What is mass?

Chapter one: Mass in Newtonian Mechanics and Lagrangian mechanics

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Contents

1 Introduction

1.1 What is physics?

Physics is the study of matter and its motion through space and time. So the fundamental concepts of physics are matter, space-time, and motion.

1.2 What are the most important questions?

The fundamental questions of physics are:

1. What is the physical world made of?

2. What is the structure of space-time?

3. How do things change through space-time?

The deeper understanding of these fundamental concepts and more profound answers to these fundamental questions, indicate the development of physical theory.

1.3 Matter and Mass

Of these fundamental concepts, mass is the modeling of matter.

On the one hand, different properties of matter may lead to different concepts of mass. So we may discover many different physical attribute of matter, and use many different concepts of mass.

On the other hand, some concepts may appear very different at first, but as we see deeper and deeper, we find a single and unified origin of these different concepts of mass.

This kind of development of physic concept is very interesting and profound.

So one may distinguish conceptually between many different attributes of mass or different physical phenomena that can be explained using the concept of mass

There is no doubt that the problem of mass is one of the key problems of modern physics.

1.4 Overview of these series of talks

“To see a World in a Grain of Sand, And a Heaven in a Wild”

We will try to see the development and the whole picture of theoretical physic through the evolution of the very fundamental concept of *mass*.

1. The inertial mass in Newtonian mechanics

2. The Newtonian gravitational mass

3. Mass in Lagrangian formulism

4. Mass in the special theory of relativity

5. $E = MC^2$

6. Mass in quantum mechanics

7. Principle of equivalence and general relativity

8. The energy momentum tensor in general relativity

9. Mass in the standard model of particle physics

10. The higgs mechanism

2 Mass in Newtonian Mechanics

2.1 Newton's three laws of motion

Newton's laws of motion are three physical laws that form the basis for classical mechanics. They describe the relationship between the forces acting on a body and its motion due to those forces. They have been expressed in several different ways over nearly three centuries, and can be summarized as follows:

1. First law: Every object continues in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by external forces acted upon it.

2. Second law: The acceleration a of a body is parallel and directly proportional to the net force F acting on the body, is in the direction of the net force, and is inversely proportional to the mass m of the body, i.e., F = ma.

3. Third law: When two bodies interact by exerting force on each other, these forces (termed the action and the reaction) are equal in magnitude, but opposite in direction.

2.2 Newton's law of universal gravitation

Newton's law of universal gravitation states that the gravitational force between two bodies of mass m and M separated by a distance r is

$$F = -G\frac{mM}{r^2}$$

.

2.3 The MASS

The term "mass' was introduced into mechanics by Newton in 1687 in his "Principia". There are the following properties of mass in Newtonian mechanics:

- Mass is a measure of the amount of matter.
- Mass of a body is a measure of its inertia.
- Masses of bodies are sources of their gravitational attraction to each other.
- Mass of a composite body is equal to the sum of masses of the bodies that constitute it; mathematically that means that mass is additive.
- Mass of an isolated body or isolated system of bodies is conserved: it does not change with time.
- Mass of a body does not change in the transition from one reference frame to another.

2.4 The amount of matter

Newton defined mass as the amount of matter. Macroscopically, mass is associated with matter. the mass of an object is somehow relate with the number and type of atoms or molecules it contains, and with the energy involved in binding it together (which contributes a negative "missing mass," or mass deficit).

But matter, unlike mass, is poorly defined in science. The generally accepted definition of matter does not exist even today. This is partially because there are so many different kinds of matters with quite different properties.

2.4.1 Can matter have zero mass?

Some may claim that matter is anything that occupies space and has rest mass. Under this definition one will not consider photons – particles of light – as particles of matter, because they are massless. For the same reason they do not consider as matter the electromagnetic field. It is not quite clear whether they consider as matter almost massless neutrinos, which usually move with velocity close to that of light.

2.4.2 We only known so little about matter

Even more strangely, the ordinary matter, in the quarks and leptons definition, constitutes about 4% of the energy of the observable universe. The remaining energy is theorized to be due to exotic forms, of which 23% is dark matter and 73% is dark energy. We have very little knowledge about the majority matter of the universe.

Anyway roughly speaking, the mass is somehow the measure of amount of matter.

2.5 The inertia mass

Inertial mass is a measure of an object's resistance to changing its state of motion when a force is applied, which arises naturally from Newton's second law of motion.

