Gravitational Waves

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Abstract

The proposed theory of gravitation is summarized, with a focus on dynamics. The linearized field equations are applied to gravitational waves. The theory predicts that longitudinal waves would be detected, which exert a force in the direction of propagation. It also explains the failure at LIGO and elsewhere to find transverse gravitational waves.

1. Introduction.

This theory emphasizes the scalar nature of time and energy. The fundamental interval of kinematics

$$ds^2 = c^2 dt^2 - d\mathbf{r}^2 \tag{1}$$

is invariant under a Lorentz transformation of the physical displacements $(dt,\, d\mathbf{r})$. The Lorentz transformation may take place at any point P, in flat or curved space-time. The vector $d\mathbf{r}$ is first projected onto an orthonormal 3-frame.[1] The projections, together with the scalar dt, are then transformed into new values, which are observed in a relatively moving frame. No coordinates are involved with this procedure.

Define the velocity $\mathbf{v} = d\mathbf{r}/dt$ in order to obtain

$$ds = c\, dt \sqrt{1 - \frac{v^2}{c^2}} \tag{2}$$

The energy and momentum

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \frac{E}{c^2}\,\mathbf{v} \tag{3}$$

form the invariant

$$m^2 c^4 = E^2 - c^2 \mathbf{p}^2 \tag{4}$$

Again, the momentum $\mathbf{p}$ may be projected onto an orthonormal 3-frame and, together with the scalar E, undergo a Lorentz transformation at point P. From (4), it follows that

$$E\,\frac{dE}{ds} = c^2\,\mathbf{p} \cdot \frac{d\mathbf{p}}{ds} \tag{5}$$

where constant rest mass is assumed. Substitute (3) to find

$$\frac{dE}{ds} = \mathbf{v} \cdot \frac{d\mathbf{p}}{ds} \tag{6}$$

This power formula and the momentum (3) make explicit use of the scalar, i.e., non-directional nature of energy.

[1] In the frame $\{\mathbf{ijk}\}$, the projections are $\mathbf{i} \cdot d\mathbf{r}$, $\mathbf{j} \cdot d\mathbf{r}$, and $\mathbf{k} \cdot d\mathbf{r}$.

2. Space, Time, Gravity. [1, 2]

An observer now introduces a coordinate system $\{x^\mu\}$, consisting of continuous space variables $\{x^i\}$ and synchronous clock readings $\{x^0\}$. In terms of these coordinates, the physical displacements are given by

$$c\,dt = e_0(x)dx^0 \qquad\qquad d\mathbf{r} = \mathbf{e}_i(x)dx^i \tag{7}$$

where $e_\mu = (e_0, \mathbf{e}_i)$ is a scalar, 3-vector basis. Substitution into the kinematic interval (1) yields

$$\begin{aligned} ds^2 &= (e_0 dx^0)^2 - \mathbf{e}_i \cdot \mathbf{e}_j\, dx^i dx^j \\ &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{8}$$

where

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} & \\ 0 & & & \end{pmatrix} \tag{9}$$

is the scalar, 3-vector metric.

The basis system changes from point to point according to the formula

$$\nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu} \tag{10}$$

This separates into scalar and 3-vector parts

$$\nabla_\nu e_0 = e_0 Q^0_{0\nu} \tag{11}$$

$$\nabla_\nu \mathbf{e}_i = \mathbf{e}_j Q^j_{i\nu} \tag{12}$$

By definition $Q^0_{j\nu} = Q^i_{0\nu} \equiv 0$, which leaves 40 components $Q^\mu_{\nu\lambda}$. The metrical functions satisfy

$$\partial_\lambda g_{\mu\nu} = g_{\mu\rho} Q^\rho_{\nu\lambda} + g_{\nu\rho} Q^\rho_{\mu\lambda} \tag{13}$$

or, in detail,

$$\partial_\lambda g_{00} = 2g_{00}Q^0_{0\lambda} \tag{14}$$
$$\partial_0 g_{ij} = g_{in}Q^n_{j0} + g_{jn}Q^n_{i0} \tag{15}$$
$$\partial_k g_{ij} = g_{in}Q^n_{jk} + g_{jn}Q^n_{ik} \tag{16}$$

The first expression may be inverted immediately

$$Q^0_{0\lambda} = \Gamma^0_{0\lambda} = \frac{1}{2}g^{00}\partial_\lambda g_{00} \tag{17}$$

