$ numbers (if $n > 2$) as

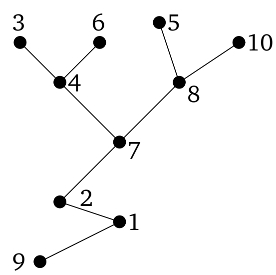

Fig. 5.14. Arbitrary numbering of the nodes of a tree

follows:

1. Take the tip with the lowest number.
2. Write down the *other* endpoint of the line incident with that tip.
3. Remove that tip and the incident line.

Repeat steps $1 - 3$ until there are only two points left.

The lowest tips for the tree in our example (with that numbering) are:

$$\begin{matrix} 3 & 5 & 6 & 4 & 9 & 1 & 2 & 7 \\ 4 & 8 & 4 & 7 & 1 & 2 & 7 & 8 \end{matrix}$$

the code is the sequence underneath.

The code that you get like that is of course unique. Furthermore, every sequence of $n-2$ numbers from $\{1,\dots,n\}$ corresponds uniquely with a tree.

The easiest way to prove this, is by giving the inverse algorithm.

code	missing elements	replace	by
$\underline{4}\,8\,4\,7\,1\,2\,7\,8$	3 5 6 9 10	4	3
$\underline{3}\,\underline{8}\,4\,7\,1\,2\,7\,8$	5 6 9 10	8	5
$3\,\underline{5}\,\underline{4}\,7\,1\,2\,7\,8$	6 9 10	4	6
$3\,5\,\underline{6}\,\underline{7}\,1\,2\,7\,8$	4 9 10	7	4
$3\,5\,6\,\underline{4}\,\underline{1}\,2\,7\,8$	9 10	1	9
$3\,5\,6\,4\,\underline{9}\,\underline{2}\,7\,8$	1 10	2	1
$3\,5\,6\,4\,9\,\underline{1}\,\underline{7}\,8$	2 10	7	2
$3\,5\,6\,4\,9\,1\,\underline{2}\,\underline{8}$	7 10	8	7
$3\,5\,6\,4\,9\,1\,2\,\underline{7}$	$\underline{8}\ \underline{10}$		

The pairs of underlined elements that appear in the same column, plus the last two missing elements, are exactly the $n-1$ lines of the tree.

How can you prove that this is the inverse algorithm? Of course by showing that every step is the inverse of the encoding algorithm. The original code started with 4, thus the tree has a line $\{\dots,4\}$. The tips of the original tree are not in the code, but all others are. The lowest one that is missing is 3, so this must be the tip that got thrown out. The removal of a tip, plus the incident line, changes the code for the inverse algorithm in

$$(\text{number of removed tip})(\text{code of the remaining tree}).$$

The lowest one that is now missing in the whole sequence is not the first item; so it is the lowest one that is missing in the remaining code.

Finally you have to show that it works for $n = 3$. Say the tree has lines $\{a,b\}$ and $\{b,c\}$. The code becomes b. Missing are a and c. The algorithm replaces b by a (if $a < c$) and marks $\{a,b\}$ as a line. Then the code becomes a. The ones missing are b and c, and this pair forms the last line. The case $c < a$ is of course similar.

This is all very easy to mechanize; in each step you only have to update one item in the code and only one item in the list of missing elements.

It is remarkable that the Prüfer encoding dates back to 1918, but the inverse algorithm dates back to only 1953 (Neville).[4]

[4] E. H. Neville, The codifying of tree-structure, *Mathematical Proceedings of the Cambridge Philosophical Society* **49** (1953), pp. 381–385.

5.5 Counting trees

We are going to count trees. We want to know how many *different* trees there are. But, as usual, we must figure out exactly what "different" means.

1. Ordinary equivalence classes. We call these topological trees (equivalent as topological spaces; homotopic embeddings in $\mathbb{R}^3$).
2. Plane trees. These are the homotopic equivalence classes in $\mathbb{R}^2$. The following figure shows two trees that are not the same when looked upon as plane trees. Speaking in combinatorial terms, this means that the branches

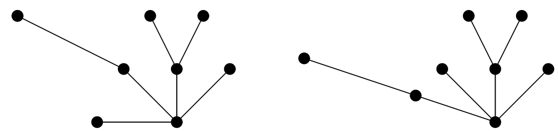

Fig. 5.15. Different plane trees

   that come together in a point P are given a cyclic order. So we have a class of bijections
$$\phi : \Gamma_P \to \{1, \ldots, \text{degree of } P\}$$
   where Γ_P is the collection of lines that meet in P. Bijections are equivalent if there is a cyclic permutation that maps one into the other.
3. Rooted trees. These are triples (G, Γ, r) where (G, Γ) is a tree and $r \in G$. In other words, rooted trees are trees with one special node, the root. The class of rooted trees can be subdivided into plane rooted trees and topologically equivalent rooted trees. Two plane rooted trees can only be equivalent if there is a bijection
$$\phi : (G, \gamma, r) \to (G', \Gamma', r')$$
   that maintains the cyclic orders (and also the roots).
4. Plane stemtrees. These are plane rooted trees, but such that there is an *order* at the root, instead of just a cyclic order.

Mathematics is a bit different than biology. In biology, a stemtree is a tree with a stem (where the root is a tip). But if the root is a tip, then the cyclic order at the other end is also "cut open." Therefore the choice of an order instead of a cyclic order is inherited through the whole tree.

In Figure 5.17 we show a topological tree and the topological rooted trees that go with it.

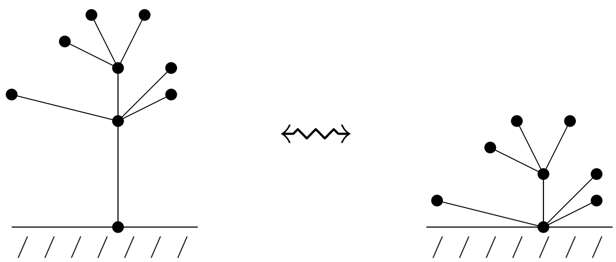

Fig. 5.16. Stemtree in biology and in combinatorics

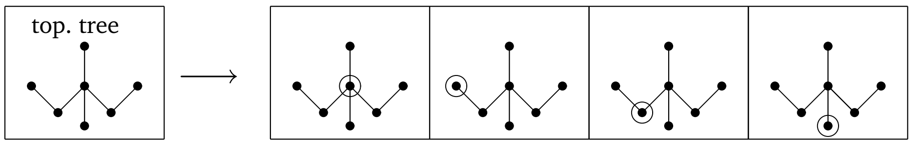

Fig. 5.17. Topological tree and topological rooted trees

The topological tree of Figure 5.17 splits into two classes of plane trees. See Figure 5.18. For each plane tree we depict the plane rooted trees. The numbers in the drawings of the plane rooted trees gives the number of plane stemtrees in which the class subdivides. This number shows in how many ways the cyclic order at the root can be cut open; so it is simply the degree of the root.

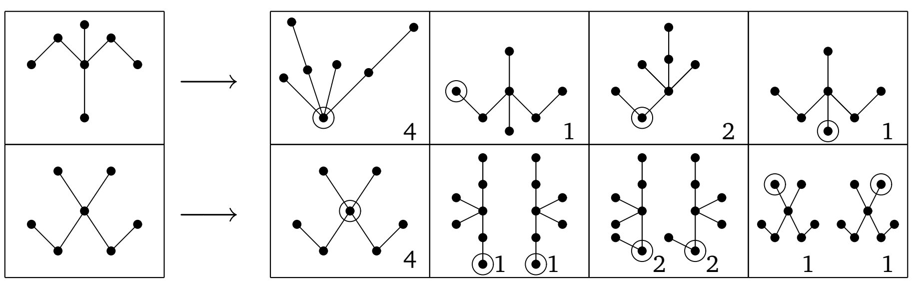

Fig. 5.18. Plane trees and plane rooted trees

In the following table we give a list of the smallest trees.

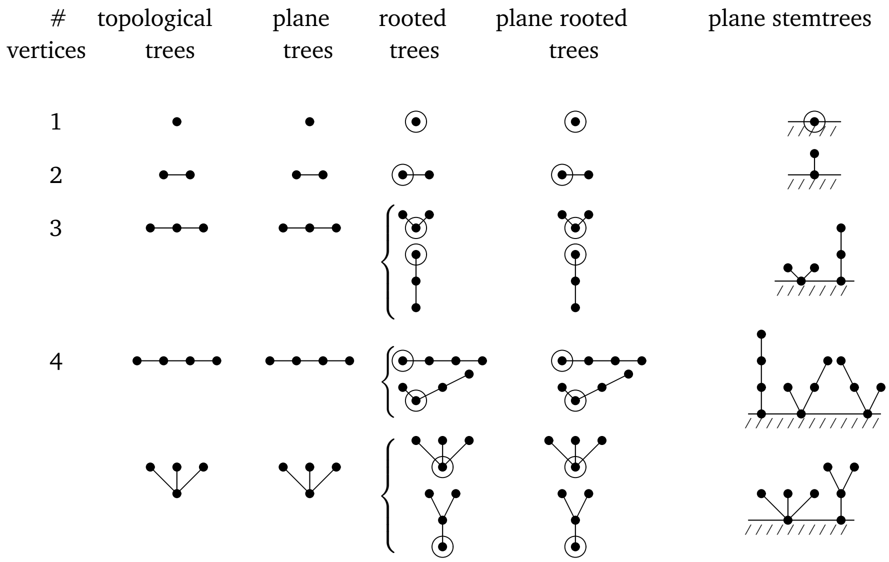

Fig. 5.19. Trees with at most 4 vertices

5.6 Cayley's functional equation

When counting topological trees we observe that a topological rooted tree consists of either one point, or a point with "something" growing out of it. That "something" is a multiset of rooted trees. Let $\mathcal{B}$ be the set of topological rooted trees. For $b \in \mathcal{B}$ we define

$$\nu(b) = \#(\text{points of } b).$$

To make a topological rooted tree with n points we choose $\phi : \mathcal{B} \to \mathbb{N}_0$ such that

$$\sum_{b \in \mathcal{B}} \nu(b)\phi(b) = n - 1.$$

For example, here is a list of $\mathcal{B}$:

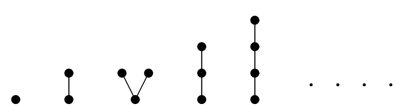

Fig. 5.20. first few elements of $\mathcal{B}$

If we define ϕ by the following table

2 1 0 3 0 0 (etc. 0)

then this ϕ corresponds to the following topological rooted tree:

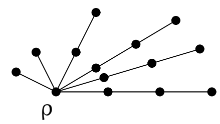

Fig. 5.21. Tree defined by ϕ

Namely, since it is a topological rooted tree, the order of the branches at the root doesn't matter.

We now do some counting that dates back more than a century.

$$F(x) = \sum_{b \in \mathcal{B}} x^{\nu(b)} = \sum_{n=1}^{\infty} c_n x^n$$

where c_n is the number of topological rooted trees with n points. So we have $F(x) = x + x^2 + 2x^3 + 4x^4 + \ldots$. By the arguments above:

$$\begin{aligned} F(x) &= x \prod_{b \in \mathcal{B}} (1 + x^{\nu(b)} + x^{2\nu(b)} + \ldots) \\ &= x \prod_{b \in \mathcal{B}} \frac{1}{1 - x^{\nu(b)}} \\ &= x \prod_{n=1}^{\infty} \left(\frac{1}{1 - x^n} \right)^{c_n} \end{aligned}$$

but also

$$\begin{aligned} &= x \exp \left(\sum_{b \in \mathcal{B}} \log \left(\frac{1}{1 - x^{\nu(b)}} \right) \right) \\ &= x \exp \left(\sum_{k=1}^{\infty} \sum_{b \in \mathcal{B}} \frac{x^{k\nu(b)}}{k} \right) \\ &= x \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} F(x^k) \right). \end{aligned}$$

This is the so-called functional equation of Cayley.

We will show that we can count also arbitrary trees with this generating function. This follows from an observation of Pólya and Otter (but it was known also to Jordan). See also Exercise 5.7. The observation is that every tree can be given an almost unique root.

Theorem 5.9. *Every tree has either*

i. *a unique central point, or* — *(All lines that come out of that point carry subtrees ("lobes") with* $< \frac{n}{2}$ *points.)*

ii. *a unique central line.* — *(Both lobes at that line have exactly* $\frac{n}{2}$ *points.)*

Proof. First we prove the uniqueness of the central point c, if it exists.

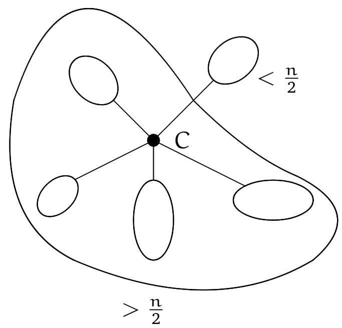

Fig. 5.22. Any other point has a lobe with more than $\frac{n}{2}$ points

Any other point has c in one of its lobes. That lobe has more than $\frac{n}{2}$ points.

The uniqueness of the central line, if it exists, follows by a similar argument.

We now use induction on n. For $n = 1, 2$ the claim is obvious. Assume that n is odd. A tree with $n + 1$ points, say B, becomes a tree with n points if we prune a tip together with its incident line. That tree has a unique point, say c. If the modified lobe has exactly $\frac{n-1}{2}$ points, then the line of c into that lobe is a central line of B. If the modified lobe has fewer than $\frac{n-1}{2}$ points then the point c is central in B.

Now assume that n is even. If there is a central line in the pruned B, then the endpoint of the central line that carries that pruned lobe is a central point of B. If there is a central point in the pruned B then this is also central in B. □

We continue our counting of trees. Next we count plane stemtrees. We order them according to the number of points, n.

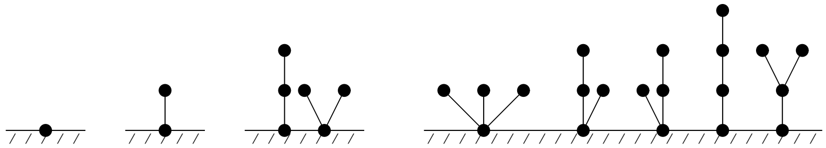

Fig. 5.23. Plane stemtrees with $n = 1$, $n = 2$, $n = 3$, and $n = 4$ points

The generating function

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \# \begin{pmatrix} \text{equivalence classes of plane} \\ \text{stemtrees with } n \text{ points} \end{pmatrix} x^n \\ &= x + x^2 + 2x^3 + 5x^4 + \ldots \end{aligned}$$

The coefficients are called the Catalan numbers (shifted one position; c_n is the coefficient of x^{n+1}):

$$c_n = \frac{(2n)!}{n!(n+1)!}, \quad \text{for example} \quad c_3 = \frac{6!}{3!4!} = \frac{5 \cdot 6}{1 \cdot 2 \cdot 3} = 5.$$

In the following we do our counting a bit informally. If you want to do things exactly then you have to use the theory of grammars. That is a different course and there they also don't do things that precisely, anyway. You'll just have to excuse us for doing things not in the neatest Bourbaki-way.

A plane stemtree looks like this:

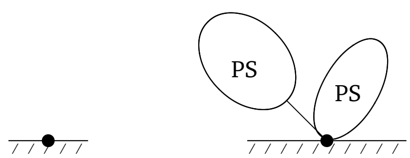

Fig. 5.24. A plane stemtree is either a point or it has a leftmost nonempty branch

If the plane stemtree has n points then it looks like this:

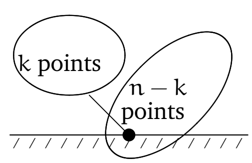

Fig. 5.25. The leftmost branch has k points and the rest has $n - k$ points

$$\text{Thus: } f(x) = x + f(x)f(x) \tag{5.1}$$

$$\text{and this gives } f(x) = \frac{1-\sqrt{1-4x}}{2} = \sum_{n=1}^{\infty} t_n x^n, \tag{5.2}$$

$$\text{with } t_n = c_{n-1}, \tag{5.3}$$

which follows easily from the power series expansion of $\sqrt{1-4x}$.

In another way you find the same result:

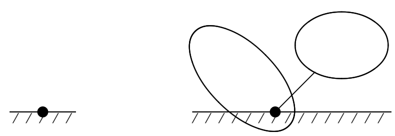

Fig. 5.26. This is just the mirror image

A third method gives the same result only after some calculations:

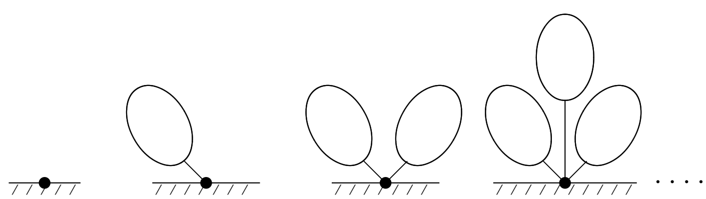

Fig. 5.27. A series expansion of plane stemtrees

This gives

$$f(x) = x + xf(x) + x\,(f(x))^2 + x\,(f(x))^3 + \ldots$$

In other words, $f(x) = \frac{x}{1-f(x)}$, from which Equation 5.1 follows again.

Let's consider a similar exercise with binary plane stemtrees, which we shall call simply binary trees for short. In every point we have either 0 or 2 branches growing up.

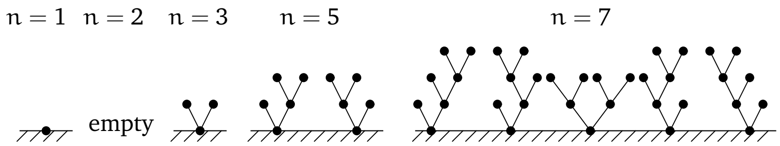

Fig. 5.28. Binary trees

The generating function starts like:

$$g(x) = x + x^3 + 2x^5 + 5x^7 + \dots$$

You can "make" a binary tree as follows:

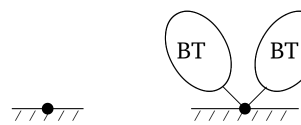

Fig. 5.29. A binary tree is either a point, or it has a root, and two binary trees are growing out of that root

$$\text{Thus: } g(x) = x + x\,(g(x))^2 \tag{5.4}$$

$$\text{and this reduces to } g(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x} \quad \text{etc.,} \tag{5.5}$$

but if we substitute

$$h(x) = \frac{f\left(x^2\right)}{x}$$

then we can find h from Equation 5.1:

$$\begin{aligned} h(x) &= \frac{1}{x}\left(x^2 + f\left(x^2\right) f(\left(x^2\right)\right) \\ &= \frac{1}{x}\left(x^2 + x^2 h^2(x)\right) = x\left(1 + (h(x))^2\right). \end{aligned}$$

Thus the solution of Equation 5.4 is $g(x) = \frac{f\left(x^2\right)}{x}$.

You can also see this as follows. We establish a 1-1 correspondence between binary trees and plane stemtrees. We draw a binary tree such that all the lines grow either in a positive x-direction or in a positive y-direction.

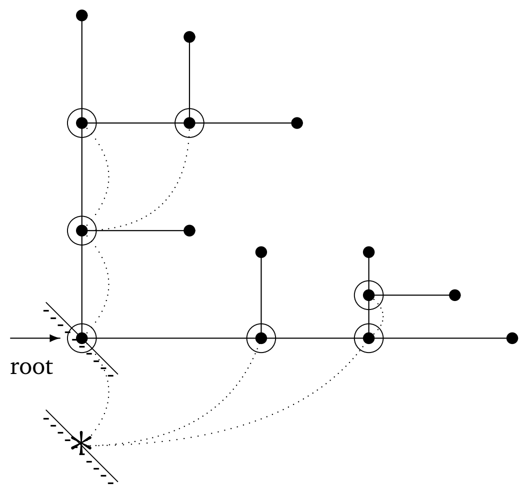

Fig. 5.30. Orthogonal drawing of a binary tree

From $*$, straight under the "root," we draw connections to the nontips of the tree (circled •s) on the horizontal branch directly above it (punctuated lines). From those points we repeat the procedure until there are only tips left.

The dotted lines form a tree; a plane stemtree.

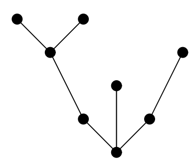

Fig. 5.31. The plane stemtree related to the binary tree above

To prove that this is a 1-1 correspondence is not straightforward. This becomes a little bit easier when we encode the trees in a formal way. We talk about the UD-encoding for plane stemtrees first; in the next section.

5.7 UD-encoding

Let's just do it by example.

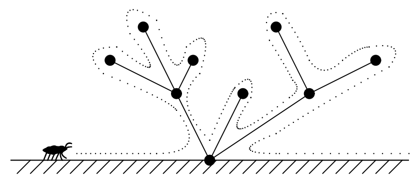

Fig. 5.32. A louse traversing a plane stemtree

This tree is planted in a two-dimensional universe and it is passed by a crawling louse. The louse has to follow the stipulated path to get to the other side of the tree. We encode the path of the louse by letting him (or her) shout at every line whether he or she is traversing this line in upward direction or in downward direction. The louse produces the following code when passing the tree in the example.

U U D U D U D D U D U U D U D D

Not all the UD-words are codewords. Which ones are?

We make a plot of the UD-word: every U corresponds with a unit linesegment going up at 45° and every D with a unit linesegment going down at 45°. Our diagram looks like this:

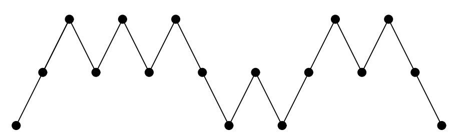

Fig. 5.33. UD-diagram

The diagrams of the codewords have the following properties:

(1) They start and end at 0, and
(2) they are always above ground: $\geqslant 0$.

The first requirement is especially important: the animal *must* land with both feet on the ground; otherwise our poor louse keeps hanging in the air.

The UD-requirements can be described in a number of ways.

Method 1.

We can make use of the “BNF-grammars” (Backus-Naur Form, developed for the description of the ALGOL 60 programming language).

$$< \text{UD-code} > ::= \ | \ \mathrm{U} < \text{UD-code} > \mathrm{D} < \text{UD-code} >$$

This corresponds with the first way to determine a generating function of a plane stemtree. Or also:

$$< \text{UD-code} >::= |\mathrm{U} < \text{UD-code} > \mathrm{D} \mid \mathrm{U} < \text{UD-code} > \mathrm{DU} < \text{UD-code} > \mathrm{D} \mid$$
$$\mid \mathrm{U} < \text{UD-code} > \mathrm{DU} < \text{UD-code} > \mathrm{DU} < \text{UD-code} > \mathrm{D} \mid \ldots$$

By the way, how do you get the tree back? Well, that's pretty simple. Take the diagram of the code and put some glue at the underside of the drawing. Let it dry a little bit (otherwise it won't stick) and then push it together.

