Linear and angular momentum spaces for Majorana spinors

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Abstract

In a Majorana basis, the Dirac equation for a free spin one-half particle is a 4x4 real matrix differential equation. The solution can be a Majorana spinor, a 4x1 real column matrix, whose entries are real functions of the space-time.

Can a Majorana spinor, whose entries are real functions of the space-time, describe the energy, linear and angular momentums of a free spin one-half particle? We show that it can.

We show that the Majorana spinor is an irreducible representation of the double cover of the proper orthochronous Lorentz group and of the full Lorentz group. The Fourier-Majorana and Hankel-Majorana transforms are defined and related to the linear and angular momentums of a free spin one-half particle.

Contents

1 Introduction

In 1928 Paul Dirac published *"The Quantum Theory of the Electron"* [1], in which he introduced a relativistic equation for the electron in interaction with an electromagnetic potential, consisting of a complex 4x4 matrix differential equation whose solution is a complex 4x1 column matrix, called Dirac spinor, whose entries are complex functions of the space-time. Using the algebra of the 4x4 matrices, he related the electron's spin with the Lorentz group. He also noticed the existence of negative-energy solutions which he used later in the prediction of the existence of the anti-electron, the positron.

In 1937 Ettore Majorana published *"A symmetric theory of electrons and positrons"* [2], in which he noted that *"it is perfectly, and most naturally, possible to formulate a theory of elementary neutral particles which do not have negative (energy) states"*. His work was based on the fact that there is a basis where the Dirac equation for the free electron is a real, instead of complex, 4x4 matrix differential equation whose solution can be a real 4x1 column matrix, called Majorana spinor, whose entries are real functions of the space-time. The existence of both positive and negative energy solutions is a consequence of the extension, through the use of complex numbers, of the free Dirac equation to include the electromagnetic interaction. For neutral particles, the free Dirac equation do not have to be extended in the same way it is when including the electromagnetic interaction and, therefore, it is possible to have a theory without negative energy solutions. Ettore Majorana disappeared in 1938.

There are applications of the Majorana's discovery in theories trying to explain phenomena in neutrino physics, dark matter searches, the fractional quantum Hall effect and superconductivity [3]. There are good references on spinors [4–6] and on its relation with the Lorentz group [7]. It is known (section 5 of [8]) that the Majorana spinor is an irreducible representation of the double cover of the proper orthochronous Lorentz group. However, we could not find a study (without second quantization operators) of the Majorana spinor solutions of the free Dirac equation.

In the context of Clifford Algebras, the generalization of the Dirac matrices algebra to other dimensions and metrics, there is work on the geometric square roots of -1 [9,10] and on the generalizations of the Fourier transform [11], with applications to image processing.

Our goal is to show that (without second quantization operators) all the kinematic properties of a free spin 1/2 particle are present in the real solutions of the real free Dirac equation. In chapter 2 we define the Majorana matrices and spinors. In chapter 3 we show that the Majorana spinor is an irreducible representation of the double cover of the proper orthochronous Lorentz group and of the full Lorentz group. In chapter 4 we show the invariance of the free Dirac equation under the action of the Lorentz group. In 5 and 6 we define the Fourier-Majorana and Hankel-Majorana transforms of a Majorana spinor whose entries are Lebesgue square integrable real functions of the space coordinates. In 7, by comparison with the particle/anti-particle solutions of the free Dirac equation, we show that the Majorana transforms are related with the linear and angular momentums of a free spin 1/2 particle. In 8, we extend the Majorana transforms to include the energy.

2 Majorana Matrices and Spinors

The Majorana matrices, $i\gamma^\mu$ with $\mu = 0, 1, 2, 3$, are the Dirac Gamma matrices, γ^μ, times the imaginary unit. The notation maintains explicit the relation between the Majorana and Dirac Gamma matrices.

Definition 2.1. $\mathbf{M}(m, n, \mathbb{F})$ is the set of $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$.

Definition 2.2. The Majorana matrices, $i\gamma^\mu \in \mathbf{M}(4, 4, \mathbb{C})$, are 4×4 complex matrices with anti-commutator $\{i\gamma^\mu, i\gamma^\nu\}$:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2g^{\mu\nu},\ \ \mu, \nu = 0, 1, 2, 3 \tag{2.1}$$

Where $g = diag(1, -1, -1, -1)$ is the Minkowski metric. The pseudo-scalar is $i\gamma^5 \equiv -\gamma^0\gamma^1\gamma^2\gamma^3$.

The product of 2 Dirac Gamma matrices is minus the product of 2 corresponding Majorana matrices: $\gamma^\mu\gamma^\nu = -i\gamma^\mu i\gamma^\nu$.

Definition 2.3. $\Gamma_- \equiv \{i\gamma^0, i\gamma^5, \gamma^0\gamma^5, i\gamma^5\gamma^0\gamma^j :\ j = 1, 2, 3\}$
$\Gamma_+ \equiv \{1, \gamma^0\gamma^j, i\gamma^j, \gamma^5\gamma^j :\ j = 1, 2, 3\}$
$\Gamma \equiv \Gamma_- \cup \Gamma_+$

From the anti-commutator of the Majorana matrices, the matrices in $\Gamma_\pm$ square respectively to ± 1, and all matrices in Γ either commute or anti-commute with each other.

Definition 2.4. The sets of matrices that (anti-)commute with a matrix $A \in \Gamma$ are: $\Omega_\pm(A) = \{B \in \Gamma : AB = \pm BA\}$.

Proposition 2.5. *The sets $\Omega_\pm(A) \cap \Gamma_+$ and $\Omega_\pm(A) \cap \Gamma_-$ are not empty for all $A \in \Gamma \backslash \{1\}$.*

Corollary. *The matrices in $\Gamma \setminus \{1\}$ have null trace and determinant 1.*

Proof. If $A \in \Gamma \setminus \{1\}$. Since there is $B \in \Omega_-(A) \cap \Gamma_+$, we have $tr(A) = tr(BAB) = -tr(A)$.

Let $A \in \Gamma_-$. Since $A^2 = -1$, then $A = e^{\frac{\pi}{2}A}$ and $det(A) = e^{\frac{\pi}{2}tr(A)} = 1$.

Let $A \in \Gamma_S \setminus \{1\}$. Since there is $B \in \Omega_+(A) \cap \Gamma_-$, we have $(AB) \in \Gamma_-$ and so $det(AB) = 1$. Since $det(B) = 1$, then $det(A) = 1$. □

Proposition 2.6. *Γ is a basis for the space of 4×4 complex matrices.*

Proof. There are only 16 linearly independent 4×4 complex matrices.

Let $B \equiv \sum_{i=1}^{16} a_i A_i$, where $a_i \in \mathbb{C}$ and $A_i \in \Gamma$ are different elements of the set for each i. We have $tr(A_j^\dagger B) = 4a_j$, for $j = 1, ..., 16$. Then, $B = 0$ implies that all the scalars a_i are null and so all the elements in Γ are linearly independent. □

Proposition 2.7. *For all commuting matrices $A, B \in \Gamma \backslash \{1\}$, $AB = BA$: all matrices in $\Gamma \backslash \{1, A, B, AB\}$ anti-commute with A or B. That is, $\Omega_-(A) \cup \Omega_-(B) = \Gamma \backslash \{1, A, B, AB\}$.*

Definition 2.8. Γ_2 is the group of 32 Majorana matrices products:

$$\Gamma_2 \equiv \{\pm 1,\ \pm i\gamma^\mu,\ \pm\gamma^0\gamma^j,\ \pm i\gamma^5\gamma^0\gamma^j,\ \pm\gamma^\mu\gamma^5,\ \pm i\gamma^5 : \mu = 0,1,2,3,\ j = 1,2,3\} \tag{2.2}$$

Definition 2.9. A 4×4 representation of the Majorana matrices, M, is a map from the Majorana matrices to the space of 4×4 complex matrices, verifying:

$$\{M(i\gamma^\mu), M(i\gamma^\nu)\} = -2g^{\mu\nu},\ \mu,\nu = 0,1,2,3 \tag{2.3}$$

Proposition 2.10. *Two 4×4 representations of the Majorana matrices are related by a similarity transformation, unique up to a complex factor [12].*

Proof. Given a 4×4 representation of the Majorana matrices, M, we extend the domain from the Majorana matrices to Γ, recursively, in such a way that for $k_1, k_2 \in \Gamma_2$, if we know $M(k_1)$ and $M(k_2)$, then $M(k_1 k_2) \equiv M(k_1)M(k_2)$.

Let A and B be 4×4 representations of the Majorana matrices.

We define the matrix S as:

$$S \equiv \sum_{g\in\Gamma_2} B(g^{-1})S'A(g) \tag{2.4}$$

Where S' will be defined later.

For all $h \in \Gamma$, it verifies $SA(h) = B(h)S$:

$$SA(h) = \sum_{g\in\Gamma_2} B(g^{-1})S'A(gh) \tag{2.5}$$

$$= \sum_{l\in\Gamma_2} B(hl^{-1})S'A(l) = B(h)S \tag{2.6}$$

We define the matrix T as:

$$T \equiv \sum_{g\in\Gamma_2} A(g^{-1})T'B(g) \tag{2.7}$$

Where T' will be defined later. For all $h \in \Gamma$, it verifies $TB(h) = A(h)T$. Consequently, $TSA(h) = A(h)TS$.

Since γ^μ and $A(\gamma^\mu)$ obey to the same commutation relations, the set $\{A(k), k \in \Gamma\}$ is also a basis for the space of 4×4 matrices. Therefore, TS is equal to the identity matrix times a coefficient. To check what the coefficient is:

$$TS = \sum_{g\in\Gamma_2} A(g^{-1})T'S'A(g) \tag{2.8}$$

We choose T', S' such that TS is non-null. Suppose that such T',S' do not exist, then for all indexes i, j $\sum_{g\in\Gamma_2} A_{ij}(g^{-1})A(g) = 0$ which implies that the matrices $A(g)$ are linear dependent, in contradiction with Proposition 2.6. With a proper normalization, we can make $T = S^{-1}$.