It is determined by applying a force to an object and measuring the acceleration that results from that force. An object with small inertial mass will accelerate more than an object with large inertial mass when acted upon by the same force. One says the body of greater mass has greater inertia.

2.6 How to measure the inertia

We can measure its acceleration (assuming only that we can readily measure length and time) and hence the force acting on it in terms of its mass. The relation between mass and force is given by $\vec{F} = m_i\ddot{\vec{r}}$.

For any arbitrary body, the inertial mass is defined as follows: we take the body and let it interact, somehow, with a standard inertial mass (one kilogram).

Both the body and the standard will accelerate towards or away from one another. Designating the acceleration of the body and of the standard by $a$and a_s, respectively, we can define the inertial mass by

$$m_i = \frac{a_s}{a}.$$

2.7 The gravitational mass

The gravitational mass can be described as a measure of magnitude of the gravitational force which is

- exerted by an object (active gravitational mass), and
- experienced by an object (passive gravitational force)

when interacting with a second object.

Newton's law of universal gravitation states that the gravitational force between two bodies of mass m_G and M_G separated by a distance r is

$$F = -G\frac{m_G M_G}{r^2}.$$

2.8 How to measure the gravitational mass

The gravitational mass can be defined in a similar manner. We take a standard object and define its gravitational mass to be one unit; for convenience we will use the standard kilogram as a standard for both inertial and gravitational mass.

We now place the body at some distance r and let it interact gravitationally with the standard. The body will accelerate towards the standard. We define the gravitational mass of the body in terms of this acceleration and the distance between the objects:

$$m_G \equiv \lim_{r\to\infty} \frac{a m_i r^2}{G}.$$

Alternatively, using $m_i = \frac{a_s}{a}$ we can write this as

$$m_G \equiv \lim_{r\to\infty} \frac{a_s r^2}{G}.$$

This gives the gravitational mass in kilograms.

The limiting procedure $r \to \infty$ is needed in order to eliminate the effects of multipole fields, which depend on the mass distribution of the two bodies. Also, proceeding to the limit $r \to \infty$ eliminates the effect of shorter range forces (nuclear force, Van der Waals force etc.).

At large distances only the gravitational and electrostatic forces will remain, but the latter can be eliminated by taking the precaution of keeping the standard body neutral.

If we take two identical copies of the standard mass and let them fall towards each other, the acceleration of each serves to define the constant G:

$$G = \lim_{r\to\infty} \left(a_{ss} r^2\right).$$

2.9 Put together

With these precise definitions of m_i and m_G it is clear that the gravitational force between two particles is

$$F = -G\frac{m_G M_G}{r^2},$$

and the equation of motion is

$$m_i \frac{d^2 r}{dt^2} = -G\frac{m_G M_G}{r^2}.$$

2.10 Two conceputure differnet mass

Whether all particles fall in the gravitational field of the particle of mass M_G with the same acceleration depends on whether all particles have the same value of

$$\frac{m_i}{m_G}$$

If this ratio is a universal constant, it must have the value one (the standard body has this value by definition). The question is then, is the equation

$$m_i = m_G \quad ,$$

satisfied for all bodies?

This is a question which can be answered only by experimental means.

2.11 The additive of mass

In Newtonian Mechanics, mass of a composite body is equal to the sum of masses of the bodies that constitute it; mathematically that means that mass is additive.

Together with the believe that matter cannot be destroyed or eliminated, the mass is believe to be always a positive real number.

2.12 Mass is conserved quantity

Mass of an isolated body or isolated system of bodies is conserved: it does not change with time.

This is the reflection of the believe that matter cannot be destroyed or created. Things will not be so obviously in relativity.

2.13 Mass is invariant

Mass of a body does not change in the transition from one inertial reference frame to another in Newtonian Mechanics. Things will not be so obviously in relativity.