The second may be inverted, if the two terms on the right-hand side are equal

$$Q^i_{j0} = \Gamma^i_{j0} = \frac{1}{2}g^{in}\partial_0 g_{nj} \tag{18}$$

Finally, Q^i_{jk} is assumed to be symmetric, in which case

$$Q^i_{jk} = \Gamma^i_{jk} = \frac{1}{2}g^{in}\left(\partial_k g_{jn} + \partial_j g_{nk} - \partial_n g_{jk}\right) \tag{19}$$

The additional conditions leave 28 independent $Q^\mu_{\nu\lambda}$. The Christofel coefficients

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}\right) \tag{20}$$

are symmetric in $\{\nu\lambda\}$, while the $Q^\mu_{\nu\lambda}$ are not. The following formula holds good

$$Q^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + g^{\mu\rho}g_{\lambda\eta}Q^\eta_{[\nu\rho]} \tag{21}$$

where

$$Q^\mu_{[\nu\lambda]} \equiv Q^\mu_{\nu\lambda} - Q^\mu_{\lambda\nu} \tag{22}$$

There are 9 independent components $Q^\mu_{[\nu\lambda]}$, namely

$$Q^0_{0i} = \frac{1}{2}g^{00}\partial_i g_{00} \qquad\qquad Q^i_{j0} = \frac{1}{2}g^{in}\partial_0 g_{nj} \tag{23}$$

The observer is free to introduce new coordinates $\{x^{\mu'}\}$. Although the space-time in question is generally time-dependent, at any point P the new coordinates will be *at rest* with respect to the old. Thus, the physical displacements remain unchanged

$$c\,dt = e_0 dx^0 = e_{0'} dx^{0'} \qquad\qquad d\mathbf{r} = \mathbf{e}_i dx^i = \mathbf{e}_{i'} dx^{i'} \tag{24}$$

The new space coordinates are independent of clock rates, $x^{i'} = x^{i'}(x^j)$, and the new clock rates are independent of space coordinate labels, $x^{0'} = x^{0'}(x^0)$. These transformations determine the covariance of the theory. In particular, the metric tensor $g_{\mu\nu}$ transforms as

$$g_{0'0'} = \frac{\partial x^0}{\partial x^{0'}} \frac{\partial x^0}{\partial x^{0'}}\, g_{00} \qquad\qquad g_{i'j'} = \frac{\partial x^m}{\partial x^{i'}} \frac{\partial x^n}{\partial x^{j'}}\, g_{mn} \tag{25}$$

Moreover, $Q^{\mu}_{[\nu\lambda]}$ (23) transforms as a tensor

$$Q^{0'}_{0'i'} = \frac{\partial x^n}{\partial x^{i'}}\, Q^0_{0n} \qquad\qquad Q^{i'}_{j'0'} = \frac{\partial x^{i'}}{\partial x^m} \frac{\partial x^n}{\partial x^{j'}} \frac{\partial x^0}{\partial x^{0'}}\, Q^m_{n0} \tag{26}$$

This gravitational field strength tensor will play a central role in the theory to follow. Perhaps most importantly, it serves to define the gravitational energy tensor [1]

$$T^{(g)}_{\mu\nu} = \kappa \left\{ Q^{\rho}_{[\lambda\mu]} Q^{\lambda}_{[\rho\nu]} + Q_\mu Q_\nu - \frac{1}{2} g_{\mu\nu} g^{\eta\tau} (Q^{\rho}_{[\lambda\eta]} Q^{\lambda}_{[\rho\tau]} + Q_\eta Q_\tau) \right\} \tag{27}$$

where $Q_\mu = Q^{\rho}_{[\rho\mu]}$ and $\kappa = c^4/8\pi G$. For a static, Newtonian potential ψ

$$g_{00} = 1 + \frac{2}{c^2}\psi \tag{28}$$

so that $Q^{\mu}_{[\nu\lambda]}$ is given by

$$Q^0_{0i} = \frac{1}{c^2}\,\partial_i \psi \qquad\qquad Q^i_{j0} = 0 \tag{29}$$

It follows that

$$T^{(g)}_{00} = \frac{1}{8\pi G} (\nabla\psi)^2 \tag{30}$$

$$T^{(g)}_{0i} = 0 \tag{31}$$

$$T^{(g)}_{ij} = \frac{1}{4\pi G} \left\{ \partial_i \psi\, \partial_j \psi - \frac{1}{2} \delta_{ij} (\nabla\psi)^2 \right\} \tag{32}$$

which is the Newtonian stress-energy tensor.