Suppose that for all $h \in \Gamma$, S' is invertible and also verifies $S'A(h) = B(h)S'$. Then $S'^{-1}SA(h) = A(h)S'^{-1}S$ and again $S'^{-1}S$ must be proportional to the identity. Let $c \in \mathbb{C}$ be such that $S'^{-1}S = c$. Multiplying on the left by S', we get $S = cS'$. □

The Majorana matrices are themselves a 4×4 representation of the Majorana matrices. Therefore, choosing a 4×4 representation of the Majorana matrices is the same as choosing a basis.

Proposition 2.11. *Two 4×4 unitary representations of the Majorana matrices are related by an unitary similarity transformation, unique up to a phase.*

Proof. Let A and B be unitary representations of the Majorana matrices. Then there is an invertible matrix S, unique up to a complex scalar, such that $A(\gamma^\mu)S = SB(\gamma^\mu)$. Multiplying on the left by $A^\dagger$ and on the right by $B^\dagger$ and making the hermitian conjugate of the equation, we get $B(\gamma^\mu)S^\dagger = S^\dagger A(\gamma^\mu)$.

So, for some complex c, $S^\dagger = cS^{-1}$. Applying the determinant, we get $c = |det(S)|^2$ is real and positive. So, $(c^{-\frac{1}{2}}S)^\dagger(c^{-\frac{1}{2}}S) = 1$.

Let both S and $S' \equiv cS$, for some complex c, be unitary. Then, $(cS)^\dagger(cS) = |c|^2 = 1$, so $c = e^{i\theta}$ for some real θ. □

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned} i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = -\gamma^0\gamma^1\gamma^2\gamma^3 \end{aligned} \tag{2.9}$$

Proposition 2.12. *Two 4×4 real representations of the Majorana matrices are related by a real similarity transformation, unique up to a real factor.*

Proof. Let A and B be real representations of the Majorana matrices. Then there is an invertible matrix S, unique up to a complex factor, such that $A(\gamma^\mu)S = SB(\gamma^\mu)$. Conjugating the equation, we get that, for some complex c, $S^* = cS$. Applying the module of the determinant, we get $c = e^{i\theta}$ for some real θ and $(e^{i\frac{\theta}{2}}S)^* = (e^{i\frac{\theta}{2}}S)$. □

Definition 2.13. The Dirac spinor is a 4×1 complex column matrix, $\mathbf{M}(4, 1, \mathbb{C})$.

The space of Dirac spinors is a 4 dimensional complex vector space.

Definition 2.14. Let S be an invertible matrix such that $Si\gamma^\mu S^{-1}$ is real, for $\mu = 0, 1, 2, 3$.

The set of Majorana spinors, $Pinor$, is the set of Dirac spinors verifying the Majorana condition:

$$Pinor \equiv \{u \in \mathbf{M}(4, 1, \mathbb{C}) : S^{-1}S^*u^* = u\} \tag{2.10}$$

Where * denotes complex conjugation.

Remark 2.15. *Let W be a subset of a vector space V over $\mathbb{C}$. W is a real vector space iff:*

1) $0 \in W$;

2) If $u, v \in W$, then $u + v \in W$;

3) If $u \in W$ and $c \in \mathbb{R}$, then $cu \in W$.

From the previous remark, the set of Majorana spinors is a 4 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition. The Majorana spinor, in the Majorana bases, is a 4×1 real column matrix.

Definition 2.16. $End(Pinor)$ is the set of endomorphisms of Majorana spinors, that is, the set of linear maps from and to Majorana spinors.

$End(Pinor)$ is a 16 dimensional real vector space, generated by the linear combinations with real scalars of the 16 matrices in the basis Γ. In the Majorana bases, $End(Pinor) = \mathbf{M}(4, 1, \mathbb{R})$.

3 Majorana representation of the Lorentz group

3.1 Double cover of the Lorentz group

We define some symbols for the sets we will use:

Definition 3.1. $GL(n, \mathbb{F})$ is the group of $n \times n$ invertible matrices over the field $\mathbb{F}$.
$SL(n, \mathbb{F})$ is the group of $n \times n$ invertible matrices over the field $\mathbb{F}$ with determinant 1.
$O(n)$ is the group of $n \times n$ real orthogonal matrices.
$SO(n)$ is the group of $n \times n$ real orthogonal matrices with determinant 1.
$SPD(n)$ is the set of $n \times n$ real symmetric positive definite matrices.

Definition 3.2. The set of Lorentz matrices, $O(1,3) \equiv \{\Lambda \in \mathbf{M}(4,4,\mathbb{R}) : \Lambda^T g \Lambda = g\}$, is the set of real matrices that leave the metric, $g = diag(1, -1, -1, -1)$, invariant.

Definition 3.3. In a basis where the Majorana matrices are unitary, the set Maj is defined as:

$$Maj \equiv \{M \in End(Pinor) : (i\gamma^5) M (-i\gamma^5) = -M,\ (i\gamma^0) M (-i\gamma^0) = -M^\dagger\} \tag{3.1}$$

The only matrices in Γ that are also in Maj are the Majorana matrices, $i\gamma^\mu$, therefore Maj is the 4 dimensional real space of the linear combinations with real coefficients of Majorana matrices.

Definition 3.4. $Pin(3,1)$ [7] is the set of endomorphisms of Majorana spinors which leave the space Maj invariant, that is:

$$Pin(3,1) \equiv \Big\{ S \in End(Pinor) :\ |det(S)| = 1,\ S^{-1}(i\gamma^\mu)S \in Maj,\ \mu = 0,1,2,3 \Big\} \tag{3.2}$$

Proposition 3.5. *The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:*

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu \equiv S^{-1}(i\gamma^\mu)S \tag{3.3}$$

is two-to-one and surjective.

Proof. 1) Let $S \in Pin(3,1)$. Since the Majorana matrices are a basis of the real vector space Maj, there is an unique real matrix $\Lambda(S)$ such that:

$$(\Lambda(S))^{\mu}{}_{\nu} i\gamma^{\nu} = S^{-1}(i\gamma^{\mu})S \tag{3.4}$$

Therefore, Λ is a map with domain $Pin(3,1)$. Now we can check that $\Lambda(S) \in O(1,3)$:

$$(\Lambda(S))^{\mu}{}_{\alpha} g^{\alpha\beta} (\Lambda(S))^{\nu}{}_{\beta} = -\frac{1}{2}(\Lambda(S))^{\mu}{}_{\alpha}\{i\gamma^{\alpha}, i\gamma^{\beta}\}(\Lambda(S))^{\nu}{}_{\beta} = \tag{3.5}$$

$$= -\frac{1}{2}S\{i\gamma^{\mu}, i\gamma^{\nu}\}S^{-1} = Sg^{\mu\nu}S^{-1} = g^{\mu\nu} \tag{3.6}$$

We have proved that Λ is a map from $Pin(3,1)$ to $O(1,3)$.

2) Since any $\lambda \in O(1,3)$ conserve the metric, the matrices $M(i\gamma^{\mu}) \equiv \lambda^{\mu}{}_{\nu} i\gamma^{\nu}$ are a representation of the Majorana matrices:

$$\{M(i\gamma^{\mu}), M(i\gamma^{\nu})\} = -2\lambda^{\mu}{}_{\alpha} g^{\alpha\beta} \lambda^{\nu}{}_{\beta} = -2(\lambda g \lambda^{T})^{\mu\nu} = -2g^{\mu\nu} \tag{3.7}$$

In a basis where the Majorana matrices are real, from 2.12 there is a real invertible matrix S_{Λ}, unique up to a real factor, such that $\lambda^{\mu}{}_{\nu} i\gamma^{\nu} = S_{\lambda}^{-1}(i\gamma^{\mu})S_{\lambda}$. Setting $|det(S)| = 1$ we fix the real factor up to a signal ± 1. Therefore, $\pm S_{\lambda} \in Pin(3,1)$ and we proved that the map $\Lambda : Pin(3,1) \to O(1,3)$ is two-to-one and surjective. □

Lemma 3.6. *$Pin(3,1) = Pin'(3,1)$, where $Pin'(3,1)$ is, in a basis where the Majorana matrices are unitary:*

$$Pin'(3,1) \equiv \Big\{ S \in End(Pinor) : (i\gamma^5)S = aS(i\gamma^5), \tag{3.8}$$

$$(i\gamma^0)S = bS^{-1\dagger}(i\gamma^0), \tag{3.9}$$

$$|det(S)| = 1;\ a, b \in \{-1, 1\} \Big\} \tag{3.10}$$

Proof. 1) For all $S \in Pin'(3,1)$, $S^{-1}(i\gamma^{\mu})S \in Maj$ and so $Pin'(3,1) \subset Pin(3,1)$.

2) In a basis where the Majorana matrices are unitary, for all $S \in Pin(3,1)$, since $S^{-1}(i\gamma^{\mu})S \in Maj$, we have:

$$(i\gamma^5)S^{-1}(i\gamma^{\mu})S(-i\gamma^5) = -S(i\gamma^{\mu})S = S^{-1}(i\gamma^5)(i\gamma^{\mu})(-i\gamma^5)S \tag{3.11}$$

$$(i\gamma^0)S^{-1}(i\gamma^{\mu})S(-i\gamma^0) = -S^{\dagger}(i\gamma^{\mu})^{\dagger}S^{-1\dagger} = S^{\dagger}(i\gamma^0)(i\gamma^{\mu})(-i\gamma^0)S^{-1\dagger} \tag{3.12}$$

On the other hand:

$$(i\gamma^5)S^{-1}(i\gamma^{\mu})S(-i\gamma^5) = (-\Lambda(S))^{\mu}{}_{\nu}(i\gamma^{\nu}) \tag{3.13}$$

$$(i\gamma^0)S^{-1}(i\gamma^{\mu})S(-i\gamma^0) = (\Lambda(S)g)^{\mu}{}_{\nu}(i\gamma^{\nu}) \tag{3.14}$$

We can easily check that $(-\Lambda), (\Lambda g) \in O(1,3)$. From proposition 3.5, we get that the matrices in $Pin(3,1)$ corresponding to $(-\Lambda), (\Lambda g) \in O(1,3)$ are unique up to a sign. Therefore, $Pin(3,1) \subset Pin'(3,1)$. □

Definition 3.7. In a basis where the Majorana matrices are unitary, the subset $Spin^+(3,1) \subset Pin(3,1)$ is:

$$Spin^+(3,1) \equiv \Big\{ S \in End(Pinor) : (i\gamma^5)S = S(i\gamma^5), \tag{3.15}$$

$$(i\gamma^0)S = S^{-1\dagger}(i\gamma^0), \tag{3.16}$$

$$|det(S)| = 1 \Big\} \tag{3.17}$$

Proposition 3.8. *1) $Pin(3,1)$ and $Spin^+(3,1)$ are groups.*

2) In a basis where the Majorana matrices are unitary, if $S \in Pin(3,1)$ ($Spin^+(3,1)$) then $S^\dagger \in Pin(3,1)$ ($Spin^+(3,1)$).