3 Mass in Lagrangian Mechanics

3.1 Elementary Introduction of Lagrangian Dynamics

Let's consider the Newton equation

$$\vec{F} = m\vec{a}$$

in the one-dimensional case, in which all the physical parameters depend on the variable $x$only. Assume that the *force is conservative*, which means that it is given as the spatial derivative of the **potential energy** $U(x)$:

$$F = -\frac{dU}{dx}.$$

Thus for this case Newton's equation can be re-written as

$$m\ddot{x} = -\frac{dU}{dx}, \tag{1}$$

where $\dot{x} = \frac{dx}{dt} = \mathrm{v}$ and $\ddot{x} = \frac{d^2x}{dt^2} = a$. Define the **Lagrangian** $L(x, \dot{x})$as a function of two variables, the position x and the speed $\dot{x}$,

$$L(x, \dot{x}) = T(\dot{x}) - U(x) = \frac{1}{2}m\dot{x}^2 - U(x), \tag{2}$$

- the kinetic energy $T(\dot{x}) = \frac{1}{2}m\dot{x}^2$ is a function of the speed variable only,
- and the potential energy is a function of the position only.

3.1.1 The Euler-Lagrange equation

From Eq. (2) one can easily see that

$$\frac{\partial L}{\partial x} = -\frac{dU}{dx}, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p. \tag{3}$$

Then obviously

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}. \tag{4}$$

With the use of Eq. (3) Newton's Eq. (1) becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x},$$

which is called **the Euler-Lagrange equation** in one dimension.

3.1.2 The Hamiltonian formulation

To find the Hamiltonian formulation of dynamics, we define first the Hamiltonian $H(x, p)$ as a function of two new variables, the momentum p and the position x: $H(x, p) = p\dot{x} - L(x, \dot{x})$, which is just the total energy $T + U$ as

$$H(x, p) = p\dot{x} - L(x, \dot{x}) = m\dot{x}^2 - \left(\frac{1}{2}m\dot{x}^2 - U(x)\right) = T + U. \tag{5}$$

The Hamilton equations, which replace Newton's equation of motion Eq. (1) are given by

$$\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}. \tag{6}$$

3.1.3 Two ways to look at the physical motion

One way to understand physical motion is through differential equations, which describe how physical quantities change continuously with time. If we known the position and momentum for this moment, we can calculate from the equation how the object will move in the next moment.

There is an alternative approach to look at this. Supposed that there are infinite mathematically possible paths for a system. The task for physic law is to find out which one is the real path nature takes.

It's interesting that the real path actually followed by a physical system is always that for which make some quantity minimized, or, more strictly, is stationary.

3.1.4 The principle of least action

The formal definition of the principle of least action is that it is "the principle stating that the actual motion of a conservative dynamical system between two points takes place in such a way that a function of the coordinates and velocities, called the action, has a minimum value with reference to all other paths between the points which correspond to the same energy."

3.1.5 The action

In physics, action is an attribute of the dynamics of a physical system. It is a mathematical functional which takes the trajectory, also called path or history, of the system as its argument and has a real number as its result. Generally, the action takes different values for different paths

The action S of a system, which is a scalar quantity, is defined as the time integral of the Lagrangian function,

$$S = \int_{t_1}^{t_2} L\left(x(t), \dot{x}(t)\right) dt.$$

Therefore the problem of the motion of a mechanical system can be stated as that of finding the path $x(t), t_1 \leq t \leq t_2$, such that the action S is minimal. From a mathematical point of view this class of problems belongs to the field of mathematics called variational calculus and a quantity defined like the action is named functional. A functional will have its extremal value when its "variation" is equal to zero. (This is like a function having an extremal value when its derivative is equal to zero).

Consider two points A and$B$. There are many trajectories joining the points, but a mechanical system which is evolving between them is choosing the trajectory that makes the action functional extremal.

Let's assume that $x(t)$is the real trajectory of the body. Let's us also imagine a second trajectory, which is very near to the first, given by $x(t) + \varepsilon h(t)$, where $h(t)$ is an arbitrary time dependent function and ε is a constant satisfying the condition $\varepsilon << 1$. Obviously at the ends of all varied paths $h(t)$ satisfies the conditions

$$h\left(t_1\right) = h\left(t_2\right) = 0. \tag{7}$$

The value of the action integral, which is a scalar, will be necessarily different according to the path taken by the particle to go from A to$B$,

$$S\left[x(t)\right] \neq S\left[x + \varepsilon h(t)\right].$$

To find the curve or curves and the path or paths that make the action extremal, we shall use the variation of the trajectory and evaluate the action for the extremal and for the varied trajectory. For the Lagrangian along the varied path we obtain, by using a Taylor series expansion,

$$L\left(x + \varepsilon h, \dot{x} + \varepsilon \dot{h}\right) = L\left(x, \dot{x}\right) + \varepsilon \left(\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h}\right).$$

Therefore the variation of the action along the two paths is given by

$$S\left[x + \varepsilon h(t)\right] - S\left[x(t)\right] = \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h}\right) dt.$$

The second term in the integrand can be transformed by using partial integration,

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \dot{h} dt = \left[\frac{\partial L}{\partial \dot{x}} h(t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}}\right) h(t) dt = - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}}\right) h(t) dt,$$

since $h(t_1) = h(t_2) = 0$.