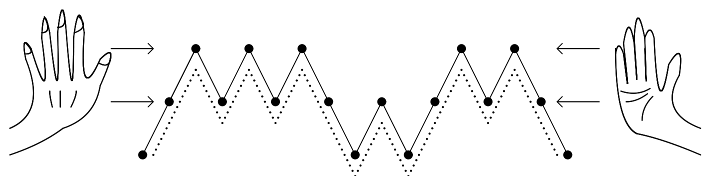

Fig. 5.34. Push the diagram together to get the plane stemtree back

Method 2.

Let d_i be the number of U's just before the i^{th} D and after the $(i-1)^{\text{th}}$ D. Then you get a sequence of numbers, all $\geqslant 0$ that satisfy (if there are 8 d's):

$$d_1 \geqslant 1;\ d_1 + d_2 \geqslant 2;\ d_1 + d_2 + d_3 \geqslant 3;\ \ldots\ d_1 + d_2 + \ldots + d_8 = 8.$$

In our example:

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
2	1	1	0	1	2	1	0

If we plot the partial sums, it looks like this:

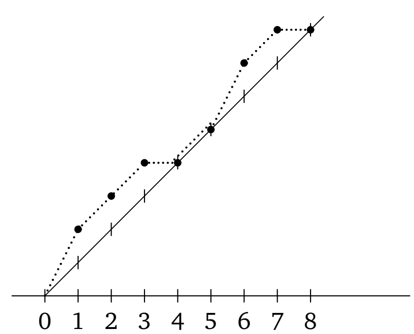

Fig. 5.35. Partial sums $d_1 + \ldots + d_i$, for $i = 1, \ldots, 8$

The Catalan numbers give the numbers of these kinds of diagrams.

The Catalan numbers turn up all over the place in combinatorics, and every time it is fun to discover the connections between those occurrences.

5.8 KE-encoding

We can encode plane stemtrees also in other ways. For example, put letters $a, b, c \ldots$ at lines that go up from a point, in clockwise order.

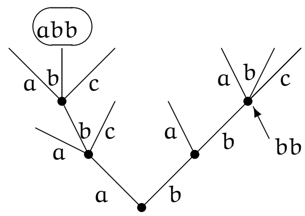

Fig. 5.36. Planar stemtree encoded with letters $a, b, c, \ldots$

We define the "track" at a point in the tree as the sequence of letters that you see when you travel from the root to that point.

Now let the louse shout the track of a point that he passes, but only when he sees that point for the first time. For the encoded tree in the example, the louse shouts the sequence:

∅
a
a a
a b
a b a
a b b
a b c
a c
b
b a
b b
b b a
b b b
b b c.

These are the words that the louse shouts are in alphabetic order.

The idea of "backtracking" is that sometimes, you can see at the start of a word that you don't need to go into that part of the tree; that you have no business there.

We are going to look at an encoding scheme for binary trees. Again the louse helps us. This time we use the KE ("knot–end") encoding. The louse, that is walking over the binary tree, notes only "knot" (internal point) or "end" (tip; leaf) and, furthermore, it only notes the *new* points.

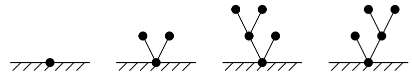

Fig. 5.37. KE-encodings of some binary trees

At the third tree the louse sees two knots, one after the other; he/she thinks: "what's the point of all this?"

Note that the KE-code ends with E. The "abbreviated KE-code," say aKE-code, is the same as the KE-code but without that last E. There are now as many K's as there are E's and the fun of it all is, that the aKE-code of a binary tree corresponds with the UD-code of the related stemtree.

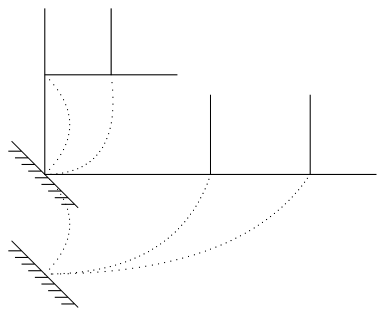

Fig. 5.38. Binary tree and plane stemtree

The stemtree with the dotted lines has the following UD-code and the binary tree has the aKE-code that's written underneath.

U U D U D D U D U D
K K E K E E K E K E

Let's see if we can generate the KE-code nicely:

$$< \mathsf{KE} - \text{code} >::= \mathsf{E} \mid \mathsf{K} < \mathsf{KE} - \text{code} >< \mathsf{KE} - \text{code} >$$

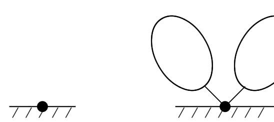

Fig. 5.39. A binary tree is either E or K followed by two binary trees

The aKE-code is defined by:

$$\begin{aligned} < aKE > &::= - \mid K < KE - \text{code} >< aKE > \\ &::= - \mid K < aKE > E < aKE > \end{aligned}$$

and now we see that the grammars for aKE and for UD are the same.

The aKE-code is related to the way to write an algebraic expression in (reversed) Polish. For example, look at the left picture first. We will explain the picture on the right in a minute.

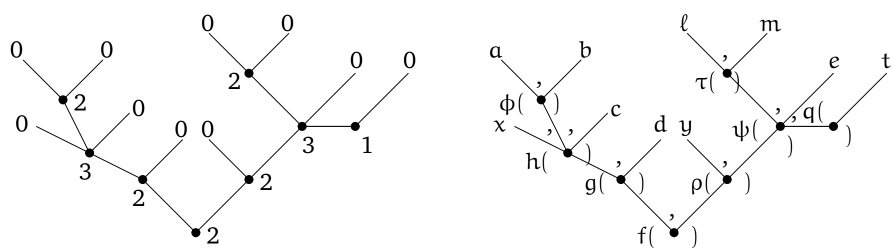

Fig. 5.40. Left: At every knot we write the number of outgoing lines. Right:Formula.

(A biologist immediately sees what kind of a tree this is by looking at the manner in which it has grown.)

At every knot we now write the number of lines that grow out of it (going up). When the tree is binary you only get zeros and twos. Instead of K or E the louse now calls the numbers:

$$2\;2\;3\;0\;2\;0\;0\;0\;0\;2\;0\;3\;2\;0\;0\;0\;1\;0.$$

Now compare this with an expression:

$$f(g(h(x, \phi(a, b), c), d), \rho(y, \psi(\tau(\ell, m), e, q(t)))).$$

This expression relates to the tree as follows. We put the variables at the points. At the internal knots we also put brackets and commas; depending whether the louse sees the knot for the first time, some intermediate time, or for the last time (see the figure). When the louse travels along the tree, it produces the formula.

You may just as well remove that mess of brackets and commas and instead put the number that the louse shouts in front of the symbol:

$${}_2\,f\;{}_2\,g\;{}_3\,h\;{}_0\,x\;{}_2\,\phi\;{}_0\,a\;{}_0\,b\;{}_0\,c \ldots \text{etc.}$$

The grammar of this kind of words is

$$\begin{aligned} < \text{code} >::= \;& 0 \mid 1 < \text{code} > \mid 2 < \text{code} >< \text{code} > \mid \\ & \mid 3 < \text{code} >< \text{code} >< \text{code} > \mid \quad \text{etc.} \end{aligned}$$

If you restrict yourself to binary trees then you get exactly the grammar of the KE-encoding.

Consider the sequence with numbers. If you subtract i from the i^{th} partial sum then you get a sequence that indicates the number of expressions that are still to come. Start with 1 for $i = 0$. The sequence of the example above changes as follows:

$$\begin{array}{r} 2\ 2\ 3\ 0\ 2\ 0\ 0\ 0\ 0\ 2\ 0\ 3\ 2\ 0\ 0\ 0\ 1\ 0 \\ 1\ 2\ 3\ 5\ 4\ 5\ 4\ 3\ 2\ 1\ 2\ 1\ 3\ 4\ 3\ 2\ 1\ 1\ 0 \\ \downarrow \\ \boxed{\text{Done!}} \end{array}$$

Russell paradox

This is the point where we can discuss the Russell-paradox[5] in its tree version. Some trees are isomorphic with a subtree at the first level, and others (for example all finite ones) are not. We call a tree special if it is isomorphic with a subtree in its first level.

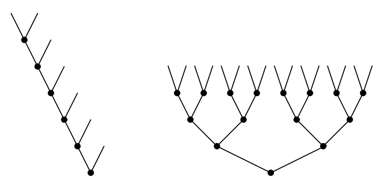

Fig. 5.41. Special trees

We now construct a tree that has as subtrees at the first level, *all* the non-special trees. Thus all the non-special trees are branches of one common root. Call this tree B. Note that B is special,

1) if and only if it does not appear in the first level
2) (by the definition of special) if and only if it does appear at the first level.

This is a contradiction.

5.9 Counting alcohols

We are going to do some special tree-counting; namely, we are going to count alcohol molecules (and you can find much more of these kind of things in that great article of Pólya from 1937). Actually, alcohol radicals: a chain of

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[5] B. Russell, *Principles of mathematics*, Cambridge University Press 1903.

carbon atoms connected to an OH-group, and the rest of the valencies filled with H-atoms. The simplest alcohol is therefore:

Fig. 5.42. The simplest alcohol (rather tasteless)

The general alcohol looks like:

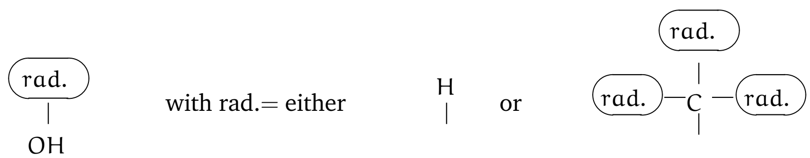

Fig. 5.43. An alcohol radical looks like the simplest one or with the H-atom replaced by three radicals

The generating function is therefore

$$\tilde{F}(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{where}$$

c_n is the number of "plane" alcohols with n C-atoms. Thus we find:

$$\tilde{F}(x) = 1 + x\left(\tilde{F}(x)\right)^3 .$$

But actually, it doesn't really matter where each radical exactly attaches, more precisely:

$$\text{an alcohol radical} ::= \mathrm{H} \mid \text{a pattern of mappings of } \{1,2,3\} \to \text{ a set of alcohol radicals.}$$

By a "pattern" we mean an equivalence class under the action of S_3. If you think about this for a minute then you see that

$$D(x) = 1 + xP_{S_3}\left(D(x), D\left(x^2\right), \ldots\right)$$

where $D(x)$ is the generating function of the topological alcohols. Thus

$$D(x) = 1 + \frac{x}{6}\left(D(x)^3 + 2D\left(x^3\right) + 3D(x)D\left(x^2\right)\right) .$$

But, when we look through some polarizing filter at chemical compounds then we can see the optical activity (sometimes), and if we want to bring this into account then we should count only the *cyclic* permutations, that is, we should do the calculations with the alternating group A_3 instead of S_3. If we do that, we find

$$E(x) = 1 + \frac{x}{3}\left((E(x))^3 + 2E\left(x^3\right)\right)$$

for the generating function of the ordinary alcohols–with–stereo-isometry.

5.10 The matrix tree theorem

A tree is a special kind of graph, we have to keep that in mind.

We can orient a tree inductively as follows.

(i) Choose a "root" P_1.
(ii) Take a tip $P \neq P_1$. There is a shortest path from P to P_1. Put an arrow along the first line of this path pointing from P towards P_1.
(iii) Remove P and the line that carries the arrow from the tree and repeat this procedure.

In this way we obtain an oriented rooted tree, with an orientation directed towards P_1.

We make a remark.

Theorem 5.10. *A finite graph with a distinguished point* P_1*, that can be oriented such that*

1) *out of every node* $\neq P_1$ *leaves exactly one arrow and out of* P_1 *leaves no arrow; and*
2) *such that there are no oriented circuits,*

is a tree.

Proof. Method (I). Consider $P \neq P_1$. Walk out of P until you cannot go any further or until you reach a node that you already saw before. Because there are no oriented circuits that node must be P_1.

Here are still a few dirty details swept under the carpet:

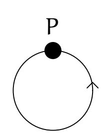

Fig. 5.44. A "loop" incident with P

(a) There can be no loops since that would be an oriented circuit.

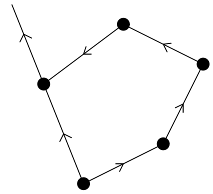

Fig. 5.45. Inconsistently oriented circuit

(b) Circuits that are oriented inconsistently are also not possible since there would be points with more than two outgoing arrows.

Disorient the graph. Then the graph is connected and there are no circuits. The result follows from Theorem 5.7 on page 83.

Method (II). Method (I) proves that the graph is connected. Furthermore we have that

$$\#(\texttt{points}) = \#(\texttt{lines}) + 1.$$

The result follows again from Theorem 5.7.

□

We start with a bit of theory developed by Kirchhof and Maxwell (ca 1850).[6] [7]

Start with a finite, oriented graph (G, Δ). A spanning tree for (G, Δ) is an oriented graph (G, E) such that $E \subseteq \Delta$ and such that (G, E) an oriented tree (oriented towards a root).

For example take Figure 5.46 (for the moment don't look at the dotted lines).

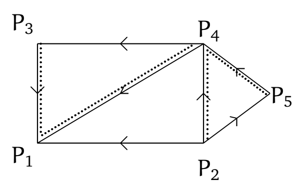

Fig. 5.46. Oriented graph and spanning tree

We ask ourselves: how many spanning trees are there in the given graph with root P_1? An example of such a spanning tree is given by the dotted lines.

[6] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird, *Ann. Phys. Chem.* **72** (1947), pp. 497–508.

[7] J. C. Maxwell, *A treatise on electricity and magnetism*, Clarendon Press 1892.

Note that the arrows from P_3 to P_1 and from P_5 to P_4 must be in any spanning tree (rooted at P_1). From P_4 we have a choice; either we choose $P_4 \to P_1$ or $P_4 \to P_3$. And from P_2 we have three choices: $P_2 \to P_1$, $P_2 \to P_4$, or $P_2 \to P_5$. All combinations are possible. Thus in our example we have 6 spanning trees in total.

In this manner you can get an algorithm that finds all spanning trees quickly. But now we will count the number of possibilities. In order to do that we associate with the graph a matrix, of size $|G| \times |G|$:

In row i we write in the j^{th} column:

$$\begin{cases} \text{if } i \neq j: & -\#(\text{arrows from } P_i \text{ to } P_j) \\ \text{if } i = j: & \text{the total out-degree of } P_i. \end{cases}$$

The total column-sum is therefore 0.

For the example above the matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Theorem 5.11 (Matrix Tree Theorem). *The number of spanning trees with root* P_k *is the* (k, k)*-minor (the determinant of the matrix that results by removing the* k^{th} *row and* k^{th} *column from the matrix).*

In our example we get the determinant of

$$\begin{vmatrix} 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix} = 6.$$

Notice that the loops in an oriented graph don't play any role; they don't appear in the matrix and for the trees they are also of no importance; you may as well leave them out.

Here's the proof.

Proof. Assume $k = 1$. (We can accomplish that by rearranging rows and columns.) Now simply leave out all the arrows that go *out of* P_1. They don't play a role in the trees that are oriented towards P_1.

Case I: Out of every point $\neq P_1$ leaves exactly one arrow.

Ia: There is no oriented circuit. Now we are done, since the graph is already a tree. Renumber the the points such that all arrows point from high to low; that is, arrows (k, ℓ) are such that $\ell < k$.

In order to do that, take a tip $\neq P_1$ and give it number n. Leave out the tip and the incident line, and go on like that. For example, the tree in figure 5.47 can be numbered as illustrated.

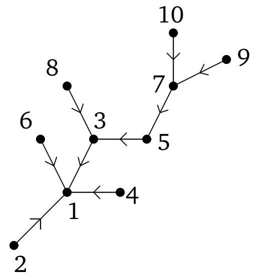

Fig. 5.47. Numbering of tree such that all arrows point from high to low

Now the matrix is lower-triangular (all elements above the diagonal are zero) and the diagonal has ones, namely the total out-degree of a point. Thus the $(1,1)$-minor is one.

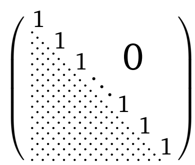

Fig. 5.48. Lower triangular matrix

Ib: There is an oriented circuit. Now the graph contains no tree, because when you land in that circuit you can't get out of it anymore.

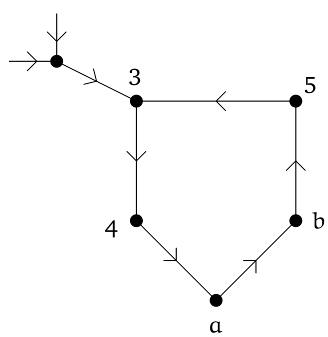

Fig. 5.49. Oriented cycle

The rows that belong to one cycle add up to zero: For every member of the cycle there is one column entry -1 (among those rows) and one column entry $+1$. Row 3 has a -1 in column 4. Row 4 has $+1$ in column 4 and otherwise zeros in rows $3, 4, a, b, \dots, 5$. The minor is therefore zero, since those rows are dependent.

Case II. Out of some point, say P_2, leaves more than one arrow. Any split of those arrows out of P_2 in two nonempty groups, say red and blue, divides the spanning trees into two kinds:

1. trees with the arrows going out of P_2 red, and
2. trees with the arrows going out of P_2 blue.

Now let $\Delta_1 = \Delta \setminus \{\text{red arrows}\}$ and let $\Delta_2 = \Delta \setminus \{\text{blue arrows}\}$. Then the total number of trees in (G, Δ) is equal to $\#(\text{in } (G, \Delta_1)) + \#(\text{in } (G, \Delta_2))$. But also, the second row in the matrix of (G, Δ) is the sum of the second rows in the matrices of (G, Δ_1) and (G, Δ_2), and thus the determinant is the sum of the determinants that go with these graphs. The claim follows by induction.

□

For a non-oriented graph you can calculate the same thing. Just change every line into two arrows, and apply the previous theorem. The undirected graph from the previous example,

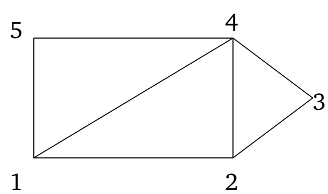

Fig. 5.50. Undirected graph

has a matrix

$$\begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and the number of trees with root 1 is therefore

$$\begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 4 & -1 \\ -2 & -2 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -2 & 7 \end{vmatrix} =$$

$$= 3 \cdot 12 + 1 \cdot (-9) - 2 \cdot 3 = 36 - 15 = 21.$$

As a special application we can count again the number of labeled trees with n points. Apply the previous theorem to the complete n-graph. The determinant that has to be calculated becomes:

$$\begin{vmatrix} (n-1) & -1 & -1 \dots & -1 \\ -1 & (n-1) & -1 \dots & -1 \\ -1 & -1 & \dots\dots & -1 \\ \dots & \dots & \dots & \dots \\ -1 & -1 & \dots\dots & (n-1) \end{vmatrix}$$

(with $n-1$ rows and columns). This is a symmetric matrix. You can see the eigenvalues blindfolded:

$\lambda = n$ $\quad$ ($\mathrm{rk}(A - nI) = 1$, thus multiplicity is $n-2$)
$\lambda = 1$ $\quad$ ($\mathrm{rk}(A - I) = n-2$ (has to be) thus multiplicity is 1.)

The determinant is the product of the eigenvalues which is n^{n-2}. This is Cayley's theorem again. NB Of course there are other ways to calculate that determinant.

Another special case occurs when the in-degree equals the out-degree for every vertex. That makes the row-sum also zero. Let n be the number of points in the graph. The number of spanning trees rooted at point 1 is the $(1,1)$-minor indicated in Figure 5.51.

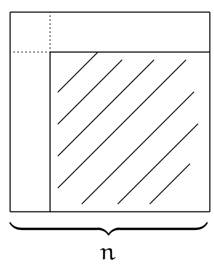

Fig. 5.51. Lower righthand minor

Adding all columns (except the first) to the rightmost column does not change the $(1,1)$-minor. If we do that, the rightmost column is -1 times the first column. We can now change the columns $(2,3,\dots,n)$ into columns $(n,2,3,\dots,n-1)$ by $n-1$ swaps. Each swap multiplies the $(1,1)$-minor by -1. This shows that the $(1,1)$-minor is equal to $(-1)^n$ times the $(1,n)$-minor shown in Figure 5.52.

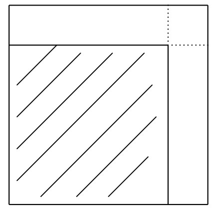

Fig. 5.52. Lower lefthand minor

But the (n, n)-minor shown in Figure 5.53 can be transformed in exactly the same manner as above into the $(1, n)$-minor of Figure 5.52.

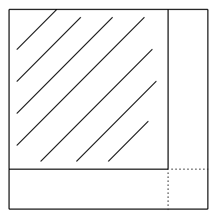

Fig. 5.53. Minor of the $(n \times n)^{\text{th}}$ element

This proves that the number of trees rooted at 1 is the same as the number of trees rooted at n and we conclude that the number of rooted trees is the same at every point.

The matrix that belongs to an ordinary graph is symmetric and by definition, the in-degree is equal to the out-degree for every point in the graph. Therefore, the number of trees in an ordinary graph does not depend on the root that you choose, which follows nicely from the calculations.

An analogous problem is that of covering a graph with dimers ("dumbbells" on the lines). For example:

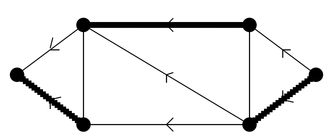

Fig. 5.54. Covering an oriented graph with dimers

You can solve this by finding a suitable, "Pfaffian" orientation Δ of the graph and by defining the $(i,j)^{\text{th}}$ element a_{ij} of a matrix A as

$$a_{ij} = \begin{cases} +1 & \text{if } (i,j) \in \Delta \\ -1 & \text{if } (j,i) \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

In this manner you get a skew-symmetric matrix and (without proof) the number of ways to cover G with dimers is exactly $\sqrt{\text{determinant}(A)}$ (see Exercise 6.16). Kasteleyn showed that every planar graph has a Pfaffian orientation.[8]

5.11 Euler graph

We come to Euler circuits. An Euler circuit is a circuit that uses all the lines of Γ exactly *one* time. This is named after the Königsberg's bridge problem.

