The Mathematical Basis of the Fine Structure Constant

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It can be shown that the Fine Structure Constant can be defined as:

$$\alpha \equiv \frac{1}{\sqrt{(-2e^{\pi} - e^{2\pi} + 2\pi e + e^{\pi^2} + \pi^2)}} = 0.007297352558.$$

The Fine Structure Constant, α, gives the magnitude of spectral line splitting (Fine Structure) due to spin-orbit coupling and the consequent difference in energy between electron spin and orbital angular momentum vectors being parallel and anti-parallel. It is also the ratio of the Hartree potential energy, Ha, or twice the Rydberg ionisation energy, Ry, to the electron rest energy, $m_e c^2$:

$$\alpha = \sqrt{\frac{Ha}{m_e c^2}} = \sqrt{\frac{2Ry}{m_e c^2}}$$

The description of the state of the electron undergoing spin-orbit coupling should show the relationship:

(total electron energy) = (electron rest energy) + ([orbit] × [spin] × [hydrogen ground state])

A complete orbit is 2π radians, a change from spin parallel to spin anti-parallel is π radians, and the hydrogen ground state is -1 Rydberg:

$$E_{total} = (m_e c^2) + ([2\pi] \times [\pi] \times [-Ry]) = m_e c^2 - 2\pi^2 Ry$$

The relationship between action S (energy × time), the reduced Planck constant ħ, and the phase of the electron wave is given by:

$$e^{iS/\hbar}$$

and the Canonical Commutation Relation is:

$$[E, t] = Et - tE = -i\hbar$$

The difference in action associated with rotating the electron wave's reference-frame by θ will be:

$$\Delta S\theta = (Et - tE)\theta$$

giving:

$$e^{i\Delta S\theta/\hbar} = e^{i(Et-tE)\theta/\hbar} = e^{i(-i)\theta} = e^{\theta}$$

The imaginary unit is therefore cancelled out. A further difference from classical rotations comes due to the Uncertainty Relations between position x and momentum p, and between energy E and time t:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

This condition can nevertheless be understood in classical terms by appealing to General Relativity Theory, in which the presence of mass-energy produces space-time curvature characterised by a metric tensor, g_{ij}, the mechanism behind gravity and gravitational waves. Since the Uncertainty Relations must also apply to uncertainty in fluctuations of the g_{ij} field, an expression of the relative error in position:

$$\eta = \frac{\Delta x}{x}$$

becomes effectively interchangeable with the equation for the strain amplitude of a gravitational wave:

$$a = \frac{\Delta l}{l}$$

That is:

$$al\Delta p \geq \frac{\hbar}{2}$$

$$\frac{\Delta E a l}{c} \geq \frac{\hbar}{2}$$

Crucially, the distortion in length identified with gravitational waves, Δl, oscillates alternately between orthogonal axes. The corollary is that a full treatment of the increment in position, Δx, should involve transposing orthogonal axes. Some considerations following from this are:

- In Feynman diagrams, the change of sign of electrical energy from electron to anti-electron,

"-" → "+", is associated with transposing the space and time axes relative to the arrows representing these particles.

- in polar coordinates and the polar form of complex numbers, a transposing of orthogonal coordinates exchanges radius and angle:

$$(r)e^{(in\theta)} \to (\theta)e^{(inr)}$$

- because the electron wave flows around the atom, and there are two switch-like states of spin alignment, the atom can be treated as a "point-circuit" and transposing orthogonal dimensions will produce a transposing of "series" and "parallel" which, in C. E. Shannon's mathematical model of switching circuits, is associated with an exchange of the operations "+" and "×".

Therefore, the $e^{iS/\hbar}$ components to be added are:

- spin alignment (π):

$$-e^{(i(Et-tE)\theta/\hbar)} \to \theta e^{(i(Et-tE)/\hbar)}$$

$$= -e^{\pi} \to \pi e$$

and:

$$-e^{(i(Et-tE)\phi/\hbar)} \to \phi e^{(i(Et-tE)/\hbar)}$$

$$= -e^{\pi} \to \pi e$$

- orbit (2π):

$$-e^{(i(Et-tE)(\pi+\pi)/\hbar)} \to e^{(i(Et-tE)(\pi\times\pi)/\hbar)}$$

$$= -e^{2\pi} \to e^{\pi^2}$$

giving:

$$-2e^{\pi}Ha + 2\pi eHa - e^{2\pi}Ha + e^{\pi^2}Ha = m_e c^2 - 2\pi^2 Ry$$

The Inverse Fine Structure Constant is:

$$\frac{1}{\alpha} = \sqrt{\frac{m_e c^2}{Ha}} = \sqrt{\frac{m_e c^2}{2Ry}}$$

so that dividing all terms by 1 Ha (or 2 Ry) gives a value for the Fine Structure Constant of:

$$\alpha \equiv \frac{1}{\sqrt{\left(-2e^{\pi} - e^{2\pi} + 2\pi e + e^{\pi^2} + \pi^2\right)}} = 0.007297352558$$