Proof. $Pin(3,1)$ and $Spin^+(3,1)$ are subsets of the group $SL(4,\mathbb{C})$. They include the identity matrix, $1 \in Pin(3,1), Spin^+(3,1)$.

Let $S_\pm \in Pin(3,1)$. Then, in a basis where the Majorana matrices are unitary, for some $a_\pm, b_\pm \in \{-1,1\}$:

$$(i\gamma^5)S_\pm = a_\pm S_\pm(i\gamma^5),\ (i\gamma^0)S_\pm = b_\pm S_\pm^{-1\dagger}(i\gamma^0) \tag{3.18}$$

Making the inverse (hermitian conjugate) of the equation on the left and the hermitian conjugate (inverse) of the equation on the right we get:

$$-S_\pm^{-1}(i\gamma^5) = -a_\pm(i\gamma^5)S_\pm^{-1},\ -S_\pm^\dagger(i\gamma^0) = -b_\pm(i\gamma^0)S_\pm^{-1} \tag{3.19}$$

$$-S_\pm^\dagger(i\gamma^5) = -a_\pm(i\gamma^5)S_\pm^\dagger,\ -S_\pm^{-1}(i\gamma^0) = -b_\pm(i\gamma^0)S_\pm^\dagger \tag{3.20}$$

Therefore, $S_\pm^\dagger \in Pin(3,1)$ and the product $S_+S_-^{-1} \in Pin(3,1)$:

$$(i\gamma^5)S_+S_-^{-1} = (a_+a_-)S_+S_-^{-1}(i\gamma^5) \tag{3.21}$$

$$(i\gamma^0)S_+S_-^{-1} = (b_+b_-)S_+^{-1\dagger}S_-^\dagger(i\gamma^0) \tag{3.22}$$

In the particular case $S_\pm \in Spin^+(3,1)$, we have $a_\pm, b_\pm = 1$. Then $S_\pm^\dagger \in Spin^+(3,1)$ and the product $S_+S_-^{-1} \in Spin^+(3,1)$. □

Definition 3.9. The discrete pin subgroup $\Delta \subset Pin(3,1)$ is:

$$\Delta \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\} \tag{3.23}$$

Lemma 3.10. *For all $S \in Pin(3,1)$, there are only two factors $\pm d \in \Delta$ and correspondingly only two $\pm S' \in Spin^+(3,1)$, such that $S = (\pm d)(\pm S')$.*

Proof. Let $S \in Pin(3,1)$ and $a, b \in \{-1,1\}$ be such that, in a basis where the Majorana matrices are unitary:

$$(i\gamma^5)S = aS(i\gamma^5),\ (i\gamma^0)S = bS^{-1\dagger}(i\gamma^0) \tag{3.24}$$

There are always only two factors $\pm d \in \Delta$, such that $d^{-1}S \in Spin^+(3,1)$:

$$a = b = 1,\ d = \pm 1 \tag{3.25}$$

$$a = -b = 1,\ d = \pm(i\gamma^5) \tag{3.26}$$

$$-a = b = 1,\ d = \pm(i\gamma^0) \tag{3.27}$$

$$-a = -b = 1,\ d = \pm(\gamma^0\gamma^5) \tag{3.28}$$

□

Remark 3.11. *1) Every real invertible matrix can be uniquely factored as the product of an orthogonal matrix and a symmetric positive definite matrix.*

2) For all real symmetric positive definite matrix Π*, there is an unique symmetric matrix* B *such that* $\Pi = e^B$*.*

3) For all real orthogonal matrix with determinant 1*,* Θ*, there is a skew-symmetric matrix* A *such that* $\Theta = e^A$*.*

Lemma 3.12. $Spin^+(3,1) = Spin'^+(3,1)$*, where:*

$$Spin'^+(3,1) \equiv \{e^{\theta^j i\gamma^5\gamma^0\gamma^j} e^{b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j = 1,2,3\} \tag{3.29}$$

Note that there is a sum in the index j*.*

Proof. In a Majorana basis, since S is invertible and real, from point 1) in remark 3.11, there is an unique $\Theta \in O(4)$ and unique $\Pi \in SPD(4)$ such that $S = \Theta\Pi$.

From point 2) in remark 3.11, there is an unique symmetric B such that $\Pi = e^B$.

Since $S, S^\dagger \in Spin^+(3,1)$, also $S^\dagger S = e^{2B} \in Spin^+(3,1)$ and we have:

$$(i\gamma^5)e^{2B} = e^{2B}(i\gamma^5),\ (i\gamma^0)e^{2B} = e^{-2B}(i\gamma^0) \tag{3.30}$$

From the uniqueness of B we get $(i\gamma^5)B = B(i\gamma^5)$ and $(i\gamma^0)B = -B(i\gamma^0)$. In a Majorana basis, the only symmetric matrices in Γ satisfying the previous equations are $\gamma^0\gamma^j$, $j = 1,2,3$. Therefore, there are unique $b^j \in \mathbb{R}$, $j = 1,2,3$, such that $\Pi = e^{b^j\gamma^0\gamma^j}$. Since $\gamma^0\gamma^j$ is traceless, $det(\Pi) = 1$ and $\Pi \in Spin^+(3,1)$.

Since $det(S) = det(\Pi) = 1$, also $det(\Theta) = 1$. We can write:

$$(i\gamma^5)\Theta e^B = \Theta e^B(i\gamma^5),\ (i\gamma^0)\Theta e^B = \Theta e^{-B}(i\gamma^0) \tag{3.31}$$

Multiplying the equations by e^{-B} on the right, also $\Theta \in Spin^+(3,1)$:

$$(i\gamma^5)\Theta = \Theta(i\gamma^5),\ (i\gamma^0)\Theta = \Theta(i\gamma^0) \tag{3.32}$$

From point 3) in remark 3.11, there is a skew-symmetric A such that $\Theta = e^A$. In a Majorana basis, the only skew-symmetric matrices in Γ are in the commuting sets $\{i\gamma^0, \gamma^0\gamma^5, i\gamma^5\}$ and $\{i\gamma^5\gamma^0\gamma^j :\ j = 1,2,3\}$. Therefore, there are $\theta^j, a^j \in \mathbb{R}$, $j = 1,2,3$, such that $C \equiv a^1 i\gamma^0 + a^2\gamma^0\gamma^5 + a^3 i\gamma^5$ and $\Theta = e^C e^{\theta^j i\gamma^5\gamma^0\gamma^j}$. We can write:

$$(i\gamma^5)e^C e^{\theta^j i\gamma^5\gamma^0\gamma^j} = e^C e^{\theta^j i\gamma^5\gamma^0\gamma^j}(i\gamma^5),\ (i\gamma^0)e^C e^{\theta^j i\gamma^5\gamma^0\gamma^j} = e^C e^{\theta^j i\gamma^5\gamma^0\gamma^j}(i\gamma^0) \tag{3.33}$$

Multiplying the equations by $e^{-\theta^j i\gamma^5\gamma^0\gamma^j}$ on the right we get:

$$(i\gamma^5)e^C = e^C(i\gamma^5),\ (i\gamma^0)e^C = e^C(i\gamma^0) \tag{3.34}$$

The last equations imply that $e^{a^1 i\gamma^0 + a^2\gamma^0\gamma^5 + a^3 i\gamma^5} = e^{-a^1 i\gamma^0 + a^2\gamma^0\gamma^5 - a^3 i\gamma^5}$ and so $(i\gamma^j)e^C = e^C(i\gamma^j)$, for $j = 1,2,3$. Then, e^C commutes with all matrices in Γ and so it must be proportional to the identity. From $det(e^C) = 1$ we get that $e^C = \pm 1$.

If $e^C = -1$, the signal can be absorbed. We define $|\theta| \equiv \sqrt{\theta^j\theta^j}$. If $|\theta|$ is null, then $\Theta = e^{\pi i\gamma^5\gamma^0\gamma^1}$; if not, then $\Theta = e^{(1+\frac{\pi}{|\theta|})\theta^j i\gamma^5\gamma^0\gamma^j}$.

Checking that $e^{\theta^j i\gamma^5\gamma^0\gamma^j}, e^{b^j\gamma^0\gamma^j} \in Spin^+(3,1)$ for all $\theta^j, b^j \in \mathbb{R},\ j = 1,2,3$, we have completed the prove. $\square$

Proposition 3.13. $O(1,3)$ *is a group, the Lorentz group, and the map* $\Lambda : Pin(3,1) \to O(1,3)$*, defined in proposition 3.5, is a group homomorphism.*

Proof. The matrix product is associative and $\Lambda(1) = 1 \in O(1,3)$.