An Euler-circuit collection is a collection of disjoint circuits that together use up all the lines exactly once. NB A Hamilton circuit is a circuit that visits all the points of a graph exactly once. We discuss Euler's famous theorem next.

Theorem 5.12. *Let (G, Γ) be a graph. Then all the degrees are even if and only if there is an Euler-circuit collection. Furthermore, there exists an Euler circuit if and only if the graph is connected and all the degrees are even.*

Proof. Obviously, an Euler-circuit collection can exist only if all the degrees are even. Assume that this is the case. We first prove that there exists an Euler-circuit collection.

(i) Assume P_1 has degree 0. Then remove P_1 from the graph.
(ii) Assume P_1 has degree $\neq 0$. Then start walking through the graph. Since all degrees are even, whenever you enter a point you can also leave it. Sooner or later you get back to vertex P_1. Then remove the lines of that circuit and start again.

Now we add the connectedness.

Assume that we have an Euler-circuit collection that is minimal. With minimal we mean that the number of elements in the collection is as small as possible. Claim: They are point-disjoint since, if they were not, then we could merge two of them together into one as illustrated in Figure 5.55.

[8] P. W. Kasteleyn, Graph theory and crystal physics. In (F. harary ed.) *Graph theory and theoretical physics*, Academic Press 1967, pp. 43–110.

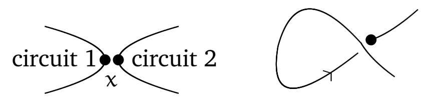

Fig. 5.55. Merge two circuits into one

□

A nice application is the beer-bottle problem.

Along a Falaise terrace there are two small paths going up. Two party members want to carry a load of beer upstairs to the party. They want to attempt this in the following way. Between them they carry a long plank that carries all the beer bottles and together they walk up. Of course they have to keep the plank horizontal, otherwise the whole load shifts right or left.

How to solve this problem?

Let the heightfunction of one path be $h_1(x) = H_1(x - x_0)$ and let the heightfuntion of the other path be $h_2(x) = H_2(x - x_1)$. Construct the set

$$V = \{(\xi, \eta) \mid h_1(\xi) = h_2(\eta)\}.$$

If the path consists of straight stretches then V is a "graph" with vertices of degrees 4 and 2 and start– and endpoint or degree 1. Connect the start and finish and notice that they are now contained in one component of the graph. That component has an Euler circuit. See Figures 5.56 and 5.57.

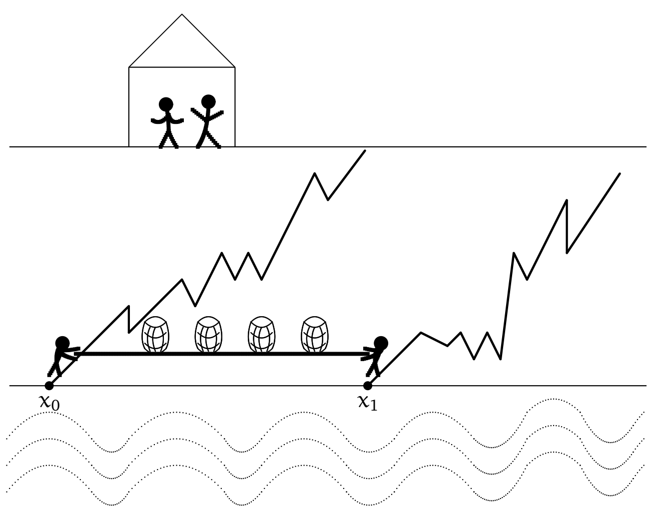

Fig. 5.56. Illustration of the beer-bottle problem

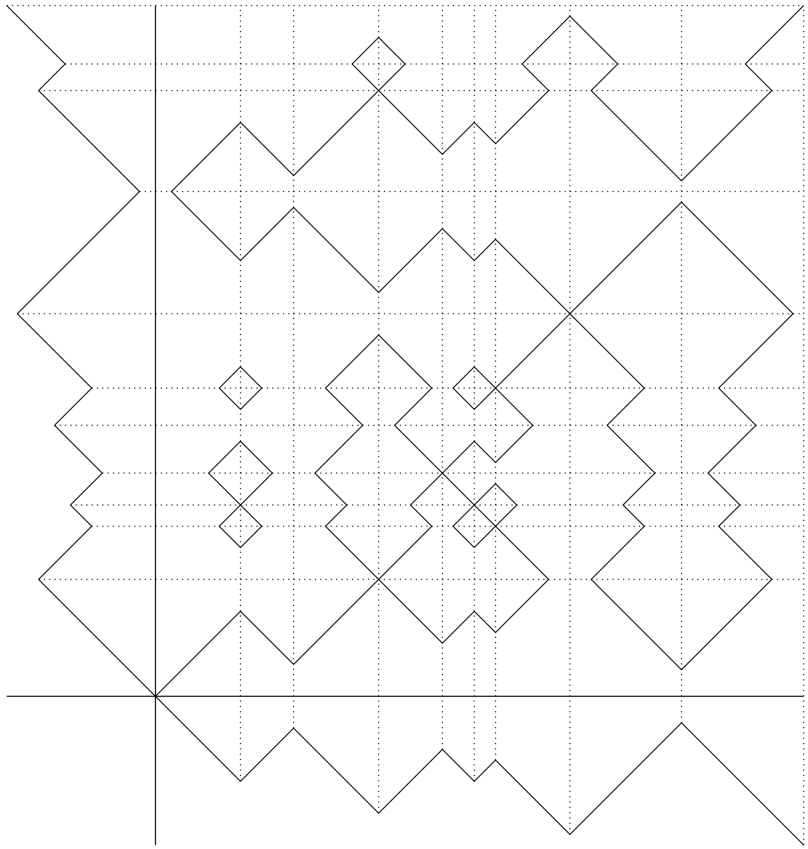

Fig. 5.57. Euler graph of the beer-bottle problem

For a brief moment we have a look at Hamilton paths. In Figure 5.58 we show a graph and a Hamilton circuit in there.

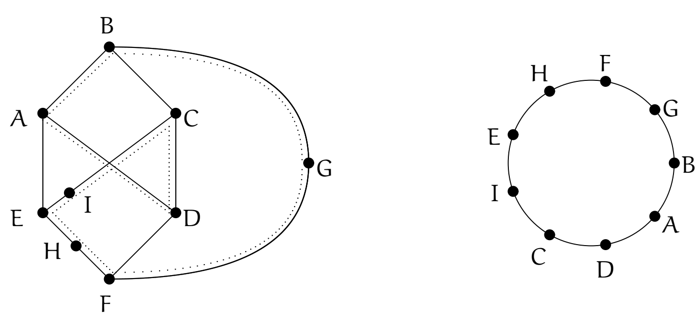

Fig. 5.58. Graph with Hamilton circuit

The theory of the Hamilton paths is boring and offers only few possibilities. Under certain conditions you can interpret the points as lines in another graph, and then there are possibilities using Euler paths and circuits.

Let us formulate Euler's theorem for oriented graphs.

Theorem 5.13. *Let* (G, Δ) *be a finite oriented graph. Assume that for every* $P \in G$ *the in-degree of* P *equals the out-degree of* P*. Then there exists a Euler-circuit collection. Furthermore, if* (G, Δ) *is "strongly connected" then there is a single Euler circuit.*

NB

1) Out-degree(P) = the number of Q such that $(P, Q) \in \Delta$.
2) In-degree(P) = the number of Q such that $(Q, P) \in \Delta$.
3) (G, Δ) is *strongly connected* if for every pair P and Q, $P \neq Q$, there is a path from P to Q:

$$P = P_0, P_1, \ldots, P_k = Q \quad \text{and} \quad (P_i, P_{i+1}) \in \Delta \quad \text{for} \quad i = 0, \ldots, k-1.$$

If we rub out the arrows of a strongly connected oriented graph and replace them by ordinary lines (and remove all double lines and loops) then we obtain a connected graph. This shows the connection between the two concepts. BTW, the concept of "strongly connectedness" is fabricated especially for this purpose.

Remark 5.14. BTW, the proof shows that for an oriented Euler graph (in-degree(P) equals out-degree(P) for all P) that, if the underlying undirected graph is connected, then the (directed) graph is strongly connected.

Proof. Simple. Start anywhere in the Euler graph, say in P_0. Lengthen the chain step by step. In every point that you enter there is an exit, except in P_0 where the exit has already been used.

Sooner or later you can't go any further because Δ is finite and so you must be back at P_0. In that way you carry on. If there are circuits that have points in common, then you can merge them into one circuit just as in the undirected case. Etc. (Finiteness is essential.) □

5.12 Linegraph

Let $\sigma \in \mathbb{N}_0$. We say that an Euler graph (G, Δ) is of type σ if for every $P \in G$

$$\text{in-degree}(P) = \sigma = \text{out-degree}(P).$$

Example 5.15. Let G be a set, then $(G, \varnothing)$ is an Euler graph of type 0.

Example 5.16. Another example:

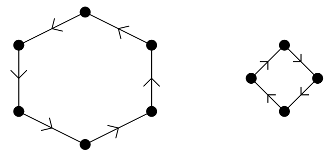

Fig. 5.59. Euler graph of type 1

We define the linegraph $\mathcal{L}(G, \Delta)$ of an oriented graph:

$$\mathcal{L}(G,\Delta) = (G_1, \Delta_1) \quad \text{where}$$

$$G_1 = \Delta \quad \text{and} \quad \begin{cases} \Delta_1 \text{ consists of all pairs of } \Delta \text{ that are in} \\ \text{a head-tail relation. Formally:} \\ \\ \Delta_1 = \{\, ((P,Q),(R,S)) \in \Delta \times \Delta \mid Q = R \,\}. \end{cases}$$

To put it even more casually; the arrows are now the points and the points now act as the connections between the lines. See Figure 5.60.

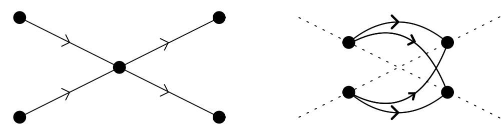

Fig. 5.60. Piece of an oriented graph transformed in a piece of the linegraph

Example 5.17. In Figure 5.61 we drew a graph and its linegraph in one picture.

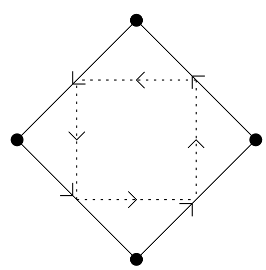

Fig. 5.61. Graph and linegraph in one picture

Remark 5.18. If (G,Δ) is an Euler graph of type σ then is $\mathcal{L}(G,\Delta)$ again an Euler graph of type σ. That is pretty obvious: The out-degree of a line d in Δ is the number of lines that go out of the head of d, etc.

Definition 5.19. $P_w(G,\Delta)$ *is the number of spanning, oriented rooted trees with a specified root in Euler graph* (G,Δ).

Remark 5.20. It does not matter which root you choose.

Definition 5.21. $P_E(G,\Delta)$ *is the number of Euler circuits in the oriented graph* (G,Δ).

Theorem 5.22. *Let* (G,Δ) *be an Euler graph of type* σ, $\sigma > 0$. *Then*

$$P_E(\mathcal{L}(G,\Delta)) = P_E(G,\Delta)\frac{1}{\sigma}(\sigma!)^{|G|(\sigma-1)}$$

Theorem 5.23. *Let* (G,Δ) *be an Euler graph. Then*

$$P_E(G,\Delta) = P_w(G,\Delta)\prod_{p\in G}(\sigma_p - 1)!$$

where

$$\sigma_p = \textit{ in-degree of } p = \textit{ out-degree of } p.$$

Example 5.24. $\sigma = 1$. Now $P_E(\mathcal{L}(G,\Delta)) = P_E(G,\Delta) = 1$. The linegraph

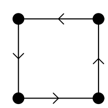

Fig. 5.62. $\sigma = 1$; 4-cycle

$\mathcal{L}(G,\Delta)$ is isomorphic to (G,Δ). Another example is illustrated in Figure 5.63.

5.13 De Bruijn sequence

An example how to use graphs and their linegraphs is the following. Consider the graph in Figure 5.64.

There are three consecutive points numbered as 000:

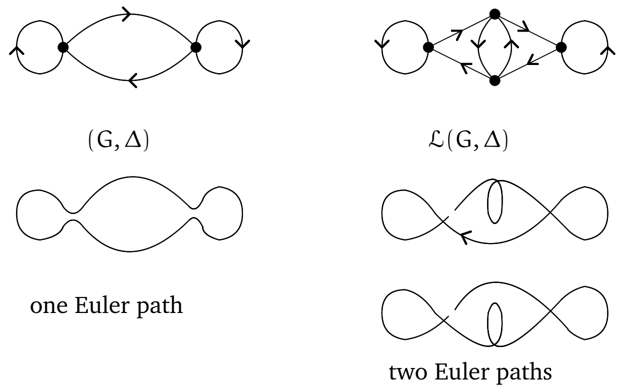

Fig. 5.63. $\sigma = 2$: very simple example

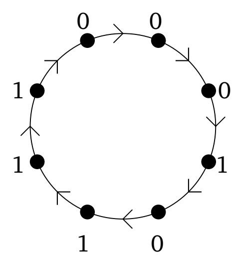

Fig. 5.64. Circuit with points labeled 0 and 1

	000
shift one position in the cycle:	001
etc.	010
	101
	011
	111
	110
	100

In this way we obtain all 3-letter words (with letters 0 and 1).

How do you construct a labeled cycle like that for an arbitrary wordlength?

A cyclic order of 0 and 1 as above is called a shift-register sequence.[9] In the old days this was an interesting problem for telex machines. Nowadays these things work totally differently.

We define the de Bruijn graph (G_n, Δ_n) as follows.

[9] A shift-register sequence is nowadays better known as a de Bruijn sequence.

$$G_n = \text{the set of all } n-1 \text{ letter words}$$

$$\Delta_n = \begin{cases} \text{consists of } F(G_{n+1})\text{: Let } (\epsilon_1, \ldots, \epsilon_n) \in G_{n+1}. \text{ Then} \\ F(\epsilon_1, \ldots, \epsilon_n) \in G_n \times G_n, \text{ namely the element:} \\ \qquad ((\epsilon_1, \ldots, \epsilon_{n-1}), (\epsilon_2, \ldots, \epsilon_n)) . \end{cases}$$

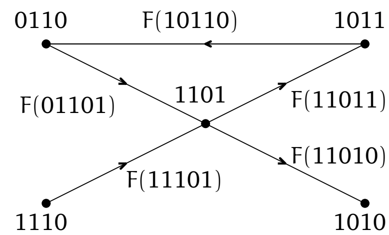

Fig. 5.65. Little piece of (G_5, Δ_5)

Example 5.25. Every vertex has two outgoing arrows: one extends the word at the head with a 0 and the other extends it with a 1. Both cut off one letter of the tail.

A shift-register sequence for 5 letters is an Euler circuit in (G_5, Δ_5).

Example 5.26. We show some examples in Figures 5.66 and 5.67.

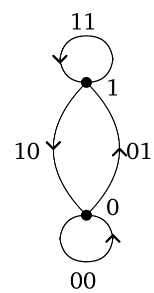

Fig. 5.66. Bit boring: (G_2, Δ_2)

We also see that $\mathcal{L}(G_n, \Delta_n) = (G_{n+1}, \Delta_{n+1})$.

We saw already that G_{n+1} can be looked upon as Δ_n. When are two points connected in $\mathcal{L}(G_n, \Delta_n)$? Point A of $\mathcal{L}(G_n, \Delta_n)$ is connected to B if the head of A equals the tail of B (Figure 5.68). That means that

$$A = \epsilon_1 a \quad \text{and} \quad B = a\epsilon_2,$$

in other words, A and B are the first part (n letters) and the last part (also n letters) of the word $\epsilon_1 a \epsilon_2$.

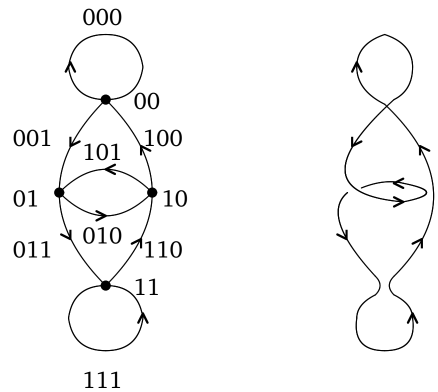

Fig. 5.67. (G_3, Δ_3): here is the circuit that we designed above as the Euler circuit

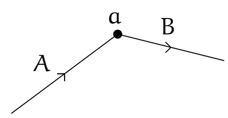

Fig. 5.68. Two lines A and B in head-tail relation; meeting in a

Let's put some of the numbers in a table:

Wordlength n	2	3	4	5	6
#(points in the cycle)	4	8	16	32	64
#(points in the graph)	2	4	8	16	32
# Euler circuits	1	2	16	2048	6710864
$^2\log$ # Euler circuits	0	1	4	11	26

Around 1945 an ingenieur[10] was doing some calculations on this table. Before he got to the last column he discovered that the numbers in the bottom row are $2^{n-1} - n$. N. G. de Bruijn added the last column. Then he discovered that a Frenchman, Flye St Marie[11] had done the same thing in 1845, and if you know how to do it, then his article is not too hard to understand.[12]

The formula is quite clear: On the basis of our general result, Theorem 5.22 on page 114, with the transition from (G_n, Δ_n) to (G_{n+1}, Δ_{n+1}) the number of Euler circuits gets multiplied by

[10] K. Posthumus.

[11] See Technical Report 75-06, Technical University Eindhoven; this contains an overview of the literature.

[12] C. Flye Sainte-Marie, Solution to question nr. 48, *L'intermédiaire des Mathématiciens* **1** (1894), pp. 107–110.

$$\frac{1}{\sigma}(\sigma!)^{|G_n|(\sigma-1)} = 2^{|G_n|-1}$$

since $\sigma = 2$. Thus (G_n consists of the 2^{n-1} words of length $n-1$)

$$\begin{aligned} {}^2\log(P_E(G_{n+1}, \Delta_{n+1})) &= {}^2\log(P_E(G_n, \Delta_n)) + 2^{n-1} - 1 \\ &= 2^{n-1} - (n-1) + 2^{n-1} - 1 = 2^n - n. \end{aligned}$$

We prove the theorem for the case that $\sigma = 2$. For more general proofs, see the remark at the end of the theorem.

First a simple case. Assume that the graph is a cycle with pimples: Figure 5.69. There is only *one* Euler circuit; a path that passes through node x

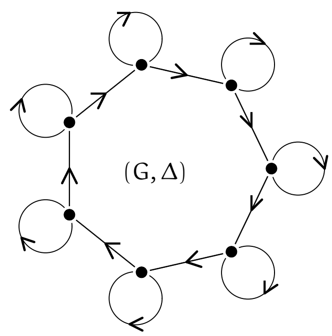

Fig. 5.69. Cycle with pimples

must also travel through the loop at x.

The linegraph looks as in Figure 5.70.

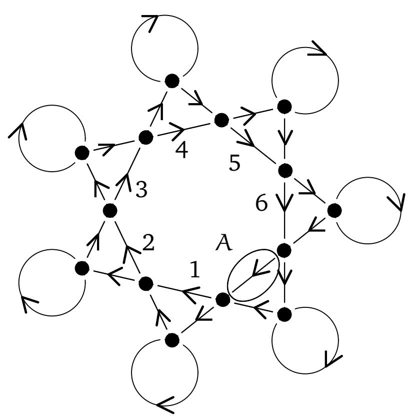

Fig. 5.70. Linegraph of the cycle with pimples

An Euler circuit can travel a vertex-plus-loop or not; alternatively it can pass by the vertex-plus-loop on the inside. If we start with the indicated line, then we are at $n-1$ steps back at A. (In this case $n = 7$). In each of those steps we have a choice between two possibilities. After one full turn the circuit is fixed; all paths that were avoided in the first turn, must now be traversed. So there are 2^{n-1} paths in $\mathcal{L}(G,\Delta)$, and this agrees with the general formula that we want to prove.

We now look into the general case. Assume that there is a point in G which is *not* a vertex-with-loop as in the picture 5.71. Note that if every vertex is a

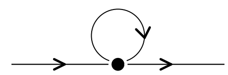

Fig. 5.71. Vertex with loop

vertex-with-loop, then we are done since this is the special pimple-case that we looked at. Thus now we may choose a point p without a loop.

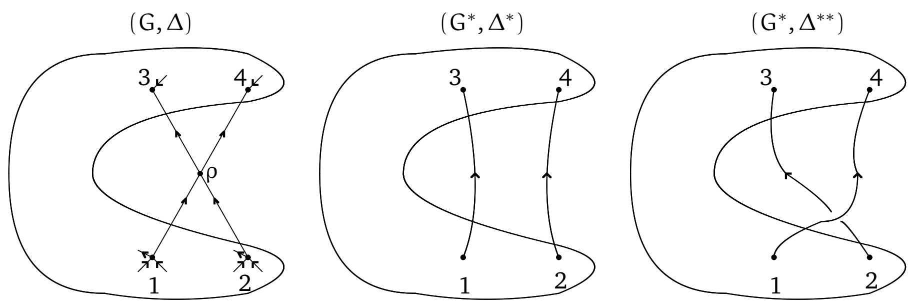

Fig. 5.72. A point p and the monster. In the second and third figure we leave $G \setminus \{p\}$ as it was, but we change something in the mouth of the monster

It'll take you a while to do this formally, but by looking at the figures you get the idea also.

Claim :

$$P_E(\mathcal{L}(G,\Delta)) = 2P_E(\mathcal{L}(G^*,\Delta^*)) + 2P_E(\mathcal{L}(G^*,\Delta^{**})). \tag{5.6}$$

Notice that this is exactly what we need for an induction step in the proof of Theorem 5.22 on page 114:

$$P_E(\mathcal{L}(G,\Delta)) = 2^{|G|-1}P_E(G,\Delta).$$

Namely, G^* and G^{**} both have one vertex less than G and

$$P_E(G,\Delta) = P_E(G^*,\Delta^*) + P_E(G^*,\Delta^{**}).$$

By induction

$$\begin{aligned}P_E(\mathcal{L}(G,\Delta)) &= 2\cdot 2^{|G^*|-1}P_E(G^*,\Delta^*) + 2\cdot 2^{|G^*|-1}P_E(G^*,\Delta^{**}) = \\ &= 2^{|G^*|}P_E(G,\Delta) = 2^{|G|-1}P_E(G,\Delta).\end{aligned}$$

Now we make the linegraph of this graph. For that we have to rub the sleep out of our eyes. Luckily most of the rubbish disappears into the monster and we don't care about that.