3. Dynamics.

The equation of planetary motion

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0 \tag{33}$$

follows from the variation

$$\delta \int \sqrt{g_{\mu\nu} u^\mu u^\nu}\, ds = 0 \tag{34}$$

where $u^\mu = dx^\mu/ds$. Several components $\Gamma^\mu_{\nu\lambda}$ transform as tensors, suggesting that real gravitational forces are at work. This may be shown explicitly by calculating the rate of change of energy and momentum. From (2, 3, 7) it follows that, in coordinate terms,

$$E = mc^2 e_0 u^0 \qquad\qquad \mathbf{p} = mc\, \mathbf{e}_i u^i \tag{35}$$

The four-velocity $e_\mu u^\mu$ changes according to the formula

$$\frac{d(e_\mu u^\mu)}{ds} = e_\mu \frac{du^\mu}{ds} + \frac{de_\mu}{ds} u^\mu = e_\mu \left\{ \frac{du^\mu}{ds} + Q^\mu_{\nu\lambda} u^\nu u^\lambda \right\} \tag{36}$$

where $de_\mu = e_\lambda Q^\lambda_{\mu\nu} dx^\nu$ is Cartan's derivative operator.[3] Substitute (21) and make use of the equation of motion (33) to obtain

$$\frac{d(e_\mu u^\mu)}{ds} = e^\mu g_{\lambda\eta} Q^\eta_{[\nu\mu]} u^\nu u^\lambda \tag{37}$$

Separate this formula into scalar and 3-vector parts, then substitute (23) to find that the planet's energy and momentum change as follows:

$$\frac{dE}{ds} = e^0 \frac{mc^2}{2} \left\{ -\partial_n g_{00}\, u^n u^0 + \partial_0 g_{mn}\, u^m u^n \right\} \tag{38}$$

$$\frac{d\mathbf{p}}{ds} = \mathbf{e}^i \frac{mc}{2} \left\{ \partial_i g_{00}\, u^0 u^0 - \partial_0 g_{in}\, u^0 u^n \right\} \tag{39}$$

These formulas are exact. They express the power and force which are exerted by the gravitational field. They are not independent, since formula (6) is satisfied. In the Newtonian limit (28), $u^0 = 1$ and $u^n = v^n/c$ so that

$$\frac{dE}{dt} = -m\, \nabla\psi \cdot \mathbf{v} \qquad\qquad \frac{d\mathbf{p}}{dt} = -m\, \nabla\psi \tag{40}$$

The conservation law is established, by performing a similar calculation with the material energy tensor $T^{\mu\nu}_{(m)}$ (which may include any field other than gravity). The four-divergence of $\mathbf{T}_{(m)} = e_\mu \otimes e_\nu \, T^{\mu\nu}_{(m)}$ is given by [4]

$$\begin{aligned} \nabla \cdot \mathbf{T}_{(m)} &= e_\mu \left\{ \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, T^{\mu\nu}_{(m)}) + Q^\mu_{\nu\lambda} T^{\nu\lambda}_{(m)} \right\} \\ &= e_\mu \left\{ \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, T^{\mu\nu}_{(m)}) + \Gamma^\mu_{\nu\lambda} T^{\nu\lambda}_{(m)} \right\} + e^\mu g_{\lambda\eta} Q^\eta_{[\nu\mu]} T^{\nu\lambda}_{(m)} \end{aligned} \tag{41}$$

The first term is zero by virtue of the equations of motion, leaving

$$\nabla \cdot \mathbf{T}_{(m)} = e^\mu g_{\lambda\eta} Q^\eta_{[\nu\mu]} T^{\nu\lambda}_{(m)} \tag{42}$$

Once again, material energy and momentum are not conserved, due to the gravitational interaction. Therefore, introduce the energy tensor $T^{\mu\nu}_{(g)}$ in order to obtain

$$\begin{aligned} \nabla \cdot \Big(\mathbf{T}_{(g)} + \mathbf{T}_{(m)} \Big) &= e_\mu \left\{ \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, T^{\mu\nu}_{(g)}) + Q^\mu_{\nu\lambda} T^{\nu\lambda}_{(g)} \right\} \\ &\qquad + e^\mu g_{\lambda\eta} Q^\eta_{[\nu\mu]} T^{\nu\lambda}_{(m)} = 0 \end{aligned} \tag{43}$$

This is the differential law of energy and momentum conservation. If gravity is negligible, then $T^{\mu\nu}_{(g)}$ and $Q^\mu_{[\nu\lambda]}$ are zero, and conservation follows from the equations of motion alone

$$\nabla \cdot \mathbf{T}_{(m)} = e_\mu \left\{ \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, T^{\mu\nu}_{(m)}) + \Gamma^\mu_{\nu\lambda} T^{\nu\lambda}_{(m)} \right\} = 0 \tag{44}$$