For all $S, S' \in Pin(3,1)$, we have:

$$(\Lambda(SS'))^{\mu}{}_{\beta} i\gamma^{\beta} = S'^{-1}S(i\gamma^{\mu})SS' = (\Lambda(S))^{\mu}{}_{\alpha} S'^{-1} i\gamma^{\alpha} S' \tag{3.35}$$

$$= (\Lambda(S))^{\mu}{}_{\alpha}(\Lambda(S'))^{\alpha}{}_{\beta} i\gamma^{\beta} \tag{3.36}$$

This implies that $\Lambda^{-1}(S) = \Lambda(S^{-1}) \in O(1,3)$ and $\Lambda(S)\Lambda(S') = \Lambda(SS') \in O(1,3)$. Since the map Λ is surjective, then $O(1,3)$ is a group and the map Λ is a group homomorphism. □

Definition 3.14. The proper orthochronous Lorentz group $SO^+(1,3)$ is:

$$SO^+(1,3) \equiv \{\Lambda(S) : S \in Spin^+(1,3)\} \tag{3.37}$$

Where Λ is the map defined in proposition 3.5.

Since there is a two-to-one surjective group homomorphism, $Pin(3,1)$ is a double cover of $O(1,3)$ and $Spin^+(3,1)$ is a double cover of $SO^+(1,3)$.

In a Majorana basis, by identifying i with $i\gamma^5$ and $\gamma^0\gamma^j$ with the Pauli matrices σ^j, we can see that $Spin^+(3,1)$ is isomorphic to $SL(2,\mathbb{C})$.

3.2 Majorana Spinor representation

Definition 3.15. A representation (M_G, V) of a group G is defined by:

1) the representation space V, which is a vector space;

2) the representation map $M : G \to GL(V)$ from the group elements to the automorphisms of the representation space, verifying for $\Lambda_1, \Lambda_2 \in G$:

$$M(\Lambda_1)M(\Lambda_2) = M(\Lambda_1\Lambda_2) \tag{3.38}$$

Definition 3.16. The Majorana spinor representation of $Pin(3,1)$ is defined by:

1) the representation space $V = Pinor$ is the space of Majorana spinors;

2) In a basis where the Majorana matrices are unitary, the representation map is:

$$M(S) = S,\ S \in Pin(3,1) \tag{3.39}$$

The Majorana spinor representation of the subgroup $Spin^+(3,1) \subset Pin(3,1)$ is obtained from the representation of $Pin(3,1)$ by restricting the domain of the representation map to the subgroup $Spin^+(3,1) \subset Pin(3,1)$.

Definition 3.17. Let W be a subspace of V. (M_G, W) is a subrepresentation of (M_G, V) if W is invariant under the group action, that is, for all $w \in W$: $(M(g)w) \in W$, for all $g \in G$.

Definition 3.18. $W^{\perp}$ is the orthogonal complement of the subspace W of the vector space V if:

1) all $v \in V$ can be expressed as $v = w + x$, where $w \in W$ and $x \in W^{\perp}$;

2) if $w \in W$ and $x \in W^{\perp}$, then $x^{\dagger}w = 0$.

Definition 3.19. The representation (M_G, V) is semi-simple if for all subrepresentation (M_G, W) of (M_G, V) , $(M_G, W^\perp)$ is also a subrepresentation of (M_G, V), where $W^\perp$ is the orthogonal complement of the subspace W.

Lemma 3.20. *Consider a representation (M_G, V) of a group G. For all $g \in G$, if there is $h \in G$ such that $M(h) = M^\dagger(g)$, then the representation (M_G, V) is semi-simple.*

Proof. Let (M_G, W) be a subrepresentation of (M_G, V). $W^\perp$ is the orthogonal complement of W.

For all $x \in W^\perp$, $w \in W$ and $g \in G$, $(M(g)x)^\dagger w = x^\dagger (M^\dagger(g) w)$.

Since W is invariant and there is $h \in G$, such that $M(h) = M^\dagger(g)$, then $w' \equiv (M^\dagger(g)w) \in W$.

Since $x \in W^\perp$ and $w' \in W$, then $x^\dagger w' = 0$.

This implies that if x is in the orthogonal complement of W ($x \in W^\perp$), also $M(g)x$ is in the orthogonal complement of W ($M(g)x \in W^\perp$), for all $g \in G$. □

Proposition 3.21. *The Majorana spinor representation of $Spin^+(3,1)$ is semi-simple.*

Proof. From point 1) in lemma 3.12 and lemma 3.20. □

Definition 3.22. A representation (M_G, V) is irreducible if their only sub-representations are the trivial sub-representations: (M_G, V) and $(M_G, \{0\})$, where $\{0\}$ is the null space.

Lemma 3.23. *Consider a semi-simple representation (M_G, V) of a group G. If the set of hermitian automorphisms of V that square to 1 and commute with $M(g)$, for all $g \in G$, is $\{+1, -1\}$, then the representation (M_G, V) is irreducible (1 is the identity matrix).*

Proof. Let (M_G, W) and $(M_G, W^\perp)$ be sub-representations of (M_G, V), where $W^\perp$, the orthogonal complement of W.

There is an automorphism $P : V \to V$, such that, for $w, w' \in W$, $x, x' \in W^\perp$, $P(w + x) = (w - x)$. $P^2 = 1$ and P is hermitian:

$$(w' + x')^\dagger (P(w + x)) = w'^\dagger w - x'^\dagger x = (P(w' + x'))^\dagger (w + x) \tag{3.40}$$

Let $w' \equiv M(g)w \in W$ and $x' \equiv M(g)x \in W^\perp$:

$$M(\Lambda)P(w + x) = M(\Lambda)(w - x) = (w' - x') \tag{3.41}$$
$$PM(\Lambda)(w + x) = P(w' + x') = (w' - x') \tag{3.42}$$

Which implies that P commutes with $M(g)$ for all $g \in G$.

If $P = +1$, then $W = V$:

$$+(w + x) = P(w + x) = (w - x) \implies x = 0 \tag{3.43}$$

If $P = -1$, then W is the null space:

$$-(w + x) = P(w + x) = (w - x) \implies w = 0 \tag{3.44}$$

□

Proposition 3.24. *The Majorana spinor representation of $Spin^+(3,1)$ is irreducible.*

Proof. The hermitian linear transformations from and to Majorana spinors, are generated by the linear combinations with real coefficients of the 10 matrices in the basis $\Gamma_S \equiv \{1, \gamma^0\gamma^j, i\gamma^j, \gamma^5\gamma^j\}$, where $j = 1, 2, 3$.

The only matrix in Γ_S commuting with all $S \in Spin^+(3,1)$ is the identity matrix. Therefore, the set of hermitian automorphisms of the Majorana spinors that square to 1 and commute with all $S \in Spin^+(3,1)$, is $\{+1, -1\}$. Applying proposition 3.21 and lemma 3.23 the proposition is proved. □

The Majorana spinor representation of the group $Pin^+(3,1)$ is also irreducible because it is already irreducible for the subgroup $Spin^+(3,1) \subset Pin^+(3,1)$.

4 Majorana spinor solutions of the free Dirac equation

Definition 4.1. $L^2(\mathbb{R}^n)$ is the Hilbert space of real functions of n real variables whose square is Lebesgue integrable in $\mathbb{R}^n$. The internal product is:

$$< f, g > \equiv \int d^n x f(x) g(x),\ \ f, g \in L^2(\mathbb{R}^n) \tag{4.1}$$

Remark 4.2. *If $f \in L^2(\mathbb{R}^n)$, then $f_s, f_c \in L^2(\mathbb{R}^n)$:*

$$f_c(p) \equiv \int d^n x\ \ \cos(p \cdot x) f(x) \tag{4.2}$$

$$f_s(p) \equiv \int d^n x\ \ \sin(p \cdot x) f(x) \tag{4.3}$$

The Dirac delta δ^n is a well defined operator of the Hilbert space $L^2(\mathbb{R}^n)$:

$$\delta^n(x) \equiv \int \frac{d^n p}{(2\pi)^n} \cos(p \cdot x) \tag{4.4}$$

$$f(0) = \int d^n x\ \delta^n(x) f(x) \tag{4.5}$$

The domain of integration is $\mathbb{R}^n$.

Remark 4.3. *The derivative ∂_i, $i = 1, ..., n$, is a skew-symmetric operator of the Hilbert space $L^2(\mathbb{R}^n)$:*

$$\int d^n x (\partial_i f(x)) g(x) = -\int d^n x f(x) (\partial_i g(x)),\ \ f, g \in L^2(\mathbb{R}^n) \tag{4.6}$$

The free Dirac equation is:

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \tag{4.7}$$

We are looking for solutions where $\Psi(x) \in Pinor \otimes L^2(\mathbb{R}^4)$ is a Majorana spinor, whose entries are square integrable functions of the space-time.

If we change $x^\mu \to (\Lambda(S))^\mu{}_\nu x^\nu$ and $\Psi(x) \to S\Psi(x)$, where $S \in Pin(3,1)$, and we multiply the equation on the left by S^{-1} we get:

$$S^{-1}(i\gamma^\mu g_{\mu\nu}(\Lambda(S))^\nu{}_\alpha g^{\alpha\beta}\partial_\beta - m)S\Psi(x) = \tag{4.8}$$
$$= (i\gamma^\delta(\Lambda(S))^\mu{}_\delta g_{\mu\nu}(\Lambda(S))^\nu{}_\alpha g^{\alpha\beta}\partial_\beta - m)\Psi(x) = \tag{4.9}$$
$$= (i\gamma^\delta\partial_\delta - m)\Psi(x) = 0 \tag{4.10}$$

The equation stays invariant.

If we multiply by $-i\gamma^0$ on the left, the equation can be rewritten as:

$$(\partial_0 + iH(\vec{x}))\Psi(x) = 0 \tag{4.11}$$
$$iH(\vec{x}) \equiv \gamma^0\gamma^j\partial_j - i\gamma^0 m,\ j = 1,2,3 \tag{4.12}$$

The solution is:

$$\Psi(x) = e^{-iH(\vec{x})x^0}\psi(\vec{x}) \tag{4.13}$$

Where $\psi(\vec{x}) \in Pinor \otimes L^2(\mathbb{R}^3)$ is a Majorana spinor, whose entries are square integrable functions of the space coordinates.