We show the linegraph of (G,Δ) in Figure 5.73. (A and B in this picture represent the incoming and outgoing arrows of 1,2,3, and 4.)

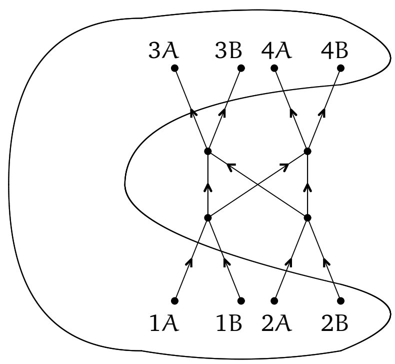

Fig. 5.73. Linegraph of (G,Δ)

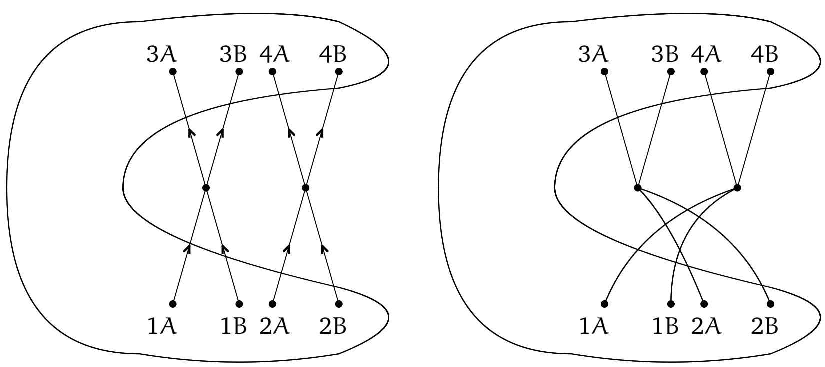

Fig. 5.74. $\mathcal{L}(G^*,\Delta^*)$ and $\mathcal{L}(G^*,\Delta^{**})$

Given the connections in the intestines of the monster, we now check the number of Euler paths in $\mathcal{L}(G,\Delta)$, in $\mathcal{L}(G^*,\Delta^*)$, and in $\mathcal{L}(G^*,\Delta^{**})$.

There are two kinds of connections in the intestines:

$$\underbrace{\begin{matrix} 1A & 1B & 2A & 2B \\ 3A & 3B & 4A & 4B \end{matrix}}_{\text{Case 1: two separate groups}} \quad \text{and} \quad \underbrace{\begin{matrix} 2A & 1B & 1A & 2B \\ 3A & 4A & 4B & 3B \end{matrix}}_{\text{case 2: two mixed groups.}}$$

Let's first look at the separate groups in Figure 5.75.

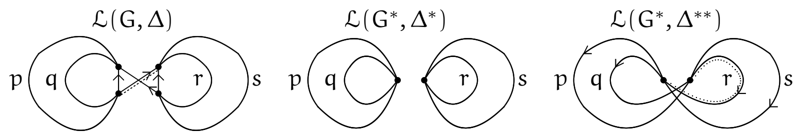

Fig. 5.75. Separate groups

If we start with the dotted line in $\mathcal{L}(G,\Delta)$ then we traverse first r, and then s, or the other way around; but r and s always come together, because if you are on the left you cannot go back, unless you use the dotted line twice.

On the left you have a choice to traverse first p and then q or the other way around. Thus in total there are 4 possibilities for traveling through $\mathcal{L}(G,\Delta)$. In $\mathcal{L}(G^*,\Delta^*)$ there are no possibilities, and in $\mathcal{L}(G^*,\Delta^{**})$ we have two:

$$r\,q\,s\,p \quad \text{or} \quad r\,p\,s\,q.$$

Thus in this case the Claim 5.6 holds.

Now let's deal with the mixed groups:

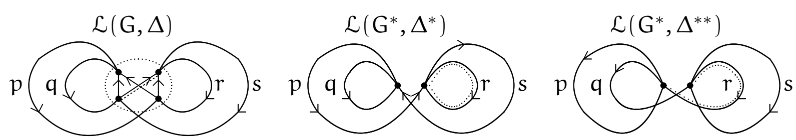

Fig. 5.76. Mixed groups

On the left we see 4 possibilities in $\mathcal{L}(G,\Delta)$:

$$\begin{matrix} r & s & p & q \\ r & q & p & s \\ r & p & s & q \\ s & p & r & q. \end{matrix}$$

On the right we see that there is exactly one possibility for $\mathcal{L}(G^*,\Delta^*)$, namely

$$r \; s \; q \; p$$

and we see that there is also exactly one possibility for $\mathcal{L}(G^*, \Delta^{**})$, namely

$$r \; p \; q \; s.$$

Actually, $\mathcal{L}(G^*, \Delta^*)$ and $\mathcal{L}(G^*, \Delta^{**})$ are both similar to Figure 5.77.

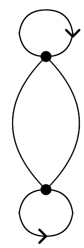

Fig. 5.77. The two figures on the righthand-side of 5.76 are the same

Summarizing, we see that in each case there are 4 Euler circuits in $\mathcal{L}(G, \Delta)$ and 2 Euler circuits in $\mathcal{L}(G^*, \Delta^*)$ and in $\mathcal{L}(G^*, \Delta^{**})$ together. This proves Claim 5.6 and so we are done.

Remark 5.27. For the general case we refer to T. van Aardenne-Ehrenfest and N. G. de Bruijn, Circuits and trees in oriented linear graphs, Simon Stevin 28, 203–217, (1951).

5.14 Spanning trees

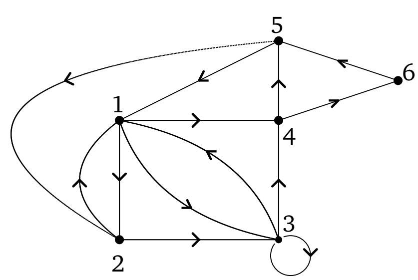

Fig. 5.78. Euler graph

Example 5.28. Take an Euler circuit in the Euler graph of Figure 5.78 that starts with the arc $(1, 2)$. For example

$$1 \quad 2 \quad 3 \quad 3 \quad 4 \quad 5 \quad 1 \quad 4 \quad 6 \quad 5 \quad 2 \quad 1 \quad 3 \quad 1.$$

Now number the lines of this Euler circuit and take with every point $\neq 1$ the last exit.

Claim: Together these lines form a tree; the last-exit-tree.

1) The number of points is one more than the number of lines.
2) The outgoing line of every point has a higher number than the incoming lines, so there are no circuits.

In our example we get the tree of Figure 5.79.

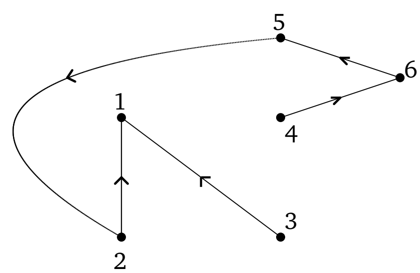

Fig. 5.79. Spanning tree

This example illustrates the proof of Theorem 5.23 on page 114 and we will discuss this next.

Theorem 5.29. *Let* (G, Δ) *be an Euler graph with a finite number of points. That means that* (G, Δ) *is an oriented graph and that for every point* $P \in G$*,*

$$\textit{in-degree}(P) = \textit{out-degree}(P).$$

Let $G = \{P_1, \ldots, P_n\}$ *where* $n = |G|$*. Let* $\sigma_i = \textit{in-degree}(P_i)$ *for* $i = 1, \ldots, n$*. Then*

$$P_E(G, \Delta) = P_w(G, \Delta, P_1) \prod_{i=1}^{n} (\sigma_i - 1)!$$

where $P_w(G, \Delta, P_1)$ *is the number of spanning trees with root* P_1 *that are oriented towards* P_1*.*

Proof. We prove that every tree with root P_1 and oriented towards P_1 occurs exactly in $\prod(\sigma_i - 1)!$ Euler circuits as a last-exit-tree.

Recall how we constructed the last-exit-tree: Color arrow (P_1, P_2) red. With every point $\neq P_1$ we color the last exit blue. Then we get a blue tree: Every point (except P_1) has an exit that has a higher number in the Euler circuit. Thus there are no circuits and the number of points is exactly one more than the number of lines. So this is a blue tree.

Vice versa, consider a spanning tree for (G, Δ). Color the lines of that tree blue. Every point has *one* outgoing blue arrow, except P_1. Give P_1 an outgoing red line, say (P_1, P_2).

With every point we now fix an ordering of the outgoing lines, such that the colored line comes last. There are

$$\prod_{i=1}^{n}(\sigma_i - 1)!$$

ways to do that.

Now take a hike (we just put it to you plain and simple). Start with the red line and choose at every point that you enter the next exit in the local order. Thus the colored arrows always come last.

Is the result an Euler circuit?

Assume that you missed some arrow when you stop in P_1. Assume that some exit at $Q \neq P_1$ was not used. Then the blue exit of Q was not used. Now follow the blue trail that starts in Q. The blue exit in $Q = Q_0$ leads to, say Q_1. In Q_1, the blue arrow that comes in is not used, and so there is also some exit of Q_1 that is not used. But then also the blue exit of Q_1 is not used. This leads to Q_2, etc.

This process ends in P_1 since the blue arrows form a tree that is oriented towards P_1. There we find a contradiction: Some blue arrow that comes into P_1 is not used. But then the exits of P_1 are not yet exhausted. $\square$

5.15 Problems

5.1. Let $T_1, \dots, T_n$ be a collection of subtrees of a tree T with the property that every pair of them has at least one point in common. Show that there is at least one point in T that is a point of every T_i.

5.2. Show that for $n \geqslant 3$ there are n^{n-3} trees with $n-1$ lines labeled with the numbers $1, \dots, n-1$.

5.3. A threshold graph is defined recursively as follows.

(1) A graph with 1 vertex is a threshold graph.

(2) The disjoint union of a single vertex and a threshold graph is a threshold graph.
(3) A graph with a vertex connected to all vertices of a threshold graph is a threshold graph.

I. Prove that the number of non-isomorphic, unlabeled threshold graphs with n vertices is 2^{n-1}.
II. Let t_n be the number of *labeled* threshold graphs on $\{1, \ldots, n\}$ and let s_n be the number of those with no isolated vertex. Define also $t_0 = s_0 = 1$. Then the first few values are:

n	0	1	2	3	4	5
t_n	1	1	2	8	46	332
s_n	1	0	1	4	23	166

Define the exponential generating functions $t(x)$ and $s(x)$ by

$$t(x) = \sum_{k=0}^{\infty} t_k \frac{x^k}{k!} \quad s(x) = \sum_{k=0}^{\infty} s_k \frac{x^k}{k!}.$$

(a) Prove that

$$t_n = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}.$$

(b) Show that

$$t_n = \sum_{k=0}^{n} \binom{n}{k} s_{n-k} \quad \Rightarrow \quad t(x) = s(x)e^x.$$

(c) All threshold graphs with at least 2 vertices come in complementary pairs; one connected and the other disconnected. This implies

$$\begin{aligned} t_n &= 2s_n \quad \text{for } n \geqslant 2 \quad \Rightarrow \\ t(x) &= 2s(x) + x - 1 \quad \text{(and with (b))} \quad \Rightarrow \\ s(x) &= \frac{1-x}{2-e^x} \quad \text{and} \quad t(x) = \frac{(1-x)e^x}{2-e^x}. \end{aligned}$$

(d) Recall the numbered partitions of Section 3.6. Let A_n be the number of numbered partitions of $\{1, \ldots, n\}$, and let $A(x)$ be the exponential generating function of the sequence A_n. Then

$$A(x) = \sum_{n=0}^{\infty} \frac{A(n)}{n!} x^n = \frac{1}{2-e^x},$$

if we let $A(0) = 1$. This implies

$$s_n = A_n - nA_{n-1}.$$

Give a combinatorial proof of this.

5.4. A cograph is defined recursively as follows.

1. A graph with 1 vertex is a cograph.
2. The disjoint union of two cographs is a cograph.
3. The complement of a cograph is a cograph.

Let g_n be the number of non-isomorphic, unlabeled cographs with n vertices and let c_n be the number of those that are connected. Define also $g_0 = 1$, $c_0 = 0$, and $g_1 = c_1 = 1$. The first few values are

n	0	1	2	3	4
g_n	1	1	2	4	10
c_n	0	1	1	2	5

Consider the generating functions

$$G(x) = \sum_{n=0}^{\infty} g_n x^n \quad \text{and} \quad C(x) = \sum_{n=0}^{\infty} c_n x^n.$$

(a) The cographs with $n \geqslant 2$ vertices come in pairs; one connected and the other disconnected. This implies

$$G(x) - x - 1 = 2(C(x) - x) \quad \text{so} \quad G(x) + x - 1 = 2C(x).$$

(b) If G is a cograph then all components are connected cographs. Use Cayley's method to show that

$$\begin{aligned} G(x) &= \prod_{k=1}^{\infty} (1 - x^k)^{-c_k} \quad \Rightarrow \\ \log(G(x)) &= \sum_{k=1}^{\infty} \frac{C(x^k)}{k} \quad \text{(and with (a))} \quad \Rightarrow \\ 2C(x) - x + 1 &= \exp\left(\sum_{k=1}^{\infty} \frac{C(x^k)}{k}\right). \end{aligned}$$

(c) Use Problem 2.2 to show that

$$g_n = \frac{1}{n} \sum_{k=1}^{n} g_{n-k} \sum_{d|k} d c_d,$$

and $g_k = 2c_k$ for $k \geqslant 2$. Check that this gives

$$g(x) = 1 + x + 2x^2 + 4x^3 + 10x^4 + \dots.$$

5.5. Let f_n be the number of labeled cographs with vertices $\{1, \dots, n\}$ and let s_n be the number of those that are connected. Define $f_0 = f_1 = s_1 = 1$ and $s_0 = 0$. The first few values are

n	0	1	2	3	4
f_n	1	1	2	8	52
s_n	0	1	1	4	26

Let $F(x)$ and $S(x)$ be the exponential generating functions

$$F(x) = \sum_{k=0}^{\infty} \frac{f_k}{k!} x^k \quad \text{and} \quad S(x) = \sum_{k=0}^{\infty} \frac{s_k}{k!} x^k.$$

(a) Cographs with at least 2 vertices come in complementary pairs; one connected and the other disconnected. Thus

$$f_n = 2s_n \quad \text{for } n \geqslant 2 \quad \Rightarrow F(x) = 2S(x) - x + 1.$$

(b) A graph is a cograph if and only if every component is a cograph. This implies

$$F(x) = 1 + \sum_{k=1}^{\infty} \frac{S^k(x)}{k!} = e^{S(x)}.$$

(c) Show that this implies

$$S'(x)\,(2S(x) - x - 1) = -1 \Rightarrow$$
$$s_{n+1} + ns_n = 2\sum_{k=0}^{n} \binom{n}{k} s_{k+1}s_{n-k} \quad \text{for } n > 0.$$

5.6. Let $TB(x) = \sum_{k=1}^{\infty} t_k x^k$ be the generating function for the *topological* binary trees. Thus the trees are equivalent under swapping of lobes.

(a) Check that

$$t_1 = t_3 = t_5 = 1 \quad t_7 = 2 \quad t_9 = 3 \quad t_{11} = 6 \quad \text{and} \quad t_{2n} = 0 \ \text{ for all } n \geqslant 1.$$

(b) Show that $TB(x)$ satisfies the functional equation

$$TB(x) = x + xP_{S_2}(TB(x), TB(x^2), \ldots) = x + \frac{x}{2}\left(TB^2(x) + TB(x^2)\right).$$

(c) Derive a recurrence relation for the coefficients t_n.

5.7. Recall Cayley's functional equation for rooted trees:

$$T(x) = x \exp\left(\sum_{k=1}^{\infty} \frac{T\left(x^k\right)}{k}\right).$$

Let (see [29]):

$$T(x) = \sum_{k=1}^{\infty} T_k x^k = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 +$$
$$+ 48x^7 + 115x^8 + 286x^9 + 719x^{10} + \ldots.$$

We first show that the coefficients T_k are determined by Cayley's formula.

(a) Let

$$\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} \frac{T\left(x^k\right)}{k},$$

then

$$a_n = \frac{1}{n} \sum_{d|n} dT_d.$$

(b) Use Problem 2.2 to show that

$$\begin{aligned} T_{n+1} &= \frac{1}{n} \sum_{k=1}^{n} k a_k T_{n-k+1} \\ &= \frac{1}{n} \sum_{k=1}^{n} \left(\sum_{d|k} dT_d \right) T_{n-k+1}. \end{aligned}$$

(c) Let $L(x)$ be the generating function for trees that are rooted at a line. To count trees rooted at a line, we may root a tree at each of its endpoints. This counts them double, except those that are symmetric at the root. This gives

$$L(x) = \frac{1}{2} \left(T^2(x) + T\left(x^2\right) \right).$$

(d) Let $S(x)$ be the generating function of trees that have a symmetry line; that is a line for which there is an automorphism of the tree which switches the two endpoints. Show that

$$S(x) = T\left(x^2\right).$$

(e) Let T be a tree with n points. Two points are equivalent if there is an automorphism of T that maps one onto the other. Similarly, two lines are equivalent if there is an automorphism that maps one onto the other.
Let T_0 be a maximal subtree for which no two vertices are equivalent. Consider a point P not in T_0 and assume that P is adjacent to a point Q of T_0. Since T_0 is maximal, there is an automorphism σ that maps P onto some point $P' \in T_0$. Let $\sigma(Q) = Q'$. Let ℓ be the line with ends P and Q. Then either ℓ is a symmetry line and the automorphism switches P and Q, or $Q = Q'$. Thus a point that is a neighbor to a point in T_0 is equivalent to a point in T_0.
Prove (by induction) that T_0 contains exactly one point of every equivalence class and exactly one line of every equivalence class, except a line of symmetry.

(f) Since T_0 is a tree, the number p^* of nonsimilar points (the number of equivalence classes) minus the number of nonsimilar lines q^* (except a symmetry line s; $s = 0$ or $s = 1$) is one:

$$p^* - q^* + s = 1.$$

(This is Otter's dissimilarity characteristic [34].)

(g) Sum this equation over all trees with n vertices. The total will be the number of trees. The sum over p^* is the number of trees rooted at a point. The sum over q^* is the number of trees rooted at a line. Let $t(x)$ be the generating function for the non-isomorphic trees. Then

$$t(x) = T(x) - L(x) + T\left(x^2\right) = T(x) - \frac{1}{2}\left(T^2(x) - T\left(x^2\right)\right).$$

6

Bipartite graphs

We are going to do something completely different now, namely bipartite graphs.

6.1 Introduction

A bipartite graph is a graph of which the set of points is partitioned into two sets, say A and B, and there are only lines between A and B. More precisely:

(G, Γ) is a bipartite graph if $G = A \cup B$ with $A \cap B = \varnothing$ and

$$\Gamma \subseteq \{\{a, b\} \in \mathcal{P}_2(G) \mid a \in A, b \in B\}.$$

Sometimes problems may look different only because A and B play different roles. We write also (a, b) instead of $\{a, b\}$ for lines with $a \in A$ and $b \in B$.

We associate a matrix with a bipartite graph, with A a set of row indices and B a set of column indices such that

$$\begin{aligned} (a, b) \in \Gamma &\Leftrightarrow \text{element in row } a \text{ and column } b \text{ is 1, and} \\ (a, b) \notin \Gamma &\Leftrightarrow \text{element in row } a \text{ and column } b \text{ is 0.} \end{aligned}$$

A problem that can be formulated as a bipartite graph problem is the well-known pentomino-game. A pentomino is a connected object of 5 squares. There are 12 of those pentominos (when we allow rotations and reflections). Together the pieces contain $12 \cdot 5 = 60$ squares. The puzzle is to put the 12

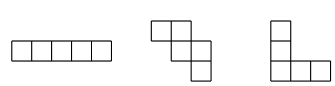

Fig. 6.1. Some pentomino pieces

pieces in a box of 6×10 squares.

We transform this into a bipartite graph problem. Let A be the set of all 60 squares and let B be the set of those subsets of A that have the shape of a pentomino-piece. We draw 5 lines from every point in B to the 5 squares in A

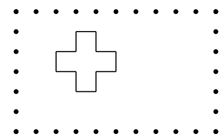

Fig. 6.2. Here is one element of B; there are $8 \times 4 = 32$ such crosses

that are occupied by the point in B. The problem is to choose some elements of B such that

1) all pieces are different, and
2) together they cover A.

We can make the first condition a part of the graph formulation with a little trick: Enlarge A with a "storeroom" of 12 elements, each named after one of the pentomino pieces. At every element of B add *one* extra line, namely to the storage-element that it represents. Then the problem becomes like this:

Given a bipartite graph (G, Γ, A, B). Find a subset $B' \subseteq B$ such that

$$\{\{a \in A \mid (a, b) \in \Gamma\} \mid b \in B'\}$$

is a partition of A.

The dual problem is the following:

Given a set T, and some subsets of T, say $T_1, \ldots, T_n$. The problem is to find $t_1, \ldots, t_\ell$ such that for all i, $1 \leqslant i \leqslant n$, there is exactly *one* j, $1 \leqslant j \leqslant \ell$, with $t_j \in T_i$.

This kind of problem occurs when a diverse group (of people) has to be represented, possibly with some side constraints. (For example the university board has to represent groups like students, coworkers, people doing stochastics, feminists, etc. Sometimes every person can represent only one group; or, maybe this is not necessary, but the condition is rather that every group is represented only by one person.)

NB The set T in this case is the set of "hats" (the set of groups, or offices; "the students," and "the feminists," etc.) and $1, \ldots, n$ are the people of the department; in the case above there can be only *one* person per hat (a group is represented by one person only).

Example 6.1. Let's get this clear.

(a) T is a deck of cards. $|T| = 52$ (that is, when it is a complete deck).
(b) T_1 consists of all Jacks,

(c) T_2 consists of all spades,
(d) T_3 consists of all red sixes,
(e) T_4 consists of all diamonds < 8, and
(f) T_5 is the Ace of clubs.