Then the variation of the action integral for a very small ε is

$$\frac{\partial S}{\partial \varepsilon} = \lim_{\varepsilon \to 0} \frac{S[x + \varepsilon h(t)] - S[x(t)]}{\varepsilon} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) h(t) dt.$$

Since $h(t) \neq 0$for $t_1 < t < t_2$, for the path $x(t)$followed by the particle between the two fixed points $A$and Bbe an extremal of the action$S$, it is necessary and sufficient that the quantity between the brackets in the integral be zero. Then this condition gives the Euler-Lagrange Equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

3.2 All are in Lagrangian

Therefore the equation of motion of a particle under the action of a conservative force can be derived from the Principle of the Least Action. So all information about the dynamics of the system is contained in the

$$L(x, \dot{x}) = T(\dot{x}) - U(x) = \frac{1}{2}m\dot{x}^2 - U(x)$$

3.3 Why such Lagrangian?

Lagrangian of free particle can be determined by space-time symmetries.

In the above, we find the Lagrangian from the newton's law of motion. Also in reverse, if we known the right Lagrangian, we could derive the equation of motion through the Euler-Lagrange Equations.

If we want do it in reverse, how could we find the right Lagrangian?

One might wonder whether one must simply guess at the form of the Lagrangian, such as

the choice $\dot{x}^2$, or if there is a more basic principle for finding the right Lagrangian.

3.4 Symmetry is fearful!

One way of doing so, is by thinking the symmetry of space-time.

The remarkable thing is that certain symmetries are so powerful and so restrictive that they entirely determine the functional form of the Lagrangian, and therefore the equations of motion.

3.5 Newtonian picture of the world

The very least we can say about our world according to Newtonian mechanics is: the Universe is made of particles, space and time. The positions of particles change with time, according to Newton's law of motion.

In other words, we try to describe the Lagrangian as a function of position x, time t and velocity $\dot{x}=\frac{dx}{dt}$.

The basic assumption about the Lagrangian of Newtonian mechanics is then it has the form

$$L_0(\mathbf{x}, \dot{\mathbf{x}}, t)$$

.

3.6 Properties of Lagrangian

The choice of a least action principle as the basis of a theory of dynamics immediately implies that the Lagrangian has certain properties.

First, we get the same equations of motion if we multiply that function with an arbitrary constant.

Second, we get the same motion if we add to the integrand the total time derivative of an arbitrary function of space-time:

$$L'{=}L+\frac{df\,(\mathrm{x,t})}{dt}$$

then

$$S'=\int L'dt{=}\int Ldt+\int\frac{df\,(\mathrm{x,t})}{dt}dt{=}S+f_B\text{-}f_A$$

The additional terms are fixed, so that they have no influence on the equations of motion.

The above two properties of L are mathematical consequences of our decision to use a least action principle.

3.7 Create Lagrangian from Symmetry principle

But further restrictions on L must be built in by using clues from Nature. These may be of any kind, but in practice the most powerful and general have proven to be symmetries of space-time.

To begin with, consider the general form Lagrangian of a free particle

$$L_0\left(\mathbf{x},\dot{\mathbf{x}},t\right)$$

.

3.7.1 Invariance under Spatial and Time Translations:

The first symmetry is the ***homogeneity of space and time***.

That is to say, the motion of a free particle does not depend on the place from which we measure its progress, nor does it depend on the time at which we start our measurement. Accordingly, L_0 cannot explicitly depend on $\mathbf{x}$ or t, so that

$$L_0\left(\mathbf{x},\dot{\mathbf{x}},t\right){=}L_0\left(\dot{\mathbf{x}}\right)$$

3.7.2 Invariance under Spatial Rotations:

The second symmetry is that of *the isotropy of space.*

That is to say, the motion of a free particle does not depend on the orientation of our coordinate system. We conclude immediately that this means that L_0 cannot depend on the direction of$\dot{\mathbf{x}}$, but only depend on the magnitude.

