A Mild Generalization of Zariski's Lemma

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Abstract. We give a mild generalization of Zariski's Lemma.

Theorem 1. *Let A be a unique factorization domain with infinitely many association classes of prime elements, let K be its field of fractions, and let L be a finite degree extension of K. Then L is* not *a finitely generated A-algebra.*

Recall that these association classes correspond bijectively to the nonzero principal prime ideals. The above assumptions are satisfied in particular by $\mathbb{Z}$ thanks to Euclid's observation that there are infinitely many prime numbers; Euclid's argument also applies to polynomial rings in one indeterminate over a field. Recall also

Theorem 2 (Zariski's Lemma)**.** *If L/k is a field extension with L finitely generated as a k-algebra, then L is a finite degree extension of k.*

Lemma 3. *Let A be a subring of a domain B, and B be integral over A. Then A is a field if and only if B is a field.*

Proof of Lemma 3. Assume B is a field, and let x be a nonzero element of A. We have

$$x^{-n} + a_{n-1}\, x^{1-n} + \cdots + a_0 = 0, \quad a_i \in A,$$

and thus

$$-x^{-1} = a_{n-1} + \cdots + a_0\, x^{n-1} \in A.$$

We won't need the converse, but let's prove it anyway. Assume A is a field, and let y be a nonzero element of B. We have

$$y^n + a_{n-1}\, y^{n-1} + \cdots + a_0 = 0, \quad a_i \in A.$$

and thus

$$y\ (y^{n-1} + a_{n-1}\, y^{n-2} + \cdots + a_1) = -a_0.$$

Assuming, as we may, that n is minimum, we have $a_0 \neq 0$, and we see that y is invertible. QED

Proof of Theorem 1. Assume by contradiction that $L = A[x_1, \ldots, x_n]$, and let a be the product of the denominators of the coefficients of the minimal polynomials of the x_i over L. Then L is integral over $A' := A[a^{-1}]$. In view of our assumptions on A, the ring A' is **not** a field, contradicting Lemma 3. QED

Proof of Theorem 2. We argue by induction on n. The result being clear if $n = 1$, assume $n \geq 2$. Form the ring $A := k[x_1]$ and its fraction field $K := k(x_1)$. By the inductive hypothesis, L is of finite degree over K, and we only need to show that x_1 is algebraic over k. But if x_1 were transcendental over k, we would get a contradiction by observing that A would satisfy the assumptions of Theorem 1. QED