$B = T$, $A = \{T_1, \ldots, T_5\}$. The question is the choose some subset of cards in B such that every T_i has exactly one card in that subset. Here is one solution,

$$\{\text{Jack of spades, Ace of clubs, 6 of diamonds}\},$$

and here is another one

$$\{\text{Jack of clubs, 2 of spades, 4 of diamonds, 6 of hearts, Ace of clubs}\}.$$

What does "duality" mean? Well, the duality is in the way the problem is formulated as a graph problem:

In the *covering problem* you draw a line from a to b if $a \in b$, (so $B \subseteq \mathcal{P}(A)$), and in the *representatives problem* when $b \in a$, (thus $A \subseteq \mathcal{P}(B)$).

You can also look at it like this. If $B \subseteq \mathcal{P}(A)$, then you speak about a covering problem when you look for a partition of A, and you speak about a representatives problem when you look for a partition of B.

6.2 Covering – and representatives problem

We come to the classic theory of Kőnig and Hall, developed in 1932–1936 by those two and by others.[1] [2]

The classic result of Kőnig and Hall concerns the representatives problem with the restriction that no representative can represent more than one group. (The Jack of spades and 6 of diamonds in the first solution of our example of the previous section wore "two hats;" they represented more than one group. In the second solution every card had only one hat.)

In terms of mappings, we can formulate it like this. Find a mapping $A \to B$ such that

$$f : A \to B \text{ is an injection, such that } \forall_{a \in A}\ [(a, fa) \in \Gamma].$$

Example 6.2. B is a set of football players (we talk about the soccer game) and A is the set of 11 functions. Thus

$$A = \{\text{goalkeeper, left back}, \ldots, \text{right wing}\}.$$

(Perhaps these terms are a bit old-fashioned; they change them so often; but they have the advantage that I, *i.e.*, de Bruijn, am familiar with them.) Not

[1] P. Hall, On representatives of subsets, *Journal of the London Mathematical Society* 10 (1935) pp. 26–30.
[2] D. König, Über Graphen and Ihre Anwendung auf Determinantentheorie und Mengenlehre, *Mathematische Annalen* 77 (1915) pp. 453–465.

every player can play in every position, for example one guy's left leg could be on the wrong side, or another is a bad goalie. We add in $A \cup B$ the allowed lines.

In the representatives-terminology, football players are members of one or more "clubs;" the goalkeepers-club, the club of left-backs, etc.

In the covering terminology every football player has a certain set of capacities.

Of course, you can't select a team from B just like that: you need a goalkeeper, and you need a left back, etc. But it won't work either if you have only two candidates for the subset

$$\{\text{left back, right back, left wing}\}.$$

6.3 Theorem of König and Hall

Another version is the marriage problem. We have $A =$ the set of women and $B =$ the set of men. Some men and women like each other and some don't. The idea is to form pairs of men and women (as you see; it's an old-fashioned problem) such that the members like each other. Even when $|A| = |B|$ and every man likes at least some woman, then this is not always possible; if there is one woman that all men like (and if they like no other), then there is no solution (at least there wasn't one in the old days).

Let's get formal again. The sort of conditions as above that make a mapping impossible, are the *only* ones that do that:

Theorem 6.3 (Hall's theorem). *Let*

$$G = A \cup B, \quad A \cap B = \varnothing, \quad \textit{and} \quad \Gamma \subseteq \{\{a, b\} \mid a \in A, b \in B\}.$$

Then the following conditions are equivalent:

$$\begin{array}{ll} (\mathrm{I}) & \forall_{K \in \mathcal{P}(A)} \mid \{b \in B \mid \exists_{a \in K} \{a, b\} \in \Gamma\} \mid \geqslant |K| \\ (\mathrm{II}) & \exists_{f: A \to B} \ f \textit{ is injective and } \forall_{a \in A} \ \{a, fa\} \in \Gamma. \end{array}$$

Proof. Introduction: $(\mathrm{II}) \Rightarrow (\mathrm{I})$ is trivial. To prove that $(\mathrm{I}) \Rightarrow (\mathrm{II})$ we produce an algorithm that either finds a K that contradicts (I) or a function f that satisfies (II). In other words, we prove $\neg(\mathrm{I}) \vee (\mathrm{II})$.

The proof goes roughly as follows: Try something. If it doesn't work then start exchanging. If you can't do that fruitfully, then you've found your counterexample to (I).

Assume k of the football players have been given a position. Try to find a player that fits the $(k+1)^{\mathrm{th}}$ position. In other words, let $k < |A|$ and assume that there is an injection $f : A' \to B$ with $|A'| = k$ and

$$\forall_{a \in A'} \{a, fa\} \in \Gamma.$$

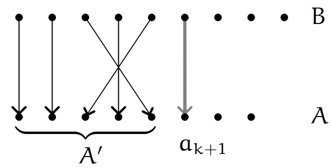

Fig. 6.3. Red arrows point from B down to A'

Write $A' = \{a_1, \ldots, a_k\}$. Choose an element $a_{k+1} \in A \setminus A'$. We color the connections $\{a, fa\}$ red (going down into A), and we color all other lines $\{a, b\}$ with $a \in A' \cup \{a_{k+1}\}$ blue (going up into B). We call points *reachable* from a_{k+1} if you can get there by going up in the graph via blue lines and down in the graph via red lines. The point a_{k+1} itself is also reachable.

Case 1. The number of reachable points in B is at least equal to the number of reachable points in A.
Case 2. Similar, but now the number of reachable points in B is *less* than the number of reachable points in A.

In the first case we take a path, along points that are reachable from a_{k+1}, to a point $\gamma \in B$ that is not incident with a red line. For every point in $f(A')$ there is a point in A' that can be reached via a red line so, by assumption, since a_{k+1} is also reachable, the point γ exists. On that path the points from A will alternate with the points from B. The path looks something like that in Figure 6.4.

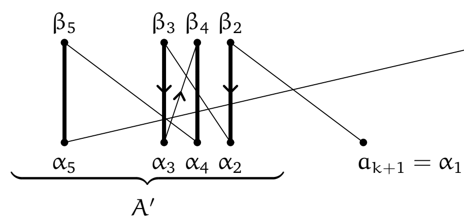

Fig. 6.4. Alternating path

Now change f by a cyclic shift of the images of α_i: $\alpha_i \to$ successor of α_i:

$$\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 & \gamma \end{array}$$

By doing that we "push" the injection one step further.

Case 2: Take K equal to the points in A that are reachable from a_{k+1}. Then, apparently

$$|\{b \in B \mid \exists_{a \in K}\, [\{a, b\} \in \Gamma]\}| < |K|,$$

and we found a counterexample. $\square$

It is a nice algorithm. It also works in practice, since if it does not work, it tells you *why* it does not work. Customers are not satisfied when you tell them that the computer can't find an answer.

6.4 Theorem of König and Egerváry

A bit more general is the theorem of König and Egerváry. [3] [4]

To introduce this: Let A and B be two sets and let M be an $|A| \times |B|$ matrix. We let Γ be a characteristic function on $A \times B$.

We say that M has a q-diagonal if there is a diagonal with q ones after a suitable permutation of rows and columns. If there exists a block of $k \times \ell$ zeros in M after a suitable permutation of rows and columns, then we call this a q-block for every $q > |A| + |B| - k - \ell$. Large blocks of zeros (with small q; q is the "defect" of the block) obstruct a q-diagonal.

If A and B have subsets A' and B' such that $(A' \times B') \cap \Gamma = \varnothing$ then $A' \times B'$ is a q-block for every $q > |A| - |A'| + |B| - |B'|$. Given A, B, and Γ, there is a largest q such that there is a q-diagonal, and there is a smallest q such that there is a q-block.

Theorem 6.4 (König-Egerváry). *For every* q *there is either a* q*-diagonal or a* q*-block (but not both).*

For a proof of this theorem we refer to Exercise 6.5. Let's look at a few special cases.

For $q = 1$ this means that the matrix is not zero or else there is a 1-block, which is an A' and B' with $|A| - |A'| + |B| - |B'| = 0$; thus $A' = A$ and $B' = B$, which is an $|A| \times |B|$-submatrix of zeros. In other words $\Gamma = \varnothing$.

If $|A| \leqslant |B|$ and $q = |A|$, then you get that there is either an injection $f : A \to B$ (a q-diagonal) or a block with $k \times \ell$ zeros with $k + \ell > |B|$, that is $k > |B| - \ell$. This is also an application of König and Hall: Let $K \subseteq A$ be a subset for which the neighborhood in B has too few elements. Let $|K| = k$. Let $L \subseteq B$ be the non-adjacencies of K and let $|L| = \ell$. Thus $\ell = |L| = |B| - |N(K)|$, where $N(K)$ is the set of elements in B that are connected to K. Now $|N(K)| < k$ implies $|L| > |B| - k$, which means that $\ell + k > |B|$. In other words, such a set K in A with too few neighbors corresponds with a q-block.

A special case of the theorem of König and Hall is where all points have the same degree ($g > 0$). (And so $|A| = |B|$.)

Now the condition is satisfied: From every point leave g lines, so in total there are $k \cdot g$ lines leaving a subset $A' \subseteq A$ with $|A'| = k$. Those $k \cdot g$ lines cannot end in less than k points of B. To see that simply count the lines between A' and $N(A')$ in two ways:

[3] E. Egerváry, On combinatorial properties of matrices, Translated by H. W. Kuhn, Logistics Papers, 11, George Washington University, paper 4 (1955), pp. 1–11.

[4] D. Kőnig, Graphen und Matrizen, *Math. Lapok* **38** (1931), pp. 116–119.

$$|A'| \cdot g = \sum_{b \in N(A')} |N(b) \cap A'| \leqslant \sum_{b \in N(A')} |N(b)| = |N(A')| \cdot g,$$

where $N(A')$ is the set of points in B that are connected by a line to A' and $N(b)$ is the set of points in A that are connected to b. Thus $|N(A')| \geqslant |A'|$. By Theorem 6.3 there is a bijection between A and B that uses only lines of the graph. The next conclusion is that the graph is a super-position of bijection graphs.

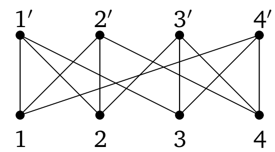

Fig. 6.5. Bipartite graph; every vertex has degree 3

Example 6.5. Remove the bijection

$$\begin{pmatrix} 3 & 1 & 2 & 4 \\ 1' & 2' & 3' & 4' \end{pmatrix}$$

and you get the following graph:

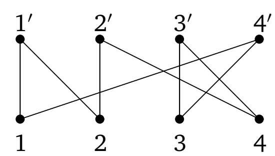

Fig. 6.6. After the removal of the bijection; every vertex has degree 2

Now we may remove another bijection, for example:

$$\begin{pmatrix} 2 & 4 & 3 & 1 \\ 1' & 2' & 3' & 4' \end{pmatrix}$$

and what is left is this bijection:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 1' & 2' & 3' & 4' \end{pmatrix}$$

Nicely formulated: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_g$ where the $(A \cup B, \Gamma_i)$ are all bipartite graphs of degree 1.

Another famous application is the following. Let G be a finite group and let H be a subgroup. Let $k = \frac{|G|}{|H|}$ be the index of H. We let A be the set of left cosets, and we let B be the set of right cosets. For every element x of G we connect xH with Hx.

An analogous theorem for multigraphs, (the joined multiplicity of the neighborhood must be no less than the joined multiplicity of the origin) says that there is a bijection: and this is a common representative system for left– and right cosets.

This fact was known by algebraic methods long before Van der Waerden noticed that this was a combinatorial exercise. See Exercise 6.13. [5]

Let's look at another application, namely doubly stochastic matrices. We call an $n \times n$-matrix doubly stochastic if the coefficients satisfy

$$\forall_{i,j}\ a_{ij} \geqslant 0 \quad \text{and} \quad \forall_j \sum_i a_{ij} = 1 \quad \text{and} \quad \forall_i \sum_j a_{ij} = 1.$$

So we have that $\sum_{i,j} a_{ij} = n$.

Theorem 6.6. *The permanent of a doubly stochastic matrix is > 0.*

(The permanent of a matrix is the same as the determinant, but without the minus-signs in the calculations.)

Proof. Assume that matrix has a $k \times \ell$-submatrix of zeros. Then $k + \ell \leqslant n$. Conclusion: there is an n-diagonal, thus the permanent has a term $\neq 0$. □

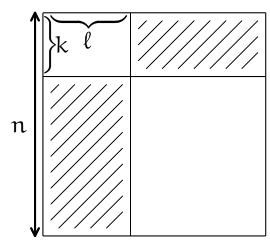

Fig. 6.7. $k \times \ell$ block of zeros

6.5 Latin squares

A latin square is an $n \times n$ square filled with n elements, every column and every row has exactly one copy of every element.

[5] B. L. van der Waerden, Ein Satz über Klasseneinteilungen von Endlichen Mengen, Abh. Math. Sem. Hamburg **5** (1927), pp. 185–188.

Example 6.7. For $n = 3$ you can have:

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$$

If we construct a 7×7 latin square, then we can start the construction easily enough,

$$\begin{matrix} 7 & 3 & 2 & 5 & 6 & 1 & 4 \\ 4 & 1 & 3 & 6 & 5 & 7 & 2 \\ 5 & 2 & 4 & 1 & 7 & 6 & 3 \\ \cdots & & & & & & \cdots \end{matrix}$$

and you can always continue.

To see this, make a bipartite graph:

$$\left.\begin{array}{l} \text{A: numbers} \\ \text{B: places in the matrix} \end{array}\right\} \begin{array}{l} \text{and connect the places with the} \\ \text{available numbers.} \end{array}$$

For every place in the matrix above there are $7-3 = 4$ numbers still available. On the other hand, every number appears 3 times (in 3 columns), and so there are also 4 places still available. We have seen that you can always partition the lines of this bipartite graph into 4 bijections. These are 4 additional rows.

6.6 Theorem of Ford-Fulkerson

In this section we will have a look at the theorem of Ford-Fulkerson (1952). [6] This concerns a flow problem in a finite oriented graph (G, Δ).

Let (G, Δ) be a finite oriented graph and let B (the source) and P (the sink) be two vertices in G. Let $\nu : \Delta \to \mathbb{R}^+$ be a function. Instead of $\nu(l)$ we will write ν_l. In Figure 6.8 we show an example (we explain the dotted line in a minute). We call ν_ℓ the capacity of the line. We want to find a function $f : \Delta \to \mathbb{R}^+$ such that

$$\forall_{\ell \in \Delta} \ \ f(\ell) \leqslant \nu_\ell$$

and

$$\forall_{x \in G \setminus \{B,P\}} \sum_{\ell \text{ towards } x} f(\ell) = \sum_{\ell \text{ out of } x} f(\ell).$$

The function f is a flow through the network (if you want to model traffic in two directions, then you can make two lines; the traffic has to use different lanes anyway). The condition means that nothing gets stashed away at crossings and that also nothing gets added at crossings.

Let's write

[6] L. R.Ford and D. R. Fulkerson, *Flows in Networks*, Princeton University Press 1962.

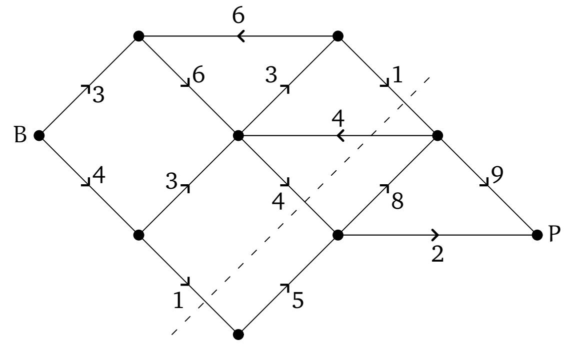

Fig. 6.8. Oriented graph and capacity function $\nu : \Delta \to \mathbb{R}^+$

$$\texttt{flow-in}(x) = \sum_{\ell \text{ towards } x} f(\ell) \quad \text{and} \quad \texttt{flow-out}(x) = \sum_{\ell \text{ out of } x} f(\ell).$$

Then the flow preservation rule says that

$$\forall_{x \in G \setminus \{B,P\}} \ \texttt{flow-out}(x) = \texttt{flow-in}(x).$$

If we now sum the flow f over all arrows in the network, we find:

$$\texttt{flow-out}(B) + \texttt{flow-out}(P) + \sum_{x \neq B, x \neq P} \texttt{flow-out}(x) =$$
$$\texttt{flow-in}(P) + \texttt{flow-in}(B) + \sum_{x \neq B, x \neq P} \texttt{flow-in}(x).$$

For points $x \in G \setminus \{P, B\}$ the total out-flow is equal to the total in-flow. So we must have

$$\texttt{flow-out}(B) - \texttt{flow-in}(B) = \texttt{flow-in}(P) - \texttt{flow-out}(P).$$

In other words:

$$\texttt{net-out-flow}(B) = \texttt{net-in-flow}(P).$$

We only consider the case where there are no arrows coming in at B and where there are no arrows leaving P. Then the identity above becomes:

$$\texttt{flow-out}(B) = \texttt{flow-in}(P).$$

We call this number the total flow of f and we write $\nu(f)$ for it. The problem is to maximize $\nu(f)$ over all f.

Looking at our example we see that there are no flows possible with total flow $\nu(f) > 6$ because no larger amount can pass across the dotted line.

We define a *cut* S of G as follows. This is a partition $S = \{S_1, S_2\}$ of G in sets S_1 and S_2 with $B \in S_1$ and $P \in S_2$. Thus

$$S_1 \cup S_2 = G, \quad S_1 \cap S_2 = \varnothing, \quad B \in S_1, \text{ and } P \in S_2.$$

If we denote the arrows that point from a point in S_1 into a point in S_2 with $\ell_1, \ldots, \ell_k$ then the *capacity* of the cut is

$$c(S) = \nu(\ell_1) + \nu(\ell_2) + \ldots + \nu(\ell_k).$$

It is pretty clear that we now have that

$$\forall_{\text{flows } f} \forall_{\text{cuts } S} \ \ \nu(f) \leqslant c(S).$$

To see that, just draw S as in Figure 6.9. If we sum the net-out-flow in every

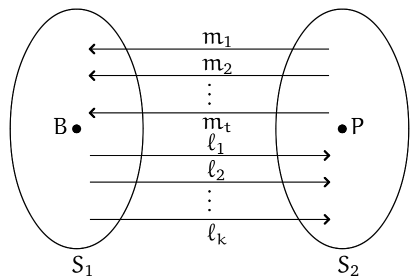

Fig. 6.9. $\nu(f) = f(\ell_1) + \ldots + f(\ell_k) - (f(m_1) + \ldots + f(m_t))$

point of S_1 then this must be the flow out of B, since the net-flow in every other point is zero. But you can also sum over the arrows that have exactly one endpoint in S_1, because the arrows that point from a point in S_1 into another point of S_1 add a net profit of zero, and left are those arrows that have only one end in S_1. Thus

$$\begin{aligned}\nu(f) = \texttt{flow-out}(B) = \sum_{x \in S_1} (\texttt{flow-out}(x) - \texttt{flow-in}(x)) = \\ \sum_{x \in S_1} \sum_{(\ell \text{ out of } x)} f(\ell) - \sum_{x \in S_1} \sum_{(\ell \text{ towards } x)} f(\ell) = \\ f(\ell_1) + \ldots + f(\ell_k) - f(m_1) - \ldots - f(m_t).\end{aligned}$$

And of course this gives

$$\nu(f) \leqslant f(\ell_1) + \ldots + f(\ell_k) \leqslant c(S).$$

The Ford-Fulkerson theorem says that there exists a cut S with $\nu(f) = c(S)$, for a maximal flow.

Theorem 6.8. *In formal language the theorem says:*

$$\max_{f \text{ is a flow}} \nu(f) = \min_{S \text{ is a cut}} c(S),$$

and in slogan it says:

$$\textit{max flow} = \textit{min cut}.$$

In our example we get Figure 6.10 (with the capacities in brackets). Notice

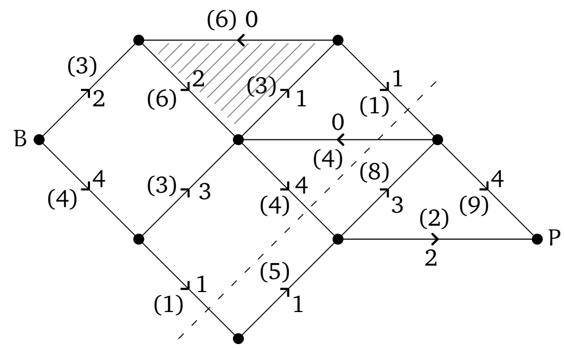

Fig. 6.10. Max flow = min cut

that the flow across the dotted cut uses the maximum capacity. Also notice that you can add 1 to the flow in every line of the shaded triangle.

Proof (Ford-Fulkerson). We associate with an oriented graph (G, Δ), with a capacity function ν and a compatible flow f, a new oriented graph (G, Θ). For simplicity we assume that the values of ν and f are integers.

$$(x, y) \in \Theta \Leftrightarrow ((x, y) \in \Delta \text{ and } f(x, y) < \nu(x, y)) \text{ or}$$
$$((y, x) \in \Delta \text{ and } f(y, x) > 0).$$

In other words: the arrows along which f can get bigger, according to ν, are also in Θ (with the same orientation), and the *reverses* of the arrows where f can be lowered are in Θ (so these last arrows point in opposite directions in Δ and in Θ). So the arrows $\ell \in \Delta$ with $0 < f(\ell) < \nu(\ell)$ are "twice" in Θ; once forward and once backward.

Let S_1 be the set of points of G that can be reached from B via Θ and let $S_2 = G \setminus S_1$. Assume $P \in S_1$. Then there exists a directed path from B to P with arrows in Θ that is completely contained in S_1. Increase f by 1 on the lines of the path that are directed in the same direction as Δ and lower f by 1 on the lines of the path that are directed opposite to Δ. The new flow f then again satisfies that

$$0 \leqslant f(\ell) \leqslant \nu(\ell).$$

We show that the new flow satisfies the flow preservation rule. Consider two arrows ℓ_1 and ℓ_2 in Δ that are on the path in (G, Θ) and assume that both ℓ_1 and ℓ_2 have vertex x as an endpoint. If both arrows are pointing towards x, or both of them are pointing away from x, then one of them has the same direction in Θ and one of them has the opposite direction in Θ. So, on one of them the flow is increased by one and on the other the flow is decreased by one. Thus the net-out-flow in x remains zero. If ℓ_1 points into x and ℓ_2 points away from x, then either they both have the same direction on the path in Θ, or they both have the opposite direction on the path in Θ. Thus the flow is either increased by one on both, or the flow is decreased by one on both. Thus $\texttt{flow-out}(x) - \texttt{flow-in}(x)$ remains zero.