Now we can write $\psi(\vec{x}) = M(\vec{x})\chi$, where $M(\vec{x}) \in End(Pinor) \otimes L^2(\mathbb{R}^3)$ is a Majorana spinor endomorphism, whose entries are square integrable functions of the space and $\chi \in Pinor$ is a Majorana spinor. Suppose that for some $E \in \mathbb{R}$, M verifies the equation:

$$iH(\vec{x})M(\vec{x}) = M(\vec{x})i\gamma^0 E \tag{4.14}$$

In the next two sections we will see that these matrices satisfying the above equation have interesting properties. Now we have:

$$\Psi(x) = M(\vec{x})e^{-i\gamma^0 E x^0}\chi \tag{4.15}$$

Before moving to the next section, we will fix notation. If p, q are Lorentz vectors, we define $\not{p} = \gamma^\mu p_\mu$ and $p \cdot q = p^\mu q_\mu$. Given a mass $m \geq 0$, we define:

$$\vec{p}^j = p^j,\ j = 1,2,3 \tag{4.16}$$
$$\vec{\not{p}} = \vec{\gamma} \cdot \vec{p} \tag{4.17}$$
$$E_p = \sqrt{\vec{p}^2 + m^2} \tag{4.18}$$
$$\not{p} = \gamma^0 p^0 - \vec{\gamma} \cdot \vec{p} \tag{4.19}$$

5 Linear Momentum of Majorana spinors

Definition 5.1. $L^2_4(\mathbb{R}^n)$ is the Hilbert space $Pinor \otimes L^2(\mathbb{R}^n)$, that is, Majorana spinors whose entries are square integrable functions of $\mathbb{R}^n$. The internal product is:

$$< \Psi, \Phi > \equiv \int d^n x\ \Psi^\dagger(x)\Phi(x),\ \Psi, \Phi \in L^2_4(\mathbb{R}^n) \tag{5.1}$$

Definition 5.2. The Fourier-Majorana Transform $\psi(\vec{p})$ of a Majorana spinor $\Psi(\vec{x}) \in L_4^2(\mathbb{R}^3)$ is the Majorana spinor:

$$\psi(\vec{p}) \equiv \int d^3\vec{x}\ O(\vec{p},\vec{x})\Psi(\vec{x}) \tag{5.2}$$

$$O(\vec{p},\vec{x}) \equiv e^{-i\gamma^0\vec{p}\cdot\vec{x}} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{5.3}$$

Where $m \geq 0$ $p^0 = E_p = \sqrt{\vec{p}^2+m^2}$.

Proposition 5.3. *The Fourier-Majorana Transform $\psi(\vec{p})$ of a Majorana spinor $\Psi(\vec{x}) \in L_4^2(\mathbb{R}^3)$ is also in the Hilbert space $L_4^2(\mathbb{R}^3)$.*

Proof. In the Majorana bases, $O(\vec{p},\vec{x})$ and $\Psi(\vec{x})$ are real and so is $\psi(\vec{p})$.

We have:

$$|[\frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}}]_{ij}|^2 \leq \frac{E_p+m}{2E_p} \leq 1,\ \ i,j = 1,2,3,4 \tag{5.4}$$

$$|\psi_i(\vec{p})|^2 \leq \sum_{j=1}^{4} |\int d^3\vec{x}\ \ \cos(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})|^2 + |\int d^3\vec{x}\ \ \sin(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})|^2 \tag{5.5}$$

From remark 4.2, we have that both $\int d^3\vec{x}\ \cos(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})$ and $\int d^3\vec{x}\ \sin(\vec{p}\cdot\vec{x})\Psi_j(\vec{x})$ are square integrable and therefore $|\psi_i(\vec{p})|^2$ is square integrable. □

Proposition 5.4. *The inverse Fourier-Majorana transform of $\psi(\vec{p})$ is:*

$$\Psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{x})\psi(\vec{p}) \tag{5.6}$$

$$O^\dagger(\vec{p},\vec{x}) = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{5.7}$$

$O^\dagger$ is the hermitian conjugate of O.

Proof. The matrix $O^\dagger(\vec{p},\vec{x})$ verifies:

$$O^\dagger(\vec{p},\vec{x}) = \frac{\not{p}}{m} O^\dagger(\vec{p},\vec{x})\gamma^0 \tag{5.8}$$

$$i\gamma^0(i\vec{\not{\partial}} - m)O^\dagger(\vec{p},\vec{x}) = -\gamma^0\vec{\not{p}}O^\dagger(\vec{p},\vec{x})i\gamma^0 - \gamma^0\not{p}O^\dagger(\vec{p},\vec{x})i\gamma^0 \tag{5.9}$$

$$= -O^\dagger(\vec{p},\vec{x})i\gamma^0 E_p \tag{5.10}$$

From remark 4.3, the operator $i\gamma^0(i\vec{\not{\partial}} - m)$ is skew-hermitian and so:

$$\Big(\int d^3\vec{x} O(\vec{p},\vec{x})i\gamma^0(i\vec{\not{\partial}} - m)O^\dagger(\vec{q},\vec{x})\Big)^\dagger = -\int d^3\vec{x} O(\vec{q},\vec{x})i\gamma^0(i\vec{\not{\partial}} - m)O^\dagger(\vec{p},\vec{x}) \tag{5.11}$$

Which implies:

$$\int d^3\vec{x} i\gamma^0 E_q O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x}) = \int d^3\vec{x} O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x})i\gamma^0 E_p \tag{5.12}$$

Noting that $E_p + E_q > 0$, this implies that:

$$\int d^3\vec{x} e^{-i\gamma^0 \vec{q}\cdot\vec{x}} \frac{\vec{\not{q}}\gamma^0(E_p+m) + \vec{\not{p}}\gamma^0(E_q+m)}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} = 0 \tag{5.13}$$

Therefore, we get:

$$\int d^3\vec{x} O(\vec{q},\vec{x}) O^\dagger(\vec{p},\vec{x}) = \int d^3\vec{x} e^{-i\gamma^0\vec{q}\cdot\vec{x}} \frac{(E_p+m)(E_q+m) + \vec{\not{q}}\gamma^0\vec{\not{p}}\gamma^0}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{5.14}$$

$$= \int d^3\vec{x} e^{-i\gamma^0(\vec{q}-\vec{p})\cdot\vec{x}} \frac{(E_p+m)(E_q+m) + \vec{\not{q}}\gamma^0\vec{\not{p}}\gamma^0}{\sqrt{E_q+m}\sqrt{2E_q}\sqrt{E_p+m}\sqrt{2E_p}} \tag{5.15}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \frac{(E_p+m)(E_p+m) + \vec{p}^2}{(E_p+m)2E_p} \tag{5.16}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \frac{(E_p+m)(E_p+m) + (E_p+m)(E_p-m)}{(E_p+m)2E_p} \tag{5.17}$$

$$= (2\pi)^3\delta^3(\vec{q}-\vec{p}) \tag{5.18}$$

The other way around:

$$\int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y}) O(\vec{p},\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{5.19}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\frac{\not{p}}{m}\vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0}{E_p} \tag{5.20}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \cos(\vec{p}\cdot(\vec{y}-\vec{x})) + \tag{5.21}$$

$$+ \int \frac{d^3\vec{p}}{(2\pi)^3} (-\cos(\vec{p}\cdot(\vec{y}-\vec{x})) \frac{\vec{\not{p}}\gamma^0}{E_p} + \sin(\vec{p}\cdot(\vec{y}-\vec{x})) \frac{mi\gamma^0}{E_p} \tag{5.22}$$

$$= \delta^3(\vec{y}-\vec{x}) \tag{5.23}$$

Note that both $\cos(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{\vec{\not{p}}\gamma^0}{E_p}$ and $\sin(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{mi\gamma^0}{E_p}$ are odd in $\vec{p}$ and therefore do not contribute to the integral. □

6 Angular momentum of Majorana spinors

6.1 Majorana Spin

Definition 6.1. The Majorana spin operators $\frac{1}{2}\sigma^k$ are defined as:

$$\frac{1}{2}\sigma^k \equiv \frac{1}{2}\gamma^k\gamma^5, \quad k = 1,2,3 \tag{6.1}$$

They verify the angular momentum algebra:

$$[\frac{1}{2}\sigma^i, \frac{1}{2}\sigma^j] = i\gamma^0\epsilon^{ijk}\frac{1}{2}\sigma^k \tag{6.2}$$

Where ϵ^{ijk} is the Levi-Civita symbol. Note that $i\gamma^0$ commutes with σ^k and squares to -1, so it plays the role of the imaginary unit in the angular momentum algebra.

The eigenstates of $\frac{1}{2}\sigma^3$ are the Majorana spinors ψ verifying:

$$\psi_\pm = \frac{1 \pm \sigma^3}{2}\psi_\pm \tag{6.3}$$

The eigenvalues are $\frac{1}{2}\sigma^3\psi_\pm = \pm\frac{1}{2}\psi_\pm$.

6.2 Majorana orbital angular momentum

Definition 6.2. A set S of elements of an Hilbert space H with internal product $<,>$, is an orthonormal basis if:

1) For all $a \in S$: $< a, a >= 1$;
2) (orthogonality) For all $a, b \in S$, with $a \neq b$: $< a, b >= 0$;
3) (completeness) For all $f, g \in H$, $< g, f >= \sum_{a \in S} < g, a >< a, f >$.

Definition 6.3. Let $\vec{x} \in \mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x} = r(\sin(\theta)\sin(\varphi)\vec{e_1} + \sin(\theta)\sin(\varphi)\vec{e_2} + \cos(\theta)\vec{e_3}) \tag{6.4}$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and $r \in [0, +\infty[$ $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$.