4. Gravitational Waves.

The gravitational field equations are derived by variation of the Einstein-Hilbert and material action

$$\delta \int \frac{\kappa}{2}\, g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\, d^4x + \delta \int L^{(m)} \sqrt{-g}\, d^4x = 0 \tag{45}$$

where

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\mu\nu} \tag{46}$$

There are seven field equations, corresponding to the seven variations $\delta g^{\mu\nu} = (\delta g^{00}, \delta g^{ij})$

$$\kappa\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big) + T^{(m)}_{\mu\nu} = 0 \tag{47}$$

Components R_{0i} and $T^{(m)}_{0i}$ do not appear.

In the weak-field approximation, the coordinate system is assumed to be nearly rectangular, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the $h_{\mu\nu}$ are small compared with unity. Substitute into $R_{\mu\nu}$ and retain only the linear terms, to find

$$R_{\mu\nu} = \frac{1}{2}\Big\{\eta^{\lambda\rho}\partial_\lambda\partial_\rho h_{\mu\nu} + \partial_\mu\partial_\nu h^\lambda_{\ \lambda} - \partial_\mu\partial_\lambda h^\lambda_{\ \nu} - \partial_\nu\partial_\lambda h^\lambda_{\ \mu}\Big\} \tag{48}$$

In regions far from the source $T^{(m)}_{\mu\nu}$, one expects to find wave solutions of the equation

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 0 \tag{49}$$

For motion along the x^3-axis, assume a solution of the form $h_{\mu\nu} = h_{\mu\nu}(x^0, x^3)$ to obtain

$$(R_{00} - \frac{1}{2}\eta_{00}R) = \frac{1}{2}\partial_3\partial_3(h^1_{\ 1} + h^2_{\ 2}) = 0 \tag{50}$$

$$(R_{33} - \frac{1}{2}\eta_{33}R) = \frac{1}{2}\partial_0\partial_0(h^1_{\ 1} + h^2_{\ 2}) = 0 \tag{51}$$

which shows that $h^1_{\ 1} + h^2_{\ 2} = 0$. Since $R = 0$, it follows that

$$R_{00} = \frac{1}{2}(\partial_0\partial_0 h^3_{\ 3} - \partial_3\partial_3 h^0_{\ 0}) = 0 \tag{52}$$

$$R_{11} = \frac{1}{2}(\partial_0\partial_0 - \partial_3\partial_3)h_{11} = 0 \tag{53}$$

$$R_{22} = \frac{1}{2}(\partial_0\partial_0 - \partial_3\partial_3)h_{22} = 0 \tag{54}$$

$$R_{33} = -\frac{1}{2}(\partial_0\partial_0 h^3_{\ 3} - \partial_3\partial_3 h^0_{\ 0}) = 0 \tag{55}$$

$$R_{12} = \frac{1}{2}(\partial_0\partial_0 - \partial_3\partial_3)h_{12} = 0 \tag{56}$$

$$R_{23} = \frac{1}{2}\partial_0\partial_0 h_{23} = 0 \tag{57}$$

$$R_{31} = \frac{1}{2}\partial_0\partial_0 h_{31} = 0 \tag{58}$$

The single condition $h^0_{\ 0} = h^3_{\ 3}$ is suggested by equations (52) and (55). This leaves three independent components which satisfy the wave equation

$$h_{00} = -h_{33} \quad h_{11} = -h_{22} \quad h_{12} = h_{21} \tag{59}$$

while $h_{23} = h_{31} = 0$.

The presence of the scalar component h_{00} is especially significant. According to (23, 27), the momentum density of a gravitational field is given by

$$\begin{aligned} T^{(g)}_{0i} &= \kappa(Q^0_{0n}Q^n_{i0} + Q^n_{n0}Q^0_{0i}) \\ &= \frac{\kappa}{4}\, g^{00}g^{mn}\left(\partial_n g_{00}\partial_0 g_{mi} + \partial_i g_{00}\partial_0 g_{mn}\right) \end{aligned} \tag{60}$$

This requires the existence of a spatially-dependent component g_{00}. In the wave field,

$$T^{(g)}_{03} = \frac{\kappa}{2}\,\partial_3 h^0_{\ 0}\,\partial_0 h^3_{\ 3} \tag{61}$$

which shows that *all* of the momentum is carried by the scalar-longitudinal field. The energy density is

$$\begin{aligned} T^{(g)}_{00} &= \frac{\kappa}{2}\left\{Q^m_{n0}Q^n_{m0} + Q^m_{m0}Q^n_{n0} - 2\eta^{mn}Q^0_{0m}Q^0_{0n}\right\} \\ &= \frac{\kappa}{4}\left\{(\partial_0 h^1_{\ 1})^2 + (\partial_0 h^1_{\ 2})^2 + (\partial_0 h^3_{\ 3})^2 + (\partial_3 h^0_{\ 0})^2\right\} \end{aligned} \tag{62}$$

There is energy in both transverse and longitudinal waves.