Note that the total flow increases; the source B has only outgoing arrows in Δ and each directed path from B to P starts with an arrow pointing out of B in Θ. On this arrow the flow is increased by one, thus the flow out of B increases.

Example 6.9. For example: suppose we have in our original diagram a flow as in Figure 6.11 (the numbers in brackets are the capacities). (Don't look at the numbers in the circles yet; we explain those in a minute.) We indicated a path

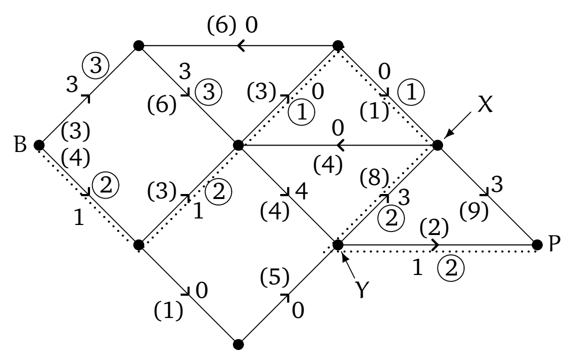

Fig. 6.11. Path in Θ

in (G, Θ) by the dotted lines. Usually the arrows in this path point in Θ in the same direction as in Δ, but in the arrow with capacity (8) it points in the opposite direction: there f can be *decreased*. The path that leaves the point in the middle of the figure in the 1 o'clock direction cannot be decreased; so that line is contained in Θ only in the same direction as in Δ. The line with flow 3 and with capacity (3) that leaves B can only be decreased, so that arrow is only present in opposite direction in Θ.

The value of f can be increased to the value given in the circles. We see at X that one in-flow is increased but that another in-flow is decreased. At the point Y one out-flow is increased and one out-flow is decreased.

You go on like that, increasing the total flow in each round, and in case we work with integers the process stops.

This proves that for a maximal flow we must have that $P \in S_2$ (even in the case where the capacities are arbitrary, positive real numbers; in that case the existence of a maximal flow follows from a compactness argument). In other words, for the cut $\{S_1, S_2\}$ in (G, Δ) now holds that the forward flows are not

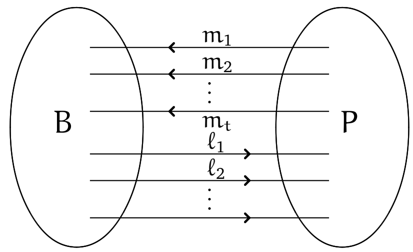

Fig. 6.12. Final cut

in Θ (so they use the full capacity) and that the backward flows do not appear in opposite direction in Θ, thus f cannot be lowered on those lines. Thus

$$f(\ell_1) = \nu(\ell_1), \ldots, f(\ell_k) = \nu(\ell_k) \quad \text{and} \quad f(m_1) = \ldots = f(m_t) = 0.$$

In other words, we have a cut, and the capacity of this cut is exactly the value of the flow.

The algorithm, as we described it above, extends to the case where the capacities are rational numbers; it is easy to see that the algorithm terminates also in that case. (With real numbers, even if you increase the flow as much as possible at every step, the algorithm does not necessarily terminate.[7]) □

Remark 6.10. Ford and Fulkerson showed that their algorithm does not necessarily terminate when the capacities are real numbers. The theorem remains true of course, but the algorithm does not necessarily converge to a maximal flow. Edmonds and Karp showed that the number of iterations can be bounded by $|G| \cdot |\Delta|$ by choosing the *shortest* path, *i.e.*, with the smallest number of arrows, in (G, Θ) to update the flow in each round.[8]

6.7 Problems

6.1. Show that a graph (G, Γ) is bipartite if and only if all cycles in G are even.

[7] M. Queyranne, Theoretical efficiency of the algorithm "Capacity" for the maximum flow problem, Mathematics of Operations Research **5** (1980), pp. 258–266.

[8] J. Edmonds and R. M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, *Journal of the Association for Computing Machinery* **19** (1972), pp. 248–264.

6.2. Let (G, Γ) be a graph. A stable set is a subset of points with no line between them. Let $\alpha(G)$ be the cardinality of a largest stable set in G. A node cover is a subset S of points (nodes) such that every line has an endpoint in S. Let $\tau(G)$ be the cardinality of a smallest node cover in G. Prove that

$$\alpha(G) + \tau(G) = n,$$

where n is the number of points in G.

6.3. Let (G, Γ) be a graph without isolated points; thus every point is the endpoint of at least *one* line. A matching is a subset of lines M such that every point is incident to at most one line of M. Let $\nu(G)$ be the cardinality of a largest matching in G. An edge cover is a set of lines L such that every point is an endpoint of a line in L. Let $\rho(G)$ be the cardinality of a smallest edge cover in G.

(a) Let M be a largest matching. Let U be the set of points that are not endpoints of lines in M. For each $u \in U$ choose one line $e(u)$. Let S be this set of lines. Prove that $M \cup S$ is an edge cover.

(b) Show that

$$|M| + |M \cup S| = n \quad \Rightarrow \quad \nu + \rho \leqslant n,$$

where $n = |G|$, $\nu = \nu(G)$, and $\rho = \rho(G)$.

(c) Let L be an edge cover with $|L| = \rho$. Show that the lines of L induce a forest; thus there is no circuit in G that consists of only lines in L. Let k be the number of components of this forest. Show that $k + \rho = n$.

(d) Each component of this forest contains at least one line. Choose one line in each component. This is a matching of cardinality k. Show that this implies

$$\nu + \rho \geqslant n \quad \text{and with (b)} \quad \nu + \rho = n$$

where $\nu = \nu(G)$, $\rho = \rho(G)$, and $n = |G|$.

6.4. Let (G, Γ) be a graph, and let ν, τ, ρ, and α be the cardinalities of a largest matching, a smallest node cover, a smallest edge cover, and a largest stable set. Then

$$\nu \leqslant \tau \quad \text{and} \quad \rho \geqslant \alpha.$$

6.5 (König-Egerváry). Let $G = (V_1 \cup V_2, \Gamma)$ be a bipartite graph. In this exercise we show that the cardinality of a largest matching in G, $\nu(G)$, is equal to the cardinality of a smallest node cover in G, $\tau(G)$:

$$\nu(G) = \tau(G) \quad \text{and, if there are no isolated points} \quad \alpha(G) = \rho(G).$$

(a) Show that this formulation is equivalent to the one in the text.
Let M be a matching and let U be a node cover.

(b) Since every line of M must have at least one endpoint in U, $|U| \geqslant |M|$.

(c) Assume that U is a node cover of cardinality τ. Let $X = U \cap V_1$ and let $Y = U \cap V_2$. There are no lines from $V_1 \setminus X$ to $V_2 \setminus Y$.

(d) Consider the bipartite subgraph H with bipartition $X \cup (V_2 \setminus Y)$. Let $S \subseteq X$ and let $N_H(S)$ be the points in $V_2 \setminus Y$ that are connected by lines to S. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in U and obtain a smaller vertex cover. Use Hall's theorem to show that H has a matching of X into $V_2 \setminus Y$.
(e) Similarly, let H' be the bipartite subgraph with bipartition $Y \cup (V_1 \setminus X)$. Then H' has a matching of Y into $V_1 \setminus X$.
(f) Since H and H' are disjoint, the two matchings together form a matching of cardinality $|U|$.

6.6 (König-Egerváry). Consider a $0,1$-matrix. A line is a row or a column. Show that the minimum number of lines that contain all the 1's is the same as the maximum number of 1's no two of which are in the same line.

6.7 (Berge's augmenting path[9]). Let (G, Γ) be a graph. Let M be a matching. A path is a chain without repeated points. An M-alternating path is a path whose lines alternate between M and $\Gamma \setminus M$. The matching saturates a point if the point is on a line of M. An M-augmenting path is an M-alternating path whose start and endpoint are not saturated. In this exercise we show that a matching M is maximum if and only if there is no M-augmenting path.

(a) Show that the condition is necessary; if there is an M-augmenting path P then replace the lines of M that are in P by the lines of P that are not in M.
(b) Suppose that M is not a maximum matching. Let M' be a matching with $|M'| > |M|$. Consider the symmetric difference $M \div M'$, defined as $(M \cup M') \setminus (M \cap M')$. Let H be the graph induced by the lines of $M \div M'$. Each point of H has degree one or two, since it is incident with at most one line of M and at most one line of M'.
(c) This implies that H is a disjoint union of paths and even cycles.
(d) Since $|M'| > |M|$, there is some path in H that starts and ends with an edge in M'. This is an M-augmenting path.

6.8 (de Caen[10]). Let (G, Γ) be a bipartite graph with at least one line. Prove that every line has at least one endpoint that is saturated by every maximum matching.

6.9 (Edmonds[11]). Let (G, Γ) be a graph. Berge's original proposal for finding an augmenting path via a simple depth-first-search approach does not lead to a polynomial-time algorithm, because the path could end up in a cycle. Edmonds came up with the idea to shrink such a cycle into a new point and

[9] C. Berge, Two theorems in graphs, *Proc. Nat. Acad. Sci.* **43** (1957), pp. 842–844.
[10] D. de Caen, On a theorem of Kőnig on bipartite graphs, *Journal of Combinatorics, Information & System Sciences* **13** (1988), pp. 127.
[11] J. Edmonds, Paths, trees, and flowers, *Canadian Journal of Mathematics* **17** (1965), pp. 449–467.

to apply recursion to the new, smaller graph. We follow the description of Schrijver. [12]

Let M be a matching in G and let X be the set of points in G that are not in any line of M. A chain

$$P = [v_0, \dots, v_t]$$

is called M-alternating if for each $i = 1, \dots, t-1$ exactly one of the lines $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ is in M.

We first show that we can find a shortest alternating chain with endpoints in X. Let (G, A) be the directed graph with arrows

$$A = \{u \to v \mid \exists_{x \in G} \{u, x\} \in \Gamma \text{ and } \{x, v\} \in M\}$$

(a) Each M-alternating chain of positive length, with start and endpoint in X, gives a directed path from X to $N(X)$ in (G, A), where as usual, $N(X)$ is the set of points of $G \setminus X$ that are connected to a point of X by a line of Γ.
(b) Each directed path in (G, A) from X to $N(X)$ corresponds with an M-alternating chain from X to X in (G, Γ).

An M-alternating chain $P = [v_0, \dots, v_t]$ is called an M-flower if

1. t is odd,
2. $v_0, \dots, v_{t-1}$ are distinct, and
3. $v_t = v_i$ for some $i < t$ where i is even.

The circuit $[v_i, \dots, v_t]$ is called the M-blossom of the M-flower.

We show that a shortest M-alternating chain $P = [v_0, \dots, v_t]$ from X to X is either an M-augmenting path or $[v_0, \dots, v_j]$ is an M-flower for some $j \leqslant t$.

(i) Assume that P is not a path. Let $i < j$ be such that $v_i = v_j$ and j as small as possible. Thus $v_0, \dots, v_{j-1}$ are distinct. If $j - i$ is even, then we can delete $v_{i+1}, \dots, v_j$ from P and obtain a shorter M-alternating chain from X to X. Thus we may assume that $j - i$ is odd.
(ii) If j is even and i is odd then $v_{i+1} = v_{j-1}$ since M is a matching. This contradicts the assumption that j is minimal.
(iii) Hence j is odd and i is even. Then $[v_0, \dots, v_j]$ is an M-flower.

We now show how to shrink a blossom to a single point. Let $B \subseteq G$. Let B be a new point; this new point replaces the points of B. The graph G/B has points $G \setminus B \cup \{B\}$. The lines of G/B are the lines of Γ in which any endpoint $b \in B$ is replaced by the new point B. Any loop that appears in the process is ignored. If M is a matching in G then we let M/B denote the corresponding set of lines in G/B; lines of M with two endpoints in B are not in M/B.

> Show that M/B is a matching in G/B if M has at most one line with one endpoint in B.

[12] A. Schrijver, [Section 24.2, Volume A] *Combinatorial optimization: Polyhedra and efficiency*, Springer Series: Algorithms and Combinatorics **24**, Berlin 2003.

Let $B = [v_i, \dots, v_t]$ be an M-blossom in G. We prove that M is a matching in G of maximum size if and only if M/B is a matching in G/B of maximum size.

(1) First assume that M/B is a matching in G/B which is not maximum. Let P be an M/B-augmenting path in G/B. If P does not contain B then P is a M-augmenting path in G.
(2) Assume that P visits B. We may assume that P enters B via a line $\{u, B\} \notin M/B$. Thus $\{u, v_j\} \in \Gamma$ for some $j \in \{i, \dots, t\}$.
(3) If j is odd, then replace B in P by $[v_j, v_{j+1}, \dots, v_t]$. If j is even, then replace B in P by $[v_j, v_{j-1}, \dots, v_i]$. In both cases we obtain an M-augmenting path in G. Thus $|M|$ is not maximal.
(4) Assume that M is not maximum. We may assume that $i = 0$, that is, $v_i \in X$. Otherwise replace M by $M \div Q$, where Q is the set of lines in the chain $[v_0, \dots, v_i]$.
(5) Let $P = [u_0, \dots, u_s]$ be an M-augmenting path in G. If P does not visit B then P is an M/B-augmenting path in G/B.
(6) If P visits B then we may assume that $u_0 \notin B$. Otherwise, replace P by its reverse. Let u_j be the first vertex of P in B.
(7) Then $[u_0, u_1, \dots, u_{j-1}, B]$ is an M/B-augmenting path in G/B. Thus $|M/B|$ is not maximal.

6.10 (Hall). Let $A_1, \dots, A_n$ be subsets of a set T. A *system of distinct representatives* is a set $\{t_1, \dots, t_n\}$ of elements of T such that $t_i \in A_i$ for all $i = 1, \dots, n$ and such that $t_i \neq t_j$ whenever $i \neq j$.

(a) Construct a bipartite graph $G = (T \cup \{1, \dots, n\}, \Gamma)$, where a line joins an element $t \in T$ with $i \in \{1, \dots, n\}$ if and only if $t \in A_i$. Show that a system of distinct representatives corresponds to a maximum matching in G.
(b) Show that the collection $A_1, \dots, A_n$ has a system of distinct representatives if and only if

$$|\cup_{i \in J} A_i| \geqslant |J| \quad \text{for all } J \subseteq \{1, \dots, n\}.$$

6.11 (Frobenius[13]). A cover with dimers is also called a perfect matching or a 1-factor. A bipartite graph $(A \cup B, \Gamma)$ has a perfect matching if and only if

$$|A| = |B| \quad \text{and} \quad \forall_{X \subseteq A}\ |X| \leqslant |N(X)|,$$

where as usual, $N(X)$ is the set of points in B that are connected by a line to some point in X.

[13] G. Frobenius, Über zerlegbare Determinanten, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1917), pp. 456–477.

6.12 (Birkhoff-von Neumann[14] [15]). An $(n \times n)$-matrix A with elements a_{ij} is doubly stochastic if

$$\forall_{i,j}\ a_{ij} \geqslant 0 \quad \text{and} \quad \forall_j \sum_i a_{ij} = 1 \quad \text{and} \quad \forall_i \sum_j a_{ij} = 1.$$

Recall Theorem 6.6 which says that A has an n-diagonal. Use this to show that A is the convex linear combination of permutation matrices; that is

$$A = \lambda_1 \cdot P_1 + \dots \lambda_k \cdot P_k$$

where each P_i is a permutation matrix, and the λ_i's are nonnegative real number with $\sum_{i=1}^{k} \lambda_i = 1$.

6.13 (van der Waerden). Let $\mathfrak{M}$ be a finite set and let $\mathfrak{U} = \{\mathfrak{U}_1, \dots, \mathfrak{U}_\mu\}$ and $\mathfrak{B} = \{\mathfrak{B}_1, \dots, \mathfrak{B}_\mu\}$ be two partitions of $\mathfrak{M}$ with $|\mathfrak{U}_i| = |\mathfrak{B}_i| = n$ for $i = 1, \dots, \mu$. In this exercise we prove that there exists a a collection $x_1, \dots, x_\mu$ of elements of $\mathfrak{M}$ that represents each $\mathfrak{B}$-class and each $\mathfrak{U}$-class.

(a) Let $x_1, \dots, x_\nu$ be a maximal system of elements such that no two x_i are in the same $\mathfrak{U}$-class or in the same $\mathfrak{B}$-class. Thus $\nu \leqslant \mu$, and we may assume that $\nu < \mu$ since otherwise we are done.
(b) We can assume that x_i is in $\mathfrak{U}_i$ and in $\mathfrak{B}_i$, for $i = 1, \dots, \nu$.
(c) Let $y \in \mathfrak{B}_{\nu+1}$. Then y is also in $\mathfrak{U}_i$ for some $i \leqslant \nu$, otherwise we could add y to the collection $x_1, \dots, x_\nu$.
Let's say that a class $\mathfrak{U}_h$ is *connected to* y if there is a "chain"

$$\mathfrak{U}_i = \mathfrak{U}_{i_1}, \mathfrak{B}_{1_1}, \mathfrak{U}_{i_2}, \mathfrak{B}_{i_2}, \dots, \mathfrak{U}_{i_{\omega-1}}, \mathfrak{B}_{i_{\omega-1}}, \mathfrak{U}_{i_\omega} = \mathfrak{U}_h,\ (\therefore i_1 = i,\ i_\omega = h)$$

which is a sequence in which every consecutive pair $\mathfrak{B}_{i_\lambda}, \mathfrak{U}_{i_{\lambda+1}}$ have an element y_λ in common.
(d) If the chain has two copies of the same $\mathfrak{U}_i$, then we can omit the part of the sequence from one copy to the next. Thus we may assume that in a chain all $\mathfrak{U}$-classes are different.
(e) Assume that one of the indices $i_1, \dots, i_\omega$ of a chain is $> \nu$. Without loss of generality, assume that $i_1, \dots, i_{\omega-1} \leqslant \nu$ and that $i_\omega > \nu$. Then we can replace the elements $x_{i_1}, \dots, x_{i_{\omega-1}}$ in $\{x_1, \dots, x_\nu\}$ by $y, y_1, \dots, y_{\omega-1}$; those elements are contained, in that order, in the $\mathfrak{U}$-classes

$$\mathfrak{U}_{i_1}, \mathfrak{U}_{i_2}, \dots, \mathfrak{U}_{i_\omega}$$

and in the $\mathfrak{B}$-classes

[14] Birkhoff, G., Tres observaciones sobre el algebra lineal, Univ. Nac. Tucumán Rev., Ser. A, no. 5 (1946), pp. 147–151.

[15] J. von Neumann, A certain zero-sum two-person game equivalent to the optimal assignment problem. In: *Contributions to the theory of games*, Vol. 2, Princeton University Press, Princeton, N.J. USA, 1953, pp. 5–12.

$$\mathfrak{B}_{\nu+1}, \mathfrak{B}_{i_1}, \ldots, \mathfrak{B}_{i_{\omega-1}}.$$

However, this contradicts the maximality of the set $\{x_1, \ldots, x_\nu\}$; the new set satisfies the condition that no element is in more than one $\mathfrak{U}$-class or in more that one $\mathfrak{B}$-class as well, and the cardinality of the new set is one greater than that of the old set.

To recapitulate; we now have that $i_1, \ldots, i_\omega \leqslant \nu$, and if $\mathfrak{U}_h$ is connected to y then $h \leqslant \nu$.

(f) The union of all $\mathfrak{U}$-classes that are connected to y contains all the corresponding $\mathfrak{B}$-classes: Let $z \in \mathfrak{B}_h$, for some h for which y is connected to $\mathfrak{U}_h$. Let $\mathfrak{U}_j$ be the class that contains z. Consider the extension

$$\mathfrak{U}_{i_1}, \mathfrak{B}_{i_1}, \ldots, \mathfrak{U}_{i_\omega}, \mathfrak{B}_{i_\omega}, \mathfrak{U}_j, \ (i_1 = i, \ i_\omega = h).$$

Then y is connected to $\mathfrak{U}_j$, and thus z is in the union of the $\mathfrak{U}$-classes that are connected to y.

(g) Since all classes have the same number of elements, we have that the union of $\mathfrak{U}$-classes that are connected to y is the same as the union of the corresponding $\mathfrak{B}$-classes. Since y is in the y-connected class $\mathfrak{U}_i$, y must also be in some $\mathfrak{B}_h$ for which the corresponding $\mathfrak{U}_h$ is connected to y.

(h) If $\mathfrak{U}_h$ is connected to y then $h \leqslant \nu$, thus y must be in

$$\mathfrak{B}_1 \cup \ldots \cup \mathfrak{B}_\nu.$$

This is a contradiction, since $y \in \mathfrak{B}_{\nu+1}$.

Bartel L. van der Waerden mentions in his paper, that we outlined here, the application of a common system of representatives for the left and right cosets of a subgroup of a finite group. An extension of the theorem to the infinite case was given by de Bruijn.[16]

6.14 (Tutte[17]). Let (G, Γ) be a graph. A component is a maximal subset of vertices which induces a connected subgraph. Thus $C \subseteq G$ is a component if $|C|$ is maximal under the condition that the graph (C, Γ'), where

$$\Gamma' = \{\, \{x, y\} \mid \{x, y\} \in \Gamma \text{ and } x, y \in C\}$$

is connected. If (G, Γ) is connected then G is its only component; in all other cases G has at least two components.

A component C is odd if $|C|$ is odd. For a graph (G, Γ) we write $\theta(G)$ for the number of odd components in G. If $X \subseteq G$, then we write $G - X$ for the subgraph induced by $G - X$; thus this is the graph with $G \setminus X$ as its set of points and with $\Gamma' \subseteq \Gamma$ as its set of lines, where Γ' is the set of those lines that connect two points of $G \setminus X$.

[16] N. G. de Bruijn, Gemeenschappelijke representantensystemen van twee klassenindeelingen van een verzameling, *Nieuw archief voor wiskunde* **22** (1943), pp. 48–52.

[17] W. T. Tutte, The factorisation of linear graphs, *J. London Math. Soc.* **22** (1947), pp. 107–111.

In this exercise we prove Tutte's theorem: (G, Γ) has a perfect matching if and only if

$$\forall_{X \subseteq G}\ \theta(G - X) \leqslant |X|. \tag{6.1}$$

(a) First we show that (6.1) is necessary. Let (G, Γ) be a graph and let M be a perfect matching. Let $X \subseteq G$. If C is an odd component in $G - X$ then at least one line of M must join a point in C with a point in X. If J is the set of all those lines then

$$\theta(G - X) \leqslant |J| \leqslant |X|,$$

since M is a matching. This proves necessity.