It must be invariant under any rotations:

$$L_0\left(\mathbf{R}\dot{\mathbf{x}}\right)=L_0\left(\dot{\mathbf{x}}\right)=L_0\left(|\dot{\mathbf{x}}|\right)$$

where $\mathbf{R}$ is a rotation operator and $|\dot{\mathbf{x}}|=\sqrt{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}$

3.7.3 Invariance under Galilean Transformations:

There is yet another symmetry essential to the Newtonian mechanics: Galilean relativity.

Coordinates of S and S', moving with relative velocity $\mathbf{v}$, are related by

$\mathbf{x}' = \mathbf{x} - \mathbf{v}t$ and $t' = t$

If S is an inertial system in which Newton's 1^{st} law holds, so is S'.

Physics law should be invariant under Galilean transformations, so does the Lagrangian.

How will this symmetry restrict the form of Lagrangian? We will see through the following trick.

We denote $X = \frac{1}{2}|\dot{\mathbf{x}}|^2$, using this new variable to express the free particle with Lagrangian $L_0(|\dot{\mathbf{x}}|) = L_0(X)$.

Thus the Euler-Lagrange equations are

$$0 = \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{\mathbf{x}}}\right) = \frac{d}{dt}\left(\frac{\partial X}{\partial \dot{\mathbf{x}}}\frac{dL_0}{dX}\right) = \frac{d}{dt}\left(\dot{\mathbf{x}}\frac{dL_0}{dX}\right)$$

where $\frac{\partial X}{\partial \dot{\mathbf{x}}} = \frac{\partial}{\partial \dot{\mathbf{x}}}\left(\frac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\right) = \dot{\mathbf{x}}$

$$\frac{d}{dt}\left(\frac{dL_0}{dX}\right) = \ddot{\mathbf{x}}\cdot\frac{\partial}{\partial \dot{\mathbf{x}}}\left(\frac{dL_0}{dX}\right) = \ddot{\mathbf{x}}\cdot\dot{\mathbf{x}}\frac{d^2L_0}{dX^2}$$

$$0 = \ddot{\mathbf{x}}\frac{dL_0}{dX} + \dot{\mathbf{x}}\frac{d}{dt}\left(\frac{dL_0}{dX}\right) = \ddot{\mathbf{x}}\frac{dL_0}{dX} + \dot{\mathbf{x}}\left(\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}}\right)\frac{d^2L_0}{dX^2}$$

$$\ddot{\mathbf{x}}\frac{dL_0}{dX} + \dot{\mathbf{x}}\left(\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}}\right)\frac{d^2L_0}{dX^2} = 0$$

Under the Galilean transformation

$$\dot{\mathbf{x}} \to \dot{\mathbf{x}}' = \dot{\mathbf{x}} - \mathbf{v} \quad \ddot{\mathbf{x}} \to \ddot{\mathbf{x}}' = \ddot{\mathbf{x}}$$

$$X \to X' = \frac{1}{2}\left(\dot{\mathbf{x}} - \mathbf{v}\right)\cdot\left(\dot{\mathbf{x}} - \mathbf{v}\right) = X - \dot{\mathbf{x}}\cdot\mathbf{v} + \frac{1}{2}\mathbf{v}^2$$

The Euler-Lagrange equation can remain unchanged iff

$$\frac{d^2L_0}{dX^2} = 0$$

which imply that

$$\frac{dL_0}{dX} = const \equiv m$$

So

$$L_0 = \frac{1}{2}m\ \dot{\mathbf{x}}^2 + const$$

And the mass is nothing but just a integral constant in the Lagrangian.

3.8 Symmetry determine the dynamic

The conclusion is that Lagrangian of free particle is determined by spacetime symmetries.

Note that the above mechanism of using symmetries to nail down the functional form of L works wonderfully in other cases, too. In fact, all known forces of Nature can be derived from a symmetry principle which, remarkably enough, not only prescribes the Lagrangian of a free particle (like the way in which we obtained above), but the interactions as well.

4 Summary: what is mass then?

- a quantitative measure of an object's resistance to acceleration
- or a measure of magnitude of the gravitational force
- or just a parameter in the Lagrangian?
- ...even stranger in the next lecture...coming soon...

Thanks!