Angular precession of elliptic orbits. Mercury

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Abstract. The relativistic precession of Mercury -43.1 seconds of arc per century-, is the result of a secular addition of 5.02 x10^{-7} rad. at the end of every orbit around the Sun. The question that arises in this paper, is to analyse the angular precession at each single point of the elliptic orbit and determine its magnitude and oscillation around the mean value, comparing key theoretical proposals. Underline also that, this astronomical determination has not been yet achieved, so it is considered that MESSENGER spacecraft, now orbiting the planet, should provide an opportunity to perform it. That event will clarify highlight issues, now that we are close to reach the centenary of the formulation and first success of General Relativity.

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1.-The theoretical G. R. angular precession.

In nearly all G.R. textbooks and articles, the trajectory of a target around a massive object (*M*) , is defined starting from the Schwarzschild solution, in a geometry and a space-time with spherical symmetry. The G.R. equation of motion with *u=1/r* is [1],[2]:

$$\frac{d^2u}{d\phi^2}+u=\frac{GM}{h^2}+\frac{3GM}{c^2}u^2$$

We can write the relativistic orbit as a slight perturbation of the newtonian ellipse as :

$$u=\frac{GM}{h^2}(1+e\ \cos\phi+\alpha(\phi))$$

h = angular momentum per unit of mass; e = eccentricity; ϕ = true anomaly; p = semi-latus;
$\alpha(\phi)$ is a very small function that produces the G.R. orbit differences, from the newtonian-kepler ellipse: an orbit precession.

On that basis, a first approximation and particular solution of this differential equation, neglecting second order terms, assuming a geodesic orbit and PPN formalism, is presented in the classic relativity textbook "Gravitation" by W. Misner [3] :

$$r=\frac{p}{1+e\ \cos[(1-\delta\phi o/2\pi)\phi]}=\frac{p}{1+e\ \cos\phi+Ke\ \phi\sin\phi}=\frac{p}{1+e\ \cos[(1-K)\phi]} \tag{1}$$

$$\alpha(\phi)=\frac{3GM}{c^2p}e\ \phi\ \sin\phi=K\ e\ \phi\ \sin\phi$$

were $\delta\ \phi_0/2\pi$ = constant angular precession = *K*.

As result of it, angular instantaneous precession in each point of the trajectory -$\delta(\phi)$-, is constant, so that the gradual addition along the orbit -$\Delta(\phi)$- , has a linear accumulation till its final value $\Delta(2\pi)$: a one complete orbital precession (*Fig*-1).

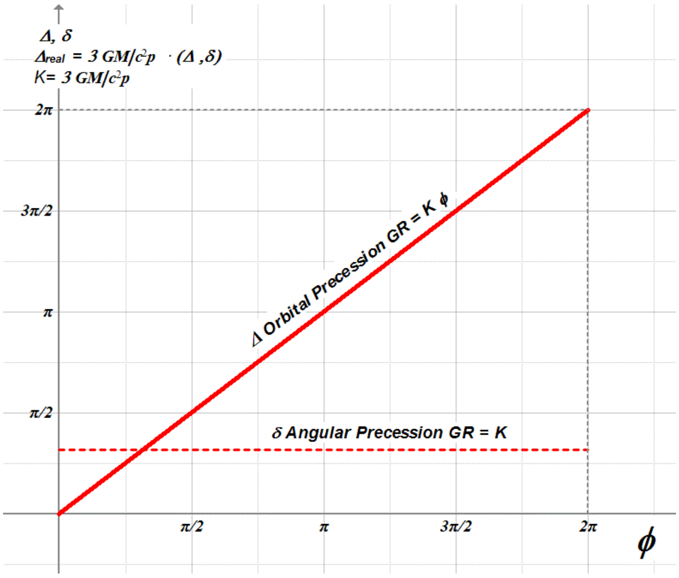

Fig-1. Angular(δ) and Orbital (Δ) precession.