(b) Suppose (6.1) holds. We proceed by induction on the size of G. Consider $X = \varnothing$. It implies that every component of G must be even. We may assume that (G, Γ) is connected and that $|G|$ is even.
Also, $|X|$ and $\theta(G - X)$ have the same parity for all X.

(c) Assume that

$$\forall_{\substack{X \subseteq G \\ X \neq \varnothing}}\ |X| > \theta(G - X).$$

Delete two points u and v that are on a line $\{u, v\} \in \Gamma$. Then, by induction $G - \{u, v\}$ has a perfect matching M', and thus $M = M' \cup \{\{u, v\}\}$ is a perfect matching for (G, Γ).

(d) Suppose

$$\exists_{\substack{X \subseteq G \\ X \neq \varnothing}}\ |X| = \theta(G - X).$$

Then choose such a subset X of maximal cardinality. Let S be the set of points of $G \setminus X$ that belong to even components of $G - X$. The graph $G[S]$ induced by S is the graph with S as its point-set and those lines of Γ that connect points in S. We claim that $G[S]$ has a perfect matching. Indeed, if there exists a subset $X' \subseteq S$ such that $G[S] - X'$ has more than $|X'|$ odd components, then $X \cup X'$ violates (6.1) for the graph (G, Γ).

(e) Let K be the set of points in odd components $G - X$. Let $\kappa \in K$. We claim that $G[K \setminus \{\kappa\}]$ has a perfect matching.
Otherwise there exists a $\bar{X} \subseteq K \setminus \{\kappa\}$ such that $G[K \setminus (\bar{X} \cup \{\kappa\})]$ has at least $|\bar{X}| + 2$ odd components. Then delete $X \cup \bar{X} \cup \{\kappa\}$ from (G, Γ). This gives at least $|X \cup \bar{X} \cup \{\kappa\}|$ odd components, which contradicts our choice of X.

(f) Now consider the bipartite graph G^b obtained by deleting
  (i) all points of S, and
  (ii) all lines from Γ that have both ends in X, and
  (iii) by 'contracting' all odd components of $G - X$ to 'pseudo-nodes.'

By this contraction we mean that we delete all the points of any odd component C and replace it with a single pseudo-node c. All lines that have one end in C are given the same endpoint c.
Let W be the set of pseudo-nodes. If G^b has a perfect matching, then we can combine a perfect matching of $G[S]$ with perfect matchings in each $G[C - v]$, where C is an odd component and v a suitable point in C.

(g) Finally, assume that G^b has no perfect matching. By Hall's, or Frobenius' theorem, since G^b is bipartite:

$$\exists_{W' \subseteq W} \, |N_{G^b}(W')| < |W'|.$$

Now $X^* = N_{G^b}(W') \subseteq X$ violates (6.1).

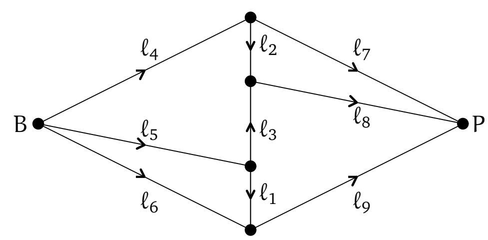

Fig. 6.13. A flow network

6.15 (Edmonds-Karp[18 19]). In this exercise we look at the maximal flow algorithm of Edmonds and Karp, which guarantees termination after at most $|G| \cdot |\Delta|$ augmentations. Thus this algorithm runs in $O\left(|G||\Delta|^2\right)$ time.

Before we go into that, let's first look at an example which shows that the Ford-Fulkerson algorithm does not need to converge to a maximal flow.

Consider the flow network in Figure 6.13. It consists of a finite oriented graph (G, Δ), two special vertices B (the source) and P (the sink), and a capacity function $\nu : \Delta \to \mathbb{R}^+$. In this example we take

$$\begin{aligned} &\nu(\ell_1) = \nu(\ell_3) = 1, \\ &\nu(\ell_2) = \frac{\sqrt{5}-1}{2}, \text{ and} \\ &\nu(\ell_4) = \nu(\ell_5) = \nu(\ell_6) = \nu(\ell_7) = \nu(\ell_8) = \nu(\ell_9) = M \geqslant 4. \end{aligned}$$

(i) Show that the maximum flow in this network is $2M + 1$.
(ii) Suppose we use the augmenting path from B to P along ℓ_5, ℓ_3, and ℓ_8. A flow of 1 is sent along this path and ℓ_3 becomes saturated. The residual capacities of ℓ_1, ℓ_2, and ℓ_3 are now 1, $\frac{\sqrt{5}-1}{2}$, and 0, respectively.
(iii) We now consider the following augmenting paths:
 (a) $\mathcal{P}_1$ is an augmenting path from B along ℓ_4, ℓ_2, ℓ_3, ℓ_1, and ℓ_9 to P.
 (b) $\mathcal{P}_2$ is an augmenting path from B along ℓ_5, ℓ_3, ℓ_2, and ℓ_7 to P.

[18] J. Edmonds and R. M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, *Journal of the ACM* **19** (1972), pp. 248–264.
[19] U. Zwick, The smallest networks on which the Ford-Fulkerson maximum flow procedure may fail to terminate, *Theoretical Computer Science*, **148** (1995), pp. 165–170.

(c) $\mathcal{P}_3$ is an augmenting path from B along ℓ_6, ℓ_1, ℓ_3, and ℓ_8 to P.

Suppose that we choose the following sequence of augmenting paths.

$$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1, \mathcal{P}_3, \ldots$$

(Thus repeatedly choose the augmenting paths $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1, \mathcal{P}_3$.)
Show that the total flow in the network at any stage is at most 3.

(iv) This proves that the Ford-Fulkerson algorithm may fail to terminate. Furthermore, it may not even converge to a maximal flow.

Let us now have a look at the Edmonds-Karp algorithm.

Suppose (G, Δ) is a flow network with capacity function $\nu : \Delta \to \mathbb{R}^+$, a source B, and a sink P.

(1) Define
$$\Delta^{-1} = \{\, \ell^{-1} \mid \ell \in \Delta \,\}$$
where ℓ^{-1} is defined as
$$\ell = (u, v) \in \Delta \quad \Rightarrow \quad \ell^{-1} = (v, u).$$

(2) For a flow $f : \Delta \to \mathbb{R}^+$ satisfying $0 \leqslant f(\ell) \leqslant \nu(\ell)$ for all $\ell \in \Delta$, define the residual network (G, Θ_f) by
$$\Theta_f = \{\, \ell \mid \ell \in \Delta \text{ and } f(\ell) < \nu(\ell) \,\} \cup \{\, \ell^{-1} \mid \ell \in \Delta \text{ and } f(\ell) > 0 \,\}.$$

(3) Let $\mu(G, \Theta_f)$ denote the minimal length of a directed $B - P$ path in (G, Θ_f). Let $\alpha(G, \Theta_f)$ denote the set of arrows that are contained in at least one shortest $B - P$ path.

(4) Suppose we augment a flow f along a shortest $B - P$ path $\mathcal{P}$ in (G, Θ_f) as much as possible by the bottleneck capacity in $\mathcal{P}$. Let f' be the new flow. Then $(G, \Theta_{f'})$ is a subgraph of (G, Θ') where
$$\Theta' = \Theta_f \,\cup\, \alpha(G, \Theta_f)^{-1}.$$

(5) Show that
$$\mu(G, \Theta_{f'}) \geqslant \mu(G, \Theta') = \mu(G, \Theta_f).$$

(6) Show also that
$$\alpha(G, \Theta_{f'}) \subseteq \alpha(G, \Theta') = \alpha(G, \Theta_f).$$

(7) At least one line in $\mathcal{P}$ belongs to Θ_f but not to $\Theta_{f'}$. This implies
$$\alpha(G, \Theta_{f'}) \subset \alpha(G, \Theta_f).$$

(8) Show that $\mu(G, \Theta_f)$ increases at most $|G|$ times.

(9) As long as $\mu(G, \Theta_f)$ does not change, show that $\alpha(G, \Theta_f)$ decreases at most $|\Delta|$ times.

(10) By (8) and (9), show that, if we choose in each round a shortest $B - P$ path in (G, Θ_f) as a flow-augmenting path, then the number of rounds is at most $|G| \cdot |\Delta|$.

6.16 (de Bruijn[20]). Let (G, Γ) be a graph with $n = 2m$ points. A line $\{i, j\}$ has a weight b_{ij}. We go into the problem of covering a graph with dimers, embarked upon on Page 108.

(a) A cycle is a subset of Γ that can be written as

$$\{\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}, \{i_k, i_1\}\}, \tag{6.2}$$

where $i_1, \dots, i_k$ are $k \geqslant 2$ distinct points. The number k is the length of the cycle. Note that here we allow a cycle of length 2, which is just a line.

(b) The weight of a cycle as above, is the product

$$b_{i_1 i_2} b_{i_2 i_3} \dots b_{i_k i_1}.$$

Thus the weight of a cycle $\{\{i, j\}, \{j, i\}\}$ of length 2 is b_{ij}^2.

(c) A complete set of cycles is a set of cycles whose point sets form a partition of G. The weight of a complete set of cycles is the product of the weight of the cycles in the set. A complete set of even cycles is a complete set of cycles in which all cycles are even. Note that a complete set of cycles of length 2 is a perfect matching.

(d) If

$$\{\{i_1, i_2\}, \{i_3, i_4\}, \dots, \{i_{n-1}, i_n\}\}$$

is a perfect matching then the product

$$\rho = b_{i_1 i_2} b_{i_3 i_4} \dots b_{i_{n-1} i_n}$$

is the root weight of the matching. Note that ρ^2 is the weight of the matching (in the sense of the weight of a complete set of cycles).

(e) An orientation of (G, Γ) is a skew symmetric $n \times n$ matrix ϵ with

$$\begin{array}{lll} \epsilon_{ij} = 0 & \text{if } \{i, j\} \notin \Gamma & \\ \epsilon_{ij} = +1 \text{ or } -1 & \text{if } \{i, j\} \in \Gamma & \text{and} \quad \epsilon_{ji} = -\epsilon_{ij}. \end{array}$$

(f) We give even cycles as in (6.2) a sign

$$-\epsilon_{i_1 i_2} \epsilon_{i_2 i_3} \dots \epsilon_{i_k i_1}.$$

Note that this sign does not depend on the way the even cycle is represented; a cyclic shift or an opposite traversal does not change it. A cycle of length 2 has sign 1.

[20] N. G. de Bruijn, Counting complete matchings without using Pfaffians, *Indagationes Mathematicae* **42** (1980), pp. 145–151.

(g) An orientation is admissible, if in every complete set of even cycles all even cycles have sign $+1$. Graphs that have an admissible orientation are called Pfaffian.

(h) Let ϵ be an admissible orientation and let

$$S = \sum_{\beta} \rho(\beta)$$

where β runs through the set of all perfect matchings and $\rho(\beta)$ is the root weight of β. Then

$$S^2 = \det A,$$

where A is the skew matrix with elements $a_{ij} = \epsilon_{ij} b_{ij}$; we prove that in the following steps.

(i) Consider

$$S^2 = \sum_{\beta_1, \beta_2} \rho(\beta_1)\rho(\beta_2)$$

where the sum is taken over all pairs (β_1, β_2) of perfect matchings. If β_1 and β_2 are perfect matchings then we obtain a complete set of even circuits $\gamma(\beta_1, \beta_2)$ by taking the union $\beta_1 \cup \beta_2$; the edges of an even cycle of length > 2 in $\gamma(\beta_1, \beta_2)$ alternate between β_1 and β_2. A cycle of length 2 is a single edge that belongs to both β_1 and β_2.

(j) Conversely, let γ be a complete set of even cycles with weight $\omega(\gamma)$. Let $\nu(\gamma)$ be the number of cycles of length > 2. We can split γ in $2^{\nu(\gamma)}$ ways into pairs β_1 and β_2 with $\gamma = \gamma(\beta_1, \beta_2)$ by 2-coloring the edges of every cycle of length > 2 in γ. The weight of γ is the product of the root weights of β_1 and β_2. Thus

$$S^2 = \sum_{\gamma} 2^{\nu(\gamma)} \omega(\gamma), \tag{6.3}$$

where γ runs through the set of all complete sets of even cycles.

(k) Recall the Leibniz formula for the determinant of an $n \times n$ matrix:

$$\det A = \sum_{\pi} \operatorname{sign}(\pi) a_{1\pi(1)} \dots a_{n\pi(n)} \tag{6.4}$$

where π runs through all permutations of $\{1, \dots, n\}$. The $\operatorname{sign}(\pi)$ is $+1$ or -1 according to whether π is even or odd (Page 26). We can omit permutations with $\{i, \pi(i)\} \notin \Gamma$ for some i since those do not contribute to the sum. In particular we omit all those permutations π with $\pi(i) = i$ for some i.

(l) For the permutations π that we are considering, the cycles of π are the cycles of (G, Γ) equipped with some orientation. Cycles in (G, Γ) of length > 2 have two possible orientations, and cycles of length 2 only one.

(m) If γ is any complete set of cycles, then there are $2^{\nu(\gamma)}$ ways to orient the cycles, and this leads to $2^{\nu(\gamma)}$ permutations π.

(n) Reversing a cycle of odd length changes the sign of the term in (6.4), since the matrix is skew. Thus if γ has odd cycles, then the $2^{\nu(\gamma)}$ terms cancel in pairs. Thus we need to consider only complete sets of even cycles.

(o) We now show that (6.3) and (6.4) are equal. We claim that all the $2^{\nu(\gamma)}$ terms in (6.4) that belong to the same γ (with even cycles only) are equal to $\omega(\gamma)$. We have

$$\omega(\gamma) = b_{1\pi(1)} \dots b_{n\pi(n)}.$$

Note also that $\text{sign}(\pi)$ is the product of as many factors -1 as there are even cycles. And finally note that the product of the ϵ's in the even cycles is -1 because the orientation is admissible.

6.17 (de Bruijn[21]). Consider n-dimensional bricks $a_1 \times \dots \times a_n$ and boxes $A_1 \times \dots A_n$. A box $A_1 \times \dots \times A_n$ is a multiple of $a_1 \times \dots \times a_n$ if there are integers $q_1, \dots, q_n$ such that the numbers $q_1 a_1, \dots, q_n a_n$ are a permutation of $A_1, \dots, A_n$. Of course, if the box is a multiple of the brick, then the box can be filled trivially with all bricks in parallel.

We first prove that if a box can be filled with bricks then at least one of the A_i is a multiple of a_1, and at least one of the A_i is a multiple of a_2, etc.

(a) Any brick $a_1 \times \dots \times a_n$ can be subdivided into bricks $a_1 \times 1 \times \dots \times 1$. Assume that the box $A_1 \times \dots \times A_n$ can be filled with bricks. Then it can also be filled with bricks $a_1 \times 1 \times \dots \times 1$.

(b) Divide each $a_1 \times 1 \times \dots \times 1$ into a_1 cubes $1 \times \dots \times 1$. The box now contains $A_1 \cdots A_n$ cubes. Each cube has coordinates $(k_1, \dots, k_n)$ where $1 \leqslant k_1 \leqslant A_1, \dots, 1 \leqslant k_n \leqslant A_n$. Consider the sum

$$S(A) = \sum_{k=1}^{A} e^{2\pi i k/a_1},$$

and the multiple sum

$$\sum_{k_1=1}^{A_1} \dots \sum_{k_n=1}^{A_n} \exp\left(2\pi i(k_1 + \dots + k_n)/a_1\right) = S(A_1) \cdots S(A_n).$$

(c) Each term corresponds with a cube in the box. These cubes can be grouped together into blocks of a_1 terms each, combining terms that belong to the same brick $a_1 \times 1 \times \dots \times 1$. In each group of terms the index runs through a set of a_1 consecutive integers and the other indices remain constant. The contribution of such a group to the total sum is zero.

(d) The whole box is filled with bricks $a_1 \times 1 \times \dots \times 1$, thus the multiple sum over all cubes vanishes. Therefore one of the $S(A_j)$ is zero.

(e)

$$S(A_j) = x + x^2 + \dots + x^{A_j} = x \cdot \frac{\left(x^{A_j} - 1\right)}{x - 1}, \quad \text{where} \quad x = e^{2\pi i/a_1}.$$

[21] N. G. de Bruijn, Filling boxes with bricks, *American mathematical Monthly* **76** (1969), pp. 37–40. Apparently the problem arose from a remark by the author's son, F. W. de Bruijn, who discovered, at the age of seven, that he was unable to fill his $6 \times 6 \times 6$ box with bricks $1 \times 2 \times 4$.

Thus

$$S(A_j) = 0 \;\Rightarrow\; e^{2\pi i A_j/a_1} = 1 \;\Rightarrow\; a_1 \mid A_j.$$

A brick $a_1 \times \ldots \times a_n$ is harmonic if the numbers $a_1, \ldots, a_n$ can be permuted into $a'_1, \ldots, a'_n$ such that

$$a'_1 \mid a'_2 \quad \text{and} \quad a'_2 \mid a'_3 \quad \text{and} \quad \ldots \quad \text{and} \quad a'_{n-1} \mid a'_n.$$

We claim that if the box can be filled with harmonic bricks $a_1 \times \ldots \times a_n$ then the box is a multiple of the brick. We prove this is the next few steps.

(i) We prove this by induction on the dimension n. If $n = 1$ then the claim is trivial.

(ii) Without loss of generality assume that

$$a_1 \mid a_2, \quad a_2 \mid a_3, \quad \ldots \quad a_{n-1} \mid a_n.$$

If the box $A_1 \times \ldots \times A_n$ is filled with bricks then, by the previous result, one of the A_i is a multiple of a_n. Assume that A_n is a multiple of a_n.

(iii) Consider one $(n-1)$-dimensional face of the box $A_1 \times \ldots \times A_{n-1}$. This is filled with bricks of various sizes $a_2 \times \ldots \times a_n$, $a_1 \times a_3 \times \ldots \times a_n$, and so on, up to bricks $a_1 \times a_2 \times \ldots \times a_{n-1}$. Because the bricks are harmonic, all of them can be subdivided into bricks $a_1 \times \ldots \times a_{n-1}$.

(iv) By induction $A_1 \times \ldots \times A_{n-1}$ is a multiple of the harmonic brick $a_1 \times \ldots \times a_{n-1}$. Since we already have that $a_n \mid A_n$, we conclude that $A_1 \times \ldots \times A_n$ is a multiple of $a_1 \times \ldots \times a_n$.

Finally we show that if the brick is not harmonic, then there is a box which can be filled and which is not a multiple of the brick.

We may assume that $n > 1$ and that $a_1 \leqslant \ldots \leqslant a_n$.

(1) Let k be the largest integer, $2 \leqslant k \leqslant n$, such that

$$a_{k-1} \nmid a_k.$$

Thus $a_{k+1}, \ldots, a_n$ are multiples of a_k.

(2) A box $(a + b) \times ab$ can be filled with bricks $a \times b$. Therefore, the box

$$a_1 \times \ldots \times a_{k-2} \times (a_{k-1} + a_k) \times a_{k-1}a_k \times a_{k+1} \times \ldots \times a_n$$

can be filled with bricks $a_1 \times \ldots \times a_n$. We show that this box is not a multiple of the brick.

(3) Let j be the smallest integer such that

$$a_j = a_{j+1} = \ldots = a_{k-1}.$$

Note that $a_{k-1} + a_k$ is not divisible by a_{k-1} or by a_k or by any multiple of a_k. Since $a_{k+1}, \ldots, a_n$ are all multiples of a_k, no number of $a_1, \ldots, a_{j-1}, a_{k-1} + a_k$ is divisible by any number of

$$a_j, \ldots, a_{k-1}, a_k, \ldots, a_n.$$

(4) If a box $A_1 \times \ldots \times A_n$ is a multiple of the brick $a_1 \times \ldots \times a_n$, then there can be at most $j-1$ i's such that A_i is no multiple of any of $a_j, \ldots, a_n$. Therefore, the box above is not a multiple of the brick $a_1 \times \ldots \times a_n$.

7

Graphs and games

Because it is almost Saint Nicholas, we play some games; mathematical two-person games. Only 2-person games are Democratic; if more persons are involved then the situation is seldom democratic, due to coalitions etc. For example, marriage is a democratic game: most votes win if and only if the decision is unanimous.

7.1 Introduction

Let (G, Δ) be an oriented graph, not necessarily finite, that satisfies:

(1) there are no oriented circuits,
(2) every point has finite out-degree, and
(3)
$$\forall_{p \in G}\ \exists_{c>0}\ [\text{ every chain that starts in } p \text{ has length} < c\ ].$$

For example $(\mathbb{N}_0, \Delta)$ with

$$\Delta = \{(n+1, n) \mid n \in \mathbb{N}_0\} \cup \{(n+2, n) \mid n \in \mathbb{N}_0\}.$$

A *dead point* is a point with out-degree 0. In our example 0 is a dead point.

The game will be played by two players A and B (not necessarily men or women) and a "fiche" (a token). At the start of the game the fiche is somewhere at some point in the graph. A "move" is a movement of the fiche along an arrow of the graph to a new point. Players make alternate moves.

There is also a prize function w that maps every dead point to a real number. The player who cannot make a move gets the prize, namely the w of that dead point.

In the game above you can win if you can leave a multiple of 3 for your opponent. Then you can leave him another multiple of 3 after any move of your opponent.

Finally, we define the *run-out* of a point P. This is the supremum of the lengths of the chains that start in P. The run-out is also defined for arbitrary oriented graph, in which case the run-out can be infinite. We have König's infinity lemma.

Lemma 7.1 (König's infinity lemma). *If every vertex has finite out-degree and some vertex* $P \in G$ *has infinite run-out, then there is an infinite chain that starts in* P.

NB The condition that the out-degree is finite is of course essential, as the following example shows:

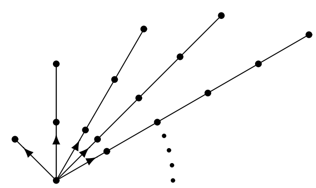

Fig. 7.1. Run out and out-degree both infinite, but there is no infinite path

Proof. The run-out of P is infinite, so there must be a point at other end of an arrow that starts in P, that has also infinite run-out. So you can start the chain in P and always choose a successor with an infinite run-out. □

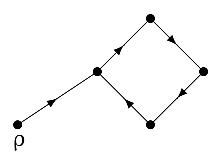

Fig. 7.2. Point with infinite run-out

Most games have a negative prize in dead points, and the objective is to maximize the prize, thus to avoid the negative prize.