Definition 6.4. $L^2(S^2)$ is the Hilbert space of real functions with domain $S^2 \equiv \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| = 1\}$, whose square is Lebesgue integrable in S^2. The internal product is:

$$< f, g >\equiv \int d(\cos\theta) d\varphi f(\theta, \varphi) g(\theta, \varphi),\ \ f, g \in L^2(S^2) \tag{6.5}$$

Definition 6.5. $L^2_4(S^2)$ is the Hilbert space of Majorana spinors whose 4 real components in the Majorana bases are in $L^2(S^2)$. The internal product is:

$$< \Psi, \Phi >\equiv \int d(\cos\theta) d\varphi\ \Psi^\dagger(\theta, \varphi)\Phi(\theta, \varphi),\ \ \Psi, \Phi \in L^2_4(S^2) \tag{6.6}$$

Definition 6.6. The Majorana angular momentum operators $\vec{L}_k$ are:

$$\vec{L}_k \equiv \sum_{i,j=1,2,3} -i\gamma^0\epsilon_{ijk}x^i\partial_j,\ \ k = 1, 2, 3 \tag{6.7}$$

Where ϵ_{ijk} is the Levi-Civita symbol.

The operators verify the angular momentum algebra:

$$[\vec{L}_i, \vec{L}_j] = i\gamma^0\epsilon_{ijk}\vec{L}_k \tag{6.8}$$

In spherical coordinates:

$$i\gamma^0\vec{L}_3 = \partial_\varphi \tag{6.9}$$

$$(\vec{L})^2 = -\sin(\theta)\partial_\theta\Big(\sin(\theta)\partial(\theta)\Big) - \frac{1}{\sin^2(\theta)}\partial^2_\varphi \tag{6.10}$$

Definition 6.7. The cosine spherical harmonics Y^c_{lm}, sine spherical harmonics Y^s_{lm} and associated Legendre functions of the first kind P_{lm} are:

$$Y^c_{lm}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\cos(m\varphi) \tag{6.11}$$

$$Y^s_{lm}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\sin(m\varphi) \tag{6.12}$$

$$P_l^m(\xi) \equiv \frac{(-1)^m}{2^l l!}(1-\xi^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}\xi^{l+m}}(\xi^2-1)^l \tag{6.13}$$

where $\theta \in [0,\pi]$, $\varphi \in [-\pi,\pi]$, $\xi \in [-1,1]$ and l, m are integer numbers $l \geq 0$, $-l \leq m \leq l$.

The spherical harmonics verify [13]:

$$\partial_\varphi Y^c_{lm}(\theta,\varphi) = -mY^s_{lm}(\theta,\varphi) \tag{6.14}$$

$$\partial_\varphi Y^s_{lm}(\theta,\varphi) = mY^c_{lm}(\theta,\varphi) \tag{6.15}$$

$$-\Big(\sin(\theta)\partial_\theta\Big(\sin(\theta)\partial_\theta\Big)+\frac{1}{\sin^2(\theta)}\partial_\varphi^2\Big)Y^a_{lm} = l(l+1)Y^a_{lm},\ a=c,s \tag{6.16}$$

Remark 6.8. *The spherical harmonics verify:*

$$< Y^s_{l'm'}, Y^c_{lm} > = 0 \tag{6.17}$$

$$< Y^s_{l'm'}, Y^s_{lm} > + < Y^c_{l'm'}, Y^c_{lm} > = \delta_{l'l}\delta_{m'm} \tag{6.18}$$

For all $f, g \in L^2(S^2)$:

$$< g, f >= \sum_{a=c,s,\ l\geq 0,\ -l\leq m\leq l} < g, Y^a_{lm} >< Y^a_{lm}, f > \tag{6.19}$$

Definition 6.9. The Majorana spherical harmonics Y_{lm} are:

$$Y_{lm}(\theta,\varphi) \equiv Y^c_{lm}(\theta,\varphi) + i\gamma^0 Y^s_{lm}(\theta,\varphi) \tag{6.20}$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{i\gamma^0 m\varphi} \tag{6.21}$$

The Majorana spherical harmonics are similar to the standard Laplace spherical harmonics definition, with $i\gamma^0$ in place of i. The properties are also similar.

They verify:

$$(\vec{L}_3 - m)Y_{lm}(\vec{x}) = 0 \tag{6.22}$$

$$(\vec{L}^2 - l(l+1))Y_{lm}(\vec{x}) = 0 \tag{6.23}$$

Proposition 6.10. *The columns of the Majorana spherical harmonics matrices form an orthonormal basis of the Hilbert space $L^2_4(S^2)$.*

Proof. We apply the remark 6.8 to directly obtain:

$$\int d(\cos\theta) d\varphi Y^{\dagger}_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{l'l}\delta_{m'm} \tag{6.24}$$

For all $\Phi, \Psi \in L^2_4(S^2)$:

$$< \Phi, \Psi > = \sum_{l\geq 0,\ -l\leq m\leq l} < \Phi, Y_{lm}\psi_{lm} > \tag{6.25}$$

$$\psi_{lm} \equiv \int d(\cos\theta) d\varphi Y^{\dagger}_{lm}(\theta,\varphi)\Psi(\theta,\varphi) \tag{6.26}$$

□

6.3 Majorana total angular momentum space

The operator $\vec{\sigma}\cdot\vec{L}$ is:

$$\vec{\sigma}\cdot\vec{L} = -i\gamma^0 \epsilon^{ij}_{k}\sigma^k x_i\partial_j \tag{6.27}$$

$$= -\frac{[\sigma^i,\sigma^j]}{2} x_i\partial_j \tag{6.28}$$

$$= \frac{\gamma^i\gamma^j - \gamma^j\gamma^i}{2} x_i\partial_j,\ \ i,j = 1,2,3 \tag{6.29}$$

In spherical coordinates:

$$i\partial\!\!\!/ = i\gamma^r(\partial_r - \frac{1}{r}\vec{\sigma}\cdot\vec{L}) \tag{6.30}$$

$$\vec{\sigma}\cdot\vec{L} = \gamma^\theta\gamma^r\partial_\theta + \gamma^\varphi\gamma^r\frac{1}{sin\theta}\partial_\varphi \tag{6.31}$$

θ and φ are the angles of $\vec{x}$ in spherical coordinates, r is the radius.

It verifies:

$$\vec{\sigma}\cdot\vec{L} = (\vec{L} + \frac{1}{2}\vec{\sigma})^2 - \vec{L}^2 - \frac{3}{4} \tag{6.32}$$

The term $\vec{L} + \frac{1}{2}\vec{\sigma}$ is the sum of two angular momentum operators of integer and one-half spin.

Remark 6.11. *Let $\vec{L}$ be an integer spin angular momentum operator, with orthonormal eigenstates $|l,m>$. Let $\frac{1}{2}\vec{\sigma}$ be a spin one-half angular momentum operator, with orthonormal eigenstates $|\frac{1}{2},s>$, where $s=\pm\frac{1}{2}$. Then, the orthonormal eigenstates of the operator $\vec{L}+\frac{1}{2}\vec{\sigma}$, are given by [13]:*

$$|j,\mu,(j+1/2)> = -\sqrt{\frac{j-\mu+1}{2j+2}}|j+1/2,\mu-1/2>|\frac{1}{2},+\frac{1}{2}> \tag{6.33}$$

$$+\sqrt{\frac{j+\mu+1}{2j+2}}|j+1/2,\mu+1/2>|\frac{1}{2},-\frac{1}{2}> \tag{6.34}$$

$$|j,\mu,(j-1/2)>=+\sqrt{\frac{j+\mu}{2j}}|j-1/2,\mu-1/2>|\frac{1}{2},+\frac{1}{2}> \tag{6.35}$$

$$+\sqrt{\frac{j-\mu-1}{2j}}|j-1/2,\mu+1/2>|\frac{1}{2},-\frac{1}{2}> \tag{6.36}$$

Where $j=\frac{1}{2},\frac{3}{2},...$ and $-j\leq\mu\leq j$. They satisfy:

$$(\vec{L}_3+\frac{\sigma^3}{2})|j,\mu,(j\pm 1/2)> = \mu|j,\mu,(j\pm 1/2)> \tag{6.37}$$

$$(\vec{L}+\frac{\vec{\sigma}}{2})^2|j,\mu,(j\pm 1/2)> = j(j+1)|j,\mu,(j\pm 1/2)> \tag{6.38}$$

$$\vec{\sigma}\cdot\vec{L}|j,\mu,(j\pm 1/2)> = -(\pm(j+1/2)+2)|j,\mu,(j\pm 1/2)> \tag{6.39}$$

$$\sigma^r\ |j,\mu,(j+1/2)> = -|j,\mu,(j-1/2)> \tag{6.40}$$

Definition 6.12. The Majorana spherical matrices are:

$$\Omega_{l\mu}(\theta,\varphi)\equiv\Big(-\sqrt{\frac{l-\mu}{2l+1}}Y_{l,\mu}(\theta,\varphi)+\sqrt{\frac{l+\mu+1}{2l+1}}Y_{l,\mu+1}(\theta,\varphi)\sigma^1\Big)\frac{1+\sigma^3}{2} \tag{6.41}$$

$$+\Big(\sqrt{\frac{l+\mu}{2l-1}}Y_{l-1,\mu}(\theta,\varphi)\sigma^1+\sqrt{\frac{l-\mu-1}{2l-1}}Y_{l-1,\mu+1}(\theta,\varphi)\Big)\frac{1-\sigma^3}{2} \tag{6.42}$$

with the integers $l\geq 1$ and $-l\leq\mu\leq l-1$. $Y_{l\mu}$ the Majorana spherical harmonics.