The scalar and longitudinal components possess energy and momentum. Therefore, the solution for h_{00} and h_{33} takes the form of a traveling wave, $a_L \cos(k^3x^3 - k^0x^0)$, where $k^0 = k^3$. It is readily shown that these components satisfy the energy conservation law (43), which is (to second order)

$$\partial_0 T^{00}_{(g)} + \partial_3 T^{03}_{(g)} = 0 \tag{63}$$

The transverse components h_{11}, h_{22}, h_{12} have no momentum, and the solution takes the form of a standing wave, $a_T \cos(k^3x^3)\cos(k^0x^0)$. In this case, conservation of energy is given by

$$\left\langle \partial_0 T^{00}_{(g)} \right\rangle = 0 \tag{64}$$

where the temporal average is taken.

The gravitational stress is given by

$$T^{(g)}_{ij} = \kappa \left\{ 2Q^0_{0i}Q^0_{0j} - \frac{1}{2}\eta_{ij} \left[\eta^{00}(Q^m_{n0}Q^n_{m0} + Q^m_{m0}Q^n_{n0}) + 2\eta^{mn}Q^0_{0m}Q^0_{0n} \right] \right\} \tag{65}$$

which is diagonal in the wave field

$$T^{(g)}_{11} = T^{(g)}_{22} = \frac{\kappa}{4}\left[(\partial_0 h^1{}_1)^2 + (\partial_0 h^1{}_2)^2 \right] \tag{66}$$

$$T^{(g)}_{33} = \frac{\kappa}{2}(\partial_3 h^0{}_0)^2 + \frac{\kappa}{4}\left[(\partial_0 h^1{}_1)^2 + (\partial_0 h^1{}_2)^2 \right] \tag{67}$$

These stresses are compressive, with the transverse standing wave exerting an equal pressure in all directions, and the longitudinal traveling wave exerting pressure along the direction of propagation. The longitudinal components satisfy the momentum conservation law

$$\partial_0 T^{30}_{(g)} + \partial_3 T^{33}_{(g)} = 0 \tag{68}$$

while the transverse components satisfy

$$\left\langle \partial_3 T^{33}_{(g)} \right\rangle = 0 \tag{69}$$

where the spatial average is taken.

Finally, the force exerted by the gravitational wave is found by substituting $h_{\mu\nu}$ into the momentum equation (39)

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= -\frac{mc^2}{2}\Big\{ \mathbf{i}_1 \Big(\partial_0 h^1{}_1 \frac{v^1}{c} + \partial_0 h^1{}_2 \frac{v^2}{c} \Big) + \mathbf{i}_2 \Big(\partial_0 h^2{}_1 \frac{v^1}{c} + \partial_0 h^2{}_2 \frac{v^2}{c} \Big) \\ &\qquad + \mathbf{i}_3 \Big(\partial_3 h^0{}_0 + \partial_0 h^3{}_3 \frac{v^3}{c} \Big) \Big\} \end{aligned} \tag{70}$$

If the detector is initially at rest, then $v^n = 0$, and the acceleration will be along the direction of propagation

$$\frac{d^2x^3}{dt^2} = -\frac{c^2}{2}\,\partial_3 h^0{}_0 \tag{71}$$

5. Concluding Remarks.

The experiments at LIGO, GEO, and Virgo have found no trace of transverse gravitational waves. This is explained by equation (70), which shows that the transverse forces are zero, if the detector is at rest. Detection is completely dominated by the longitudinal term.

In this theory, the energy and momentum are well-defined for particles and fields, including the gravitational field itself. The treatment of particle dynamics resembles that of special relativity. It begins with the Lorentz transformation, which serves to establish the relativistic expressions for energy and momentum. A coordinate basis is then introduced, which is free to vary from point to point in space-time. This non-uniformity is responsible for the energy, momentum, and stress of the gravitational field. Moreover, it is physically observed as the force of gravitation. The non-uniformity is embodied in the field strength tensor $Q^{\mu}_{[\nu\lambda]}$, which is the fundamentally new element in the theory. It appears in the gravitational energy tensor (27), the gravitational force (37), and the conservation law (43).

References

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