Chess.

Some games can be modelled with a graph.

Consider chess. There are some complications, for example the "white player" is not allowed to move black's pieces and vice versa.

Any board situation has two representatives in the "game graph:" in one "white starts," or "white plays" and in the other "black plays." A "white plays" point is always followed by a "black plays" point.

But there are more complications. Whether some moves are allowed or not may depend on the history of the game. For example, such moves are "rochade" and "en passant."

There are two types of dead points. In one the king is under attack and in the other it is not. Different types of dead points have different prizes.

Then there are rules such as the 40 moves rule, and the rule that you may claim a draw when the same situation appears on the board for a third time. Furthermore, any player may give up at any point in the game. This corresponds with a game in which *every* point P has a prize $w(P)$ attached to it. A move now either collects the prize, or moves in the graph. We can also simulate this in an ordinary game-graph by growing a tail of length 2 at every point, and attaching $w(P)$ to the end of that tail.

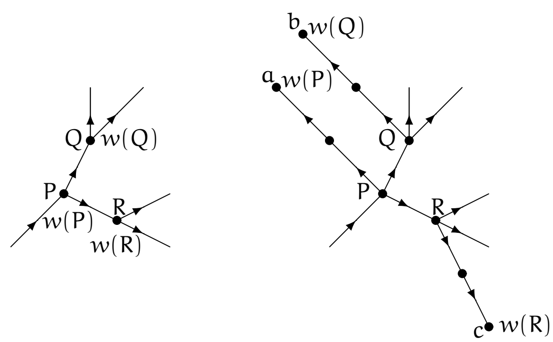

Fig. 7.3. Attaching a tail to every point

We may simulate chess by a graph but we haven't covered the situation yet where one of the players offers a draw, and where the other player then turns beet red and resigns.

7.2 Pay-off function

Let's get back to graphs.

We can partition G as $G_0 \cup G_1 \cup \ldots \cup G_k \cup \ldots$ where each

$$G_k \text{ is the set of points with run-out } k.$$

You can also allow G_∞ if there are points with infinite run-outs.

Theorem 7.2. *Let* (G, Δ) *be an oriented graph, such that every point has finite run-out and finite out-degree. Let* $w : G_0 \to \mathbb{R}$*. Then there is exactly* one *function* $\psi : G \to \mathbb{R}$ *such that*

(i) $\psi |_{G_0} = w$, *and*
(ii) *for every* $P \in G \setminus G_0$

$$\psi(P) = \max_{Q \in \Gamma(P)} -\psi(Q). \tag{7.1}$$

where $\Gamma(P)$ *is the set of points that can be reached from* P *within one move:*

$$\Gamma(P) = \{Q \in G \mid (P, Q) \in \Delta\}.$$

Proof. Evident: ψ can be defined recursively. The function-values on G_0 are determined. Assume that ψ is determined on $G_0 \cup \ldots \cup G_k$. Then for every $P \in G_{k+1}$, $\Gamma(P)$ is in $G_0 \cup \ldots \cup G_k$ and then $\psi(P)$ is determined by 7.1. □

The game strategy is now as follows. Move the fiche to a point with a ψ as small as possible. Assume that there are the following situations in a point P (we illustrate all the possibilities).

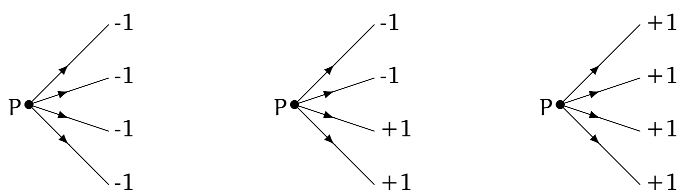

Fig. 7.4. Left: whatever you do, the opponent loses; $\psi(P) = 1$. Right: Whatever you do, the opponents wins; $\psi(P) = -1$. In the middle case: You can make a wrong move, but you win by moving to -1; $\psi(P) = 1$

This also proves the optimality of the strategy for the "1-or-2-back game" that we started with. Every two $+1$'s in a row are followed by a -1 (in that

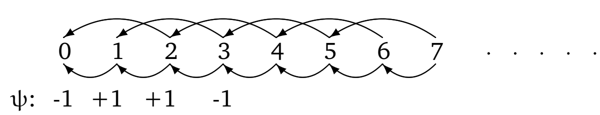

Fig. 7.5. 1-or-2-back game evaluation

situation any move goes to a $+1$ and thus it loses). A -1 is followed by two $+1$'s; in that situation you can make a move such that your opponent loses.

7.3 Nim

In this section we discuss Nim; in Chinese "jianshizi."

This is a pretty well-known game, but it is surprising how many people have never heard of it. It is played with a bunch of matches. There are a number of heaps of matches (say k). A move consists of taking away one or more matches from one of the heaps. Who can't move, loses. (So the one who takes the last match, wins.)

What are the positions in which you lose? To analyze that we take the nim-sum: Write the numbers $n_1, \ldots, n_k \in \mathbb{N}_0$ in binary, add them up modulo 2, and interpret that as a number in binary. For example:

$$\begin{array}{lr} 3 & 11 \\ 5 & 101 \\ 8 & 1000 \\ \hline 14 = & 1110 \end{array}$$

Let us write $3 \dotplus 5 \dotplus 8 = 14$; likewise $3 \dotplus 3 \dotplus 3 \dotplus 3 \dotplus 3 = 3$. The result is that position $n_1, \ldots, n_k$ loses if $n_1 \dotplus n_2 \dotplus \ldots \dotplus n_k = 0$. We prove that by showing the following. Define

$$\tilde{\psi}(n_1, \ldots, n_k) = \begin{cases} -1 & \text{if } n_1 \dotplus \ldots \dotplus n_k = 0 \\ +1 & \text{otherwise.} \end{cases}$$

We prove that $\tilde{\psi}$ has the required property.

Proof. In the 0-position it works: $\tilde{\psi}(0, \ldots, 0) = -1$.

We must now prove that

$$\tilde{\psi}(n_1, \ldots, n_k) = \max_{Q \in \Gamma(n_1, \ldots, n_k)} -\tilde{\psi}(Q).$$

If the nim-sum $\neq 0$, for example

$$\begin{array}{lr} 45 & 1\underline{0}1101 \\ 54 & 1\underline{1}0110 \\ 13 & 1101 \\ \hline 22 = & 0\underline{1}0110 \end{array}$$

then we can reach nim-sum $= 0$ in *one* move: Take the first column that sums up to 1. That column contains a 1. Pick a row that has a 1 there. Adjust the number in that row (in our case 54) as follows. Lower the 1 to zero and make sure that in every other column the total number of 1's becomes even. The middle row in our example becomes

$$32 \quad 100000$$

This means that by taking 22 matches from the middle heap we change the nim-sum into 0.

Vice versa, if the nim-sum$=0$, then any move ruins the nim-sum. $\square$

Actually, this procedure works also with *additions*, but then you need of course another stop-rule, and other rules to play the game altogether, otherwise the game doesn't end.

A game that looks like Nim is the following. Start with an infinite row of slots, in which you put 10 NT$-coins. Players alternately move coins to the

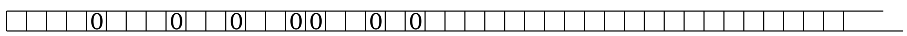

Fig. 7.6. Infinite row of slots with 10 NT$-coins

left, but not passing over another coin. Who can't move, loses. (You can also play this with sugar-rice cakes.)

N. G. de Bruijn recalls that he invented the game, some long time ago, and then discovered that a book of R. Sprague[1] contained it also. But when he looked into a first edition of that book, he saw that the Sprague alluded the game to de Bruijn!

For now, we leave the solution as an exercise; we include it at the end. Try the following hint, before looking at the back: The holes between the coins act like heaps of matches, but you need a little trick to take care of the fact that *two* "heaps" change at the same time when you make a move.

If your partner is already familiar with Nim, you may try a tree-version to fool your friends and family. Draw a number of rooted trees, and indicate the roots for example by putting a fiche on it. A move consists of moving the fiche up in the tree. Here you need to consider the Nim-position defined by

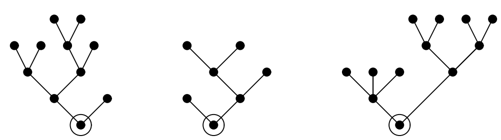

Fig. 7.7. Starting position for the game with trees

the collection of run-outs from the fiches (this is a multiset). The original Nim game corresponds to this game on trees where all the points have out-degree at most one.

[1] R. P. Sprague, *Recreations in mathematics*, Blackie and sons, London-Glasgow 1963.

7.4 Grundy function

We come to a general theory of Grundy (1939).[2]

Let (G, Δ) be an oriented graph with finite out-degree and finite run-out in every point. Let $w : G_0 \to \mathbb{R}$ be constant -1. We define $g : G \to \mathbb{R}$ as follows:

$$g(P) = \begin{cases} 0 \quad \text{if } P \in G_0 \quad \text{and, otherwise:} \\ \text{the smallest number in } \mathbb{N}_0 \text{ which does not ap-} \\ \text{pear as } g\text{-value among the vertices that can be} \\ \text{reached from } P \text{ in } \textit{one} \text{ move.} \end{cases}$$

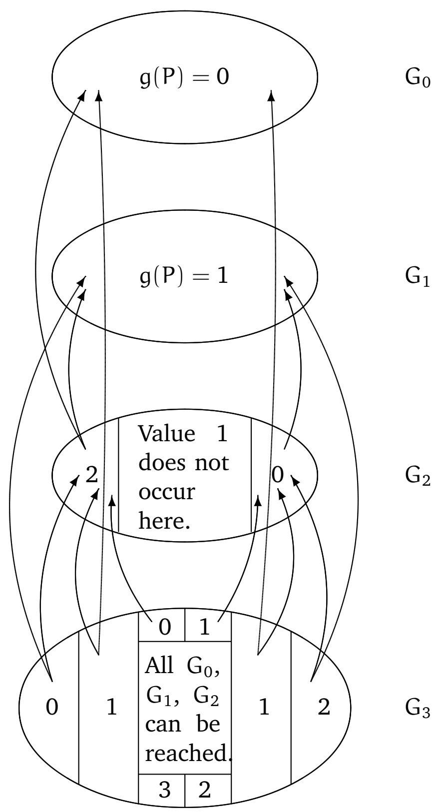

Fig. 7.8. Overview of some g-values

We now take the sum of games. First we do this for the case of a Nim-game with *one* tree as in Figure 7.9. If we have more games, for example as in Figure 7.10, then we define the product (or sum-) game as follows.

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[2] P. M. Grundy, Mathematics and games, *Eureka* **2** (1939), pp. 6–8.

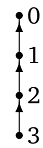

Fig. 7.9. Nim game with one tree: Grundy function is equal to the run-out

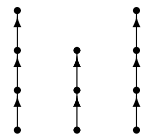

Fig. 7.10. More games $(G_i, \Delta_i), i = 1, 2, \ldots$

Choose i and make a move in $(G^{(i)}, \Delta_i)$. Define the graph (G, Δ) as follows. Let

$$G = G^{(1)} \times G^{(2)} \times \ldots \times G^{(k)}.$$

If $(g_1, \ldots g_k) \in G$ and $1 \leqslant m \leqslant k$ and $(g_m, h) \in \Delta_m$, then

$$((g_1, \ldots, g_k), (g_1, \ldots, g_{m-1}, h, g_{m+1}, \ldots, g_k)) \in \Delta.$$

The theorem of Grundy now says: The Grundy-function of the product-game is the nim-sum of the original games:

$$g(P_1, \ldots, P_k) = g_1(P_1) \dotplus \ldots \dotplus g_k(P_k)$$

The proof is a bit of a puzzle. We leave it as an exercise!

Notice that if the Grundy-funtion is $\neq 0$ then there is an exit in G_0; in other words, a losing position is one of the exits. If the Grundy-function is 0 then none of the exits belongs to G_0. So we have that

(a) Grundy-function $= 0 \Leftrightarrow$ losing position.
(b) Grundy-function $\neq 0 \Leftrightarrow$ winning position.

This classifies the winning– and losing positions of the Nim game.

7.5 Wijthoff game

A variation on Nim is the game of Wijthoff.[3] There are two heaps of matches. You can take matches of one of the two heaps or of both, but in that last case you can only take the same amount from both heaps. Thus

[3] W. A. Wijthoff, A modification of the game of Nim, *Nieuw Archief voor Wiskunde* **2** (1906-07), pp. 199–202.

$$\{5,9\} \to \begin{cases} \{2,9\} \\ \{5,2\} \\ \{3,7\} \end{cases}$$

are legal moves. Who can't move loses the game.

The losing positions are:

$$\begin{array}{rccccc} \text{difference:} & 0 & 1 & 2 & 3 & 4 \\ \text{smallest heap:} & 0 & 1 & 3 & 4 & 6 \\ \text{largest heap:} & 0 & 2 & 5 & 7 & 10 \\ & \ldots & \ldots & \ldots & & \end{array}$$

The number in the middle row is every time the smallest number that has not occurred yet. Suppose you move into a situation with a different smallest number. If you made the difference smaller, then your opponent can move to a losing position by taking away from both heaps, and otherwise he can move to a losing position my taking away from one of the two heaps. For example you move $\{6,10\} \to \{5,9\}$. Then you opponent can move to $\{5,3\}$ and you lose. The point is that all pairs that have 5 in it can be brought back to one of the positions with a dot underneath.

We prove that these are the losing positions. Say a losing position is $\{\text{largest},\ \text{smallest}\} = \{a,b\}$. Let b be the smallest number that does not occur in any losing position with a smaller difference. We have the following possibilities.

1. A move $\to \{a',b\}$ with $b \leqslant a' < a$.
   This is answered by taking away from both heaps to a losing position with a difference $a'-b$.
2. A move $\to \{a',b\}$ with $a' < b$.
   This is answered by choosing a new losing position with a' and with a partner $b' < b$. By choice of b this pair exists.
3. A move $\to \{a',b'\}$ with $a' = a-k$ and $b' = b-k$.
   This is answered by a move that lowers a' to a losing pair with a smaller difference.

What number are these? They are, for the smallest $[k\tau]$ and for the largest $[k\tau^2]$ where

$$\tau = \frac{1}{2} + \frac{1}{2}\sqrt{5} \quad \text{and} \quad \tau^2 = 1 + \frac{1}{2} + \frac{1}{2}\sqrt{5}.$$

Indeed, $k\tau$ and $k\tau^2$ differ by exactly k, and so do their integer parts $[k\tau]$ and $[k\tau^2]$.

The proof is based on the following theorem.

Theorem 7.3. *Let α and β be two positive numbers that are*

(1) *irrational, and that satisfy*
(2) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Then the sequence

$$[\alpha], [\beta], [2\alpha], [2\beta], \ldots$$

contains all the natural numbers exactly once.

Proof. We first give two proofs of the fact that no number occurs twice.

Proof (that no number occurs twice). Assume that

$$m < k\alpha < m + 1 \quad \text{and} \quad m < \ell\beta < m + 1$$

then

$$m\beta < k\alpha\beta < (m + 1)\beta \quad \text{and} \quad m\alpha < \ell\alpha\beta < (m + 1)\alpha$$

and so

$$m(\alpha + \beta) < (k + \ell)\alpha\beta < (m + 1)(\alpha + \beta)$$

and $\alpha + \beta = \alpha\beta$ thus

$$m < k + \ell < m + 1$$

which is a contradiction. □

Another (shorter) proof for the fact that no number occurs twice goes as follows:

Proof (that no number occurs twice). There are $\left[\frac{n}{\alpha}\right]$ multiples of α smaller than n and $\left[\frac{n}{\beta}\right]$ multiples of β smaller than n. Also

$$n - 2 = \frac{n}{\alpha} + \frac{n}{\beta} - 2 < \left[\frac{n}{\alpha}\right] + \left[\frac{n}{\beta}\right] < \frac{n}{\alpha} + \frac{n}{\beta} = n$$

so

$$\left[\frac{n}{\alpha}\right] + \left[\frac{n}{\beta}\right] = n - 1.$$

□

Now let $\frac{1}{\alpha}$ be the one of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ that has $\frac{1}{2} < \frac{1}{\alpha} < 1$. Then $1 < \alpha < 2$ and so the sequence

$$[\alpha], [2\alpha], \ldots$$

has only holes of length 1. We show that those holes are filled with multiples of β.

$$\begin{aligned}
\text{Assume} \quad & m < k\alpha < m+1 \\
\text{and} \quad & m+2 < (k+1)\alpha < m+3. \\
\text{Then} \quad & (m+2)\beta < (k+1)\alpha\beta = (k+1)(\alpha+\beta) \\
\text{thus} \quad & (m+1-k)\beta < (k+1)\alpha < m+3. \\
\text{Further} \quad & k\alpha\beta < (m+1)\beta \\
\Rightarrow \quad & k(\alpha+\beta) < (m+1)\beta, \\
\text{so} \quad & m < k\alpha < (m+1-k)\beta. \\
\text{Thus} \quad & [(m+1-k)\beta] \in \{m, m+1, m+2\}.
\end{aligned}$$

But m and $m+2$ are already occupied by $[k\alpha]$ and $[(k+1)\alpha]$ and so:

$$[(m+1-k)\beta] = m+1.$$

Thus all elements of $\mathbb{N}_0$ are covered. □

7.6 Other games

You can prove that the following game has a winning strategy but nobody knows what it is. Write down the numbers from 1 up to n. It does not matter; you can still analyze $n = 7$ or 8 but certainly not $n = 100$ or 200. A move consists of rubbing out a number and its divisors (that are not yet rubbed out). For example, with $n = 10$ you can get the following situations:

$$\begin{array}{cccccccccc}
\underline{1} & 2 & 3 & 4 & \underline{5} & 6 & 7 & 8 & 9 & 10 \\
\underline{1} & \underline{2} & \underline{3} & 4 & \underline{5} & \underline{6} & 7 & 8 & 9 & 10 \\
\underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & 7 & \underline{8} & 9 & 10
\end{array}$$

and the rest gets wiped out one by one. So this is a losing position: the one who has to move loses.

The starting position wins. Namely, assume that

$$\underline{1}\ 2\ \ldots\ n$$

is a losing position, then this is the winning first move. Assume that this position is winning. Then there is a winning move. But you can also start with that move.

This is analogous to a tempo problem in chess. The move that wipes out the 1 is a tempo that you can use or not.

Here's another nice game. You play it on a chess board, or a checkers board, or a 6×6-board. You start at some column that has an arrow. Players have to draw an arrow from the head of the arrow to one of the 3 other sides of the square. The person who cannot draw an arrow, *wins*. You can not draw

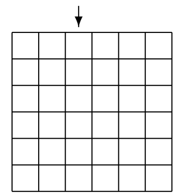

Fig. 7.11. Board with starting column

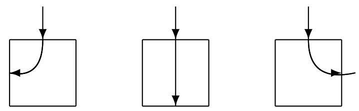

Fig. 7.12. Moves

an arrow if there is no empty square.

The strategy is simple. Cover the board with dominos. Always walk to the middle of such a domino. The opponent has always a free square to move to and he moves to a new domino. Thus the person who starts, wins.

See also: N. G. de Bruijn, Spelen op een graaf, Nieuw Tijdschrift voor Wiskunde 63 (1976), pp. 201–206.

That paper of de Bruijn also contains the solution of “Sprague’s game:” for nim-numbers take *every other hole*, starting at the end.

7.7 Problems

7.1. Two players, Jones and Alice, alternately write down numbers. Jones starts by writing down 0. Alice can add either 1, 2, or 3 to this. Then Jones can add 1, 2, or 3 to the total, and so on. The person who writes down 25 first, wins and collects NT$ 25. Numbers above 25 are not allowed. Show how Alice can win this game.

7.2. Two people play a game on a graph (G, Γ). They alternately select different points $x_0, x_1, \ldots$, such that for each $k > 0$ $\{x_{k-1}, x_k\} \in \Gamma$. The last player who can select a point wins. Show that the first player has a winning strategy if and only if (G, Γ) has no perfect matching.

7.3 (de Bruijn[4]). Consider an infinite grid. Thus the grid-corners are points (i, j), $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$, for $i, j \in \mathbb{Z}$. Two players, Alice and Jones, alternately put stones in the grid-squares. Alice plays with white stones

[4] N. G. de Bruijn, Spelen op een graaf, *Nieuw tijdschrift voor wiskunde* **63** (1976), pp. 201–208.

and Jones plays with black stones. Only one stone per square is allowed. The person who can put first 9 stones in a horizontal, vertical, or diagonal sequence wins. If Alice starts, then Jones can always make a draw.

Hint.

Jones partitions the grid into H's numbered as follows.

1		5
2	4	6
3		7

If there are 9 white stones in a sequence, somewhere in the grid, then there must be an H with 3 white stones in a sequence, and this is

(i) a sequence of the left leg of an H if the 9-sequence is vertical,
(ii) a sequence of numbers 2, 4, and 6, if the 9-sequence is horizontal, and
(iii) a sequence 1, 4, 7 or a sequence 3, 4, 5, if the 9-sequence is a diagonal.

If Jones makes sure to put his stone always in the same H as the H in which Allice put her last stone, then Jones can keep the game a draw. If Allice starts on 4, then Jones puts his stone on 2. If Allice does not put her stone on 4, then Jones puts his stone there. Allice may be able to complete a right leg, but to win the game with a vertical sequence she also needs to complete a left leg.

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[5] This is a translation and improvement of [23]. The translation is by J. W. Nienhuys.

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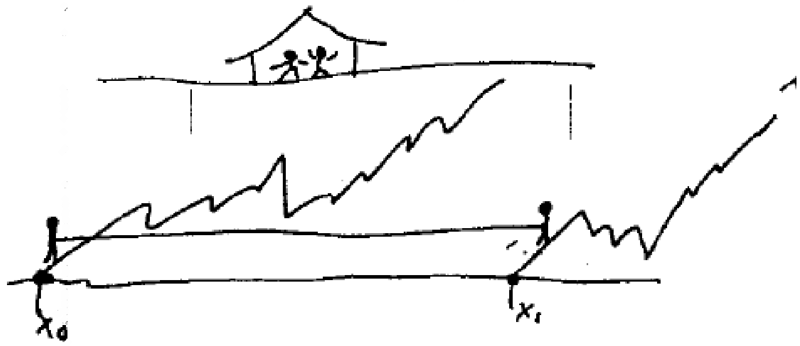
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