Proposition 6.13. *The columns of the Majorana spherical harmonics matrices form a complete orthonormal basis of the Hilbert space $L^2_4(S^2)$.*

Proof. Using remark 6.11, after some calculations, we get:

$$\int d(\cos\theta)d\varphi\Omega^\dagger_{l'\mu'}(\theta,\varphi)\Omega_{l\mu}(\theta,\varphi)=\delta_{l'l}\delta_{\mu'\mu} \tag{6.43}$$

$$\sum_{l\geq 1,\ -l\leq\mu\leq l-1}\int d(\cos\theta)d\varphi\ \Phi^\dagger(\theta,\varphi)\Omega_{l\mu}(\theta,\varphi)\psi_{l\mu}=\int d(\cos\theta)d\varphi\ \Phi^\dagger(\theta,\varphi)\Psi(\theta,\varphi) \tag{6.44}$$

For all $\Phi\in L^2_4(S^2)$. □

Using remark 6.11, the Majorana spherical matrices verify:

$$(\vec{L}^3+\frac{\sigma^3}{2})\ \Omega_{l\mu}=(\mu+\frac{1}{2})\Omega_{l\mu} \tag{6.45}$$

$$\vec{\sigma}\cdot\vec{L}\ \Omega_{l\mu}=-\Omega_{l\mu}(l\sigma^3+1) \tag{6.46}$$

$$\sigma^r\ \Omega_{l\mu}=-\Omega_{l\mu}\sigma^1 \tag{6.47}$$

$$i\gamma^r\Omega_{l\mu}=(-1)^\mu\Omega_{l,-\mu-1}i\gamma^5 \tag{6.48}$$

$$\vec{\sigma}\cdot\vec{L}i\gamma^r\ \Omega_{l\mu}=i\gamma^r\Omega_{l\mu}(l\sigma^3-1) \tag{6.49}$$

6.4 Radial Momentum Space

Remark 6.14. *The spherical Bessel functions of the first kind, $j_l : \mathbb{R}_+ \to \mathbb{R}$ with the integer $l \geq 0$, verify:*

$$(\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2})j_l(pr) = -p^2 j_l(pr) \tag{6.50}$$

$$\int_0^{+\infty} dr\ r^2 j_l(pr) j_l(p'r) = \frac{\pi\delta(p-p')}{2p^2} \tag{6.51}$$

$$\int_0^{+\infty} \frac{dp\ 2p^2}{\pi} j_l(pr) j_l(pr') = \frac{\delta(r-r')}{r^2} \tag{6.52}$$

Where the Dirac delta δ is such that for all $f \in L^2(\mathbb{R})$:

$$f(0) = \int dx\ \delta(x) f(x) \tag{6.53}$$

Definition 6.15. The Hankel-Majorana Transform $\psi(p, l, \mu)$ of a Majorana spinor $\Psi(\vec{x}) \in L_4^2(\mathbb{R}^3)$ is the Majorana spinor:

$$\psi(p, l, \mu) \equiv \int dr d(cos\theta) d\varphi r^2 \Lambda^\dagger(p, l, \mu, r, \theta, \varphi)\Psi(r, \theta, \varphi) \tag{6.54}$$

$$\Lambda(p, l, \mu, r, \theta, \varphi) \equiv \Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta, \varphi)\frac{1+\sigma^3}{2} \tag{6.55}$$

$$+ \Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta, \varphi)\frac{1-\sigma^3}{2} \tag{6.56}$$

Where Λ are the Hankel-Majorana matrices, $m, p \geq 0$, $E_p = \sqrt{p^2 + m^2}$ and the integers $l \geq 1, -l \leq \mu \leq l$.

Proposition 6.16. *Let $\psi(p, l, \mu)$ be the Hankel-Majorana Transform of a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^3)$. The inverse Hankel-Majorana Transform of $\psi(p, l, \mu)$ is:*

$$\Psi'(r, \theta, \varphi) \equiv \sum_{l \geq 1, -l \leq \mu \leq l} \int_0^{+\infty} \frac{dp\ (E_p + m)}{E_p \pi} \Lambda(p, l, \mu, r, \theta, \varphi)\psi(p, l, \mu) \tag{6.57}$$

It verifies, for all $\Phi \in L_4^2(\mathbb{R}^3)$:

$$\int d(cos\theta) d\varphi\ dr\ r^2\ \Phi^\dagger(r, \theta, \varphi)\Psi'(r, \theta, \varphi) = \int d(cos\theta) d\varphi\ dr\ r^2 \Phi^\dagger(r, \theta, \varphi)\Psi(r, \theta, \varphi) \tag{6.58}$$

Proof. The following equation is verified:

$$i\gamma^0(i\vec{\not\partial} - m)\Lambda(p, l, \mu) = E_p \Lambda(p, l, \mu) i\gamma^0 \tag{6.59}$$

Since the operator $i\gamma^0(i\vec{\not\partial} - m)$ is skew-Hermitic the equation above implies that:

$$i\gamma^0 E_{p'} I = I i\gamma^0 E_p \tag{6.60}$$

$$I \equiv \int d(cos\theta) d\varphi\ dr\ r^2\ \Lambda^\dagger(p', l', \mu', r, \theta, \varphi)\Lambda(p, l, \mu, r, \theta, \varphi) \tag{6.61}$$

As $E_p + E_{p'} > 0$, in the integral I the terms odd in $i\gamma^r$ are null. From the orthogonality of the spherical matrices, we get that the Λ matrices are orthogonal:

$$I = \delta_{l'l}\delta_{\mu'\mu} \int d(cos\theta)d\varphi\ dr\ r^2 \tag{6.62}$$

$$\Big(p' j_l(p'r) p j_l(pr) + (E_{p'} - m) j_{l-1}(p'r)(E_p - m) j_{l-1}(pr) \frac{1+\sigma^3}{2} \tag{6.63}$$

$$+ p' j_{l-1}(p'r) p j_{l-1}(pr) + (E_{p'} - m) j_l(p'r)(E_p - m) j_l(pr) \frac{1-\sigma^3}{2}\Big) \tag{6.64}$$

$$= \delta_{l'l}\delta_{\mu'\mu} \frac{\pi\delta(p-p')}{2p^2}(E_p - m)2E_p = \delta_{l'l}\delta_{\mu'\mu} \frac{\pi E_p \delta(p-p')}{E_p + m} \tag{6.65}$$

To show completeness, using $i\gamma^r \Omega_{l\mu} = (-1)^\mu \Omega_{l,-\mu-1} i\gamma^5$, we first show that:

$$\sum_{l'\mu'} \int d(cos\theta)d\varphi\ \psi^\dagger(p,l',\mu')\Lambda^\dagger(p,l',\mu',r,\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) = \tag{6.66}$$

$$= \psi^\dagger(p,l,\mu) p (j_l(pr)\frac{1+\sigma^3}{2} + j_{l-1}(pr)\frac{1-\sigma^3}{2}) \tag{6.67}$$

$$+ \psi^\dagger(p,l,-\mu-1)(-1)^\mu (E_p - m)(-j_l(pr)\frac{1-\sigma^3}{2} + j_{l-1}(pr)\frac{1+\sigma^3}{2}) i\gamma^5 \tag{6.68}$$

$$= \int d(cos\theta')d\varphi' dr' (r')^2\ \Psi^\dagger(r',\theta',\varphi')\Big(\tag{6.69}$$

$$p j_l(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1+\sigma^3}{2} \tag{6.70}$$

$$+ p j_{l-1}(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1-\sigma^3}{2} \tag{6.71}$$

$$(-1)^\mu (E_p - m) j_{l-1}(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l,-\mu-1}(\theta',\varphi')\frac{1+\sigma^3}{2} i\gamma^5 \tag{6.72}$$

$$- (-1)^\mu (E_p - m) j_l(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l,-\mu-1}(\theta',\varphi')\frac{1-\sigma^3}{2} i\gamma^5 \tag{6.73}$$

$$= \int d(cos\theta')d\varphi' dr' (r')^2\ \Psi^\dagger(r',\theta',\varphi')\Big(\tag{6.74}$$

$$p j_l(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1+\sigma^3}{2} \tag{6.75}$$

$$+ p j_{l-1}(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l\mu}(\theta',\varphi')\frac{1-\sigma^3}{2} \tag{6.76}$$

$$(E_p - m) j_{l-1}(pr')\Big(p j_l(pr) + (E_p - m) j_{l-1}(pr) i\gamma^r\Big)\Omega_{l,\mu}(\theta',\varphi')\frac{1-\sigma^3}{2} \tag{6.77}$$

$$- (E_p - m) j_l(pr')\Big(p j_{l-1}(pr) - (E_p - m) j_l(pr) i\gamma^r\Big)\Omega_{l,\mu}(\theta',\varphi')\frac{1+\sigma^3}{2} \tag{6.78}$$

$$= \int d(cos\theta')d\varphi' dr' (r')^2\ \Psi^\dagger(r',\theta',\varphi')\Omega_{l\mu}\frac{2p^2 E_p}{E_p + m}\Big(\tag{6.79}$$

$$j_l(pr') j_l(pr)\frac{1+\sigma^3}{2} + j_{l-1}(pr') j_{l-1}(pr)\frac{1-\sigma^3}{2}\Big) \tag{6.80}$$

If we integrate on p and use the completeness of the spherical Bessel functions, we get:

$$\int d(cos\theta)d\varphi\Psi'^{\dagger}(r,\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) = \int d(cos\theta)d\varphi\Psi^{\dagger}(r,\theta,\varphi)\Omega_{l\mu}(\theta,\varphi) \tag{6.81}$$

Since the columns of the spherical matrices $\Omega_{l\mu}$ are a complete basis, we have shown the completeness of the Hankel-Majorana transform:

$$\int d(cos\theta)d\varphi dr\ r^2\ \Psi'^{\dagger}(r,\theta,\varphi)\Phi(r,\theta,\varphi) = \int d(cos\theta)d\varphi dr\ r^2\Psi^{\dagger}(r,\theta,\varphi)\Phi(r,\theta,\varphi) \tag{6.82}$$

For all $\Phi \in L^2_4(\mathbb{R}^3)$. □

7 Relation between the Dirac and Majorana Momentums

The Dirac equation for the free fermion can be written as:

$$i\gamma^0(i\partial\!\!\!/ - m)\Psi(x) = 0 \tag{7.1}$$

Where Ψ is a spinor. Note that the equation contains only Majorana matrices. The Fourier or Hankel Transforms of the equation are:

$$(\partial_0 + i\gamma^0 E_p)\Psi(x^0, p) = 0 \tag{7.2}$$

The solutions can be written as:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{p\!\!\!/\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x}\psi(\vec{p}) \tag{7.3}$$

Where $p^0 = E_p$ and $\psi(\vec{p})$ is an arbitrary spinor. If $\psi(\vec{p})$ is a Majorana spinor, then the solution $\Psi(x)$ is also a Majorana spinor.

The solutions can also be written as:

$$\Psi(x^0,r,\theta,\varphi) = \sum_{l\geq 1, -l\leq\mu\leq l-1} \int_0^{+\infty} \frac{dp(E_p+m)}{E_p\pi}\Lambda(p,l,\mu,r,\theta,\varphi)e^{-i\gamma^0 E_p\cdot x^0}\psi(p,l,\mu) \tag{7.4}$$

Where $\psi(p,l,\mu)$ is an arbitrary spinor and Λ are the Hankel-Majorana matrices.

The set of quantum numbers $(\vec{p})$ and (p,l,μ) are related with the linear and spherical momentums of Dirac spinors. The Majorana spin is related with the Dirac spin. For instance, to obtain the Dirac spinor solution for the free electron, we just set $\psi_e(\vec{p}) = \frac{1+\gamma^0}{2}\psi_e(\vec{p})$ and we get:

$$\Psi_e(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{p\!\!\!/ + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{-ip\cdot x}\frac{1+\gamma^0}{2}\psi_e(\vec{p}) \tag{7.5}$$

The matrix γ^0 was replaced by the identity matrix 1, due to the presence of the projector. The same thing happens with the spherical solution and with the spin.

To obtain the Dirac spinor solution for the free positron, we just set $\psi_p(\vec{p}) = \frac{1-\gamma^0}{2}\psi_p(\vec{p})$ and the matrix γ^0 gets replaced by -1.

8 Energy-momentum space

Now we can extend our transforms to define an energy-momentum space. We will use the notation:

$$[\not p] = \gamma^0 E_p - \vec{\gamma} \cdot \vec{p} \tag{8.1}$$

Note that $\not p$ is not necessarily on-shell, while $[\not p]$ is on-shell, that is $([\not p])^2 = m^2$. Both E_p and $[\not p]$ do not depend on p^0.

Definition 8.1. Given a Majorana spinor $\Psi \in L^2_4(\mathbb{R}^4)$, the Fourier-Majorana transform in space-time is defined as:

$$\psi(p) \equiv \int d^4x O(p,x)\Psi(x) \tag{8.2}$$

Where $O(p,x)$ is:

$$O(p,x) \equiv e^{i\gamma^0 p^0 x^0} O(\vec{p},\vec{x}) = e^{i\gamma^0 p\cdot x} \frac{[\not p]\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} \tag{8.3}$$

Note that E_p and $[\not p] = \gamma^0 E_p - \vec{\gamma}\cdot\vec{p}$ don't depend on p^0, but $p\cdot x = p^0x^0 - \vec{p}\cdot\vec{x}$ does.

Proposition 8.2. *The inverse Fourier-Majorana transform in space-time is given by:*

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4} O^\dagger(p,x)\psi(p) \tag{8.4}$$

Where $O^\dagger$ is the hermitian conjugate of O, given by:

$$O^\dagger(p,x) = O^\dagger(\vec{p},\vec{x})e^{-i\gamma^0 p^0 \cdot x^0} = \frac{[\not p]\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x} \tag{8.5}$$

Proof.

$$\int \frac{d^4p}{(2\pi)^4} O^\dagger(p,y)O(p,x) = \int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y})\Big(\int \frac{dp^0}{2\pi} e^{-i\gamma^0 p^0(y^0 - x^0)}\Big)O(\vec{p},\vec{x}) \tag{8.6}$$

$$= \delta(y^0 - x^0)\int \frac{d^3\vec{p}}{(2\pi)^3} O^\dagger(\vec{p},\vec{y})O(\vec{p},\vec{x}) \tag{8.7}$$

$$= \delta^4(y - x) \tag{8.8}$$

$$\int d^4x O(q,x)O^\dagger(p,x) = \int dx^0 e^{i\gamma^0 q^0 x^0}\Big(\int d^3\vec{x} O(\vec{q},\vec{x})O^\dagger(\vec{p},\vec{x})\Big)e^{-i\gamma^0 p^0 x^0} \tag{8.9}$$

$$= (2\pi)^3\delta^3(\vec{q} - \vec{p})\int dx^0 e^{i\gamma^0(q^0 - p^0)x^0} \tag{8.10}$$

$$= (2\pi)^4\delta^4(q - p) \tag{8.11}$$

□

Definition 8.3. The Hankel-Majorana transform in space-time of a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^4)$ is:

$$\psi'(p^0, p, l, \mu) \equiv \int dx^0 e^{i\gamma^0 p^0 x^0} \psi(x^0, p, l, \mu) \tag{8.12}$$

Where $\psi(x^0, p, l, \mu)$ is the Hankel-Majorana transform in space of Ψ.

Proposition 8.4. *Let $\psi(p^0, p, l, \mu)$ be the Hankel-Majorana Transform in space-time of a Majorana spinor $\Psi \in L_4^2(\mathbb{R}^4)$. The inverse Hankel-Majorana Transform of $\psi(p^0, p, l, \mu)$ is:*

$$\Psi'(x^0, r, \theta, \varphi) \equiv \sum_{l\geq 1, -l\leq\mu\leq l} \int_0^{+\infty} \frac{dp(E_p + m)}{E_p \pi} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \Lambda(p, l, \mu, r, \theta, \varphi) e^{-i\gamma^0 p^0 \cdot x^0} \psi(p^0, p, l, \mu) \tag{8.13}$$

It verifies, for all $\Phi \in L_4^2(\mathbb{R}^4)$:

$$\int dx^0 d(cos\theta) d\varphi\ dr\ r^2\ \Phi^\dagger(x^0, r, \theta, \varphi) \Psi'(x^0, r, \theta, \varphi) = \tag{8.14}$$

$$= \int dx^0 d(cos\theta) d\varphi\ dr\ r^2 \Phi^\dagger(x^0, r, \theta, \varphi) \Psi(x^0, r, \theta, \varphi) \tag{8.15}$$

Proof. The matrices $\Lambda(p, l, \mu, r, \theta, \varphi) e^{-i\gamma^0 p^0 \cdot x^0}$ are orthogonal:

$$\int dx^0 d(cos\theta) d\varphi\ dr\ r^2\ e^{i\gamma^0 p'^0 \cdot x^0} \Lambda^\dagger(p', l', \mu', r, \theta, \varphi) \Lambda(p, l, \mu, r, \theta, \varphi) e^{-i\gamma^0 p^0 \cdot x^0} = \tag{8.16}$$

$$= \delta_{l'l} \delta_{\mu'\mu} \frac{\pi E_p \delta(p - p')}{E_p + m} \int dx^0 e^{i\gamma^0 p'^0 \cdot x^0} e^{-i\gamma^0 p^0 \cdot x^0} = \delta_{l'l} \delta_{\mu'\mu} \frac{\pi E_p \delta(p - p')}{E_p + m} 2\pi \delta(p'^0 - p^0) \tag{8.17}$$

To show completeness, we first show that:

$$\sum_{l'\mu'} \int_0^{+\infty} \frac{dp'\ (E'_p + m)}{E'_p \pi} \int d(cos\theta) d\varphi dr\ r^2 \tag{8.18}$$

$$\psi^\dagger(p^0, p', l', \mu') e^{i\gamma^0 p^0 \cdot x^0} \Lambda^\dagger(p', l', \mu', r, \theta, \varphi) \Lambda(p, l, \mu, r, \theta, \varphi) = \tag{8.19}$$

$$= \psi^\dagger(p^0, p, l, \mu) e^{i\gamma^0 p^0 \cdot x^0} \tag{8.20}$$

$$= \int dx'^0 d(cos\theta) d\varphi dr\ r^2 \Psi^\dagger(x'^0, r, \theta, \varphi) \Lambda(p, l, \mu, r, \theta, \varphi) e^{-i\gamma^0 p^0 x'^0} e^{i\gamma^0 p^0 \cdot x^0} \tag{8.21}$$

If we integrate on p^0, we get:

$$\int d(cos\theta) d\varphi dr\ r^2 \Psi'^\dagger(x^0, r, \theta, \varphi) \Lambda(p, l, \mu, r, \theta, \varphi) = \int d(cos\theta) d\varphi dr\ r^2 \Psi^\dagger(r, \theta, \varphi) \Lambda(p, l, \mu, r, \theta, \varphi) \tag{8.22}$$

Since the columns of the Hankel matrices $\Lambda(p, l, \mu, r, \theta, \varphi)$ are a complete basis, we have shown the completeness of the Hankel-Majorana transform in space-time:

$$\int dx^0 d(cos\theta) d\varphi dr\ r^2\ \Psi'^\dagger(x^0, r, \theta, \varphi) \Phi(x^0, r, \theta, \varphi) = \tag{8.23}$$

$$= \int dx^0 d(cos\theta) d\varphi dr\ r^2 \Psi^\dagger(x^0, r, \theta, \varphi) \Phi(x^0, r, \theta, \varphi) \tag{8.24}$$

For all $\Phi \in L_4^2(\mathbb{R}^4)$. □

9 Conclusion

We fulfilled our goal to show that (without second quantization operators) all the kinematic properties of a free spin 1/2 particle with mass are present in the real solutions of the real free Dirac equation.

Since we live in a world where the Lorentz symmetries are important, we hope that the Majorana transforms can have some applications. I personally think that the study of the Majorana spinor properties will be useful in our understanding of the Standard Model. In particular, since the Majorana spinors are an irreducible representation of the double cover of the proper orthochronous Lorentz group, like the Weyl spinor, as well as the full Lorentz group, unlike the Weyl spinor, I think that their study might improve our knowledge about the discrete symmetries of the Lorentz group and the interactions that violate them.

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