Final one complete orbit precession is : [$\Delta(\phi) = \mathrm{K} \times \phi$]

$$\Delta(2\pi) = K 2\pi = \frac{6\pi\, GM}{c^2 p}\ rad\,/\,orbit = 5.02 \times 10^{-7}\ rad\,/\,orbit = 43{,}1\ sec.arc./\,century$$

This particular solution with a constant angular precession was, in my opinion, the first result obtained by Einstein in 1915.[4] :

"... *That contribution from the radius vector and described angle between the perihelion and the aphelion is obtained from the elliptical integral:*

$$\Delta = \int_{\alpha_1}^{\alpha_2} \frac{dx}{\sqrt{\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3}} \qquad \left\{ = \int_{\alpha_1}^{\alpha_2} f(x)\,dx \right\} \qquad (2)$$

where α_1 *and* α_2 *(...reciprocal values of the maximal and minimal distance from the Sun...) are the corresponding first roots of the equation :*

$$\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3 \qquad (3)$$

Coefficients of equation (3), were determined by Schwarzschild and other authors :

$$\left(\frac{du}{d\phi}\right)^2 = \frac{2E}{h^2} + \frac{2GM}{c^2h^2}u - u^2 + \frac{2GM}{c^2}u^3 = (3)$$

where E = *Energy per unit.*

The coefficients for equation (3), must be also consistent with the complete orbit precession of Mercury :

$\boldsymbol{\alpha} = 2GM/c^2 = 2{,}95 \times 10^{3}$; $\quad$ $\mathbf{1/B^2} = 1/h^2 = 13.6 \times 10^{-32}$; $\quad$ $\mathbf{2A/B^2} = -1.02 \times 10^{-9}$

Equation (2) represented by function *f(x)*, has the following graphic expression :

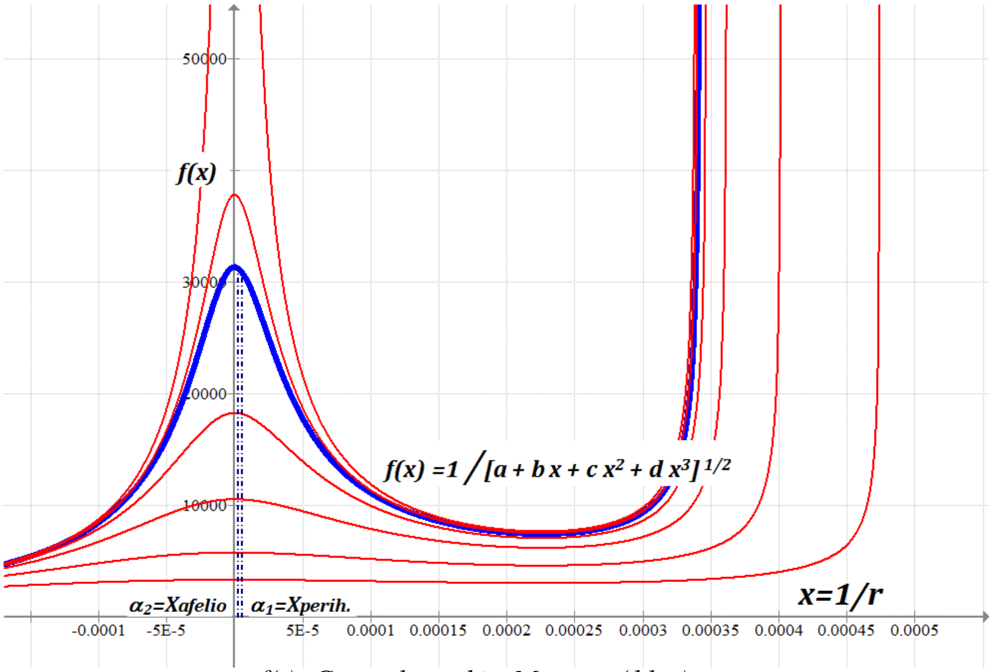

Fig-2a: *f(x): General graphic. Mercury (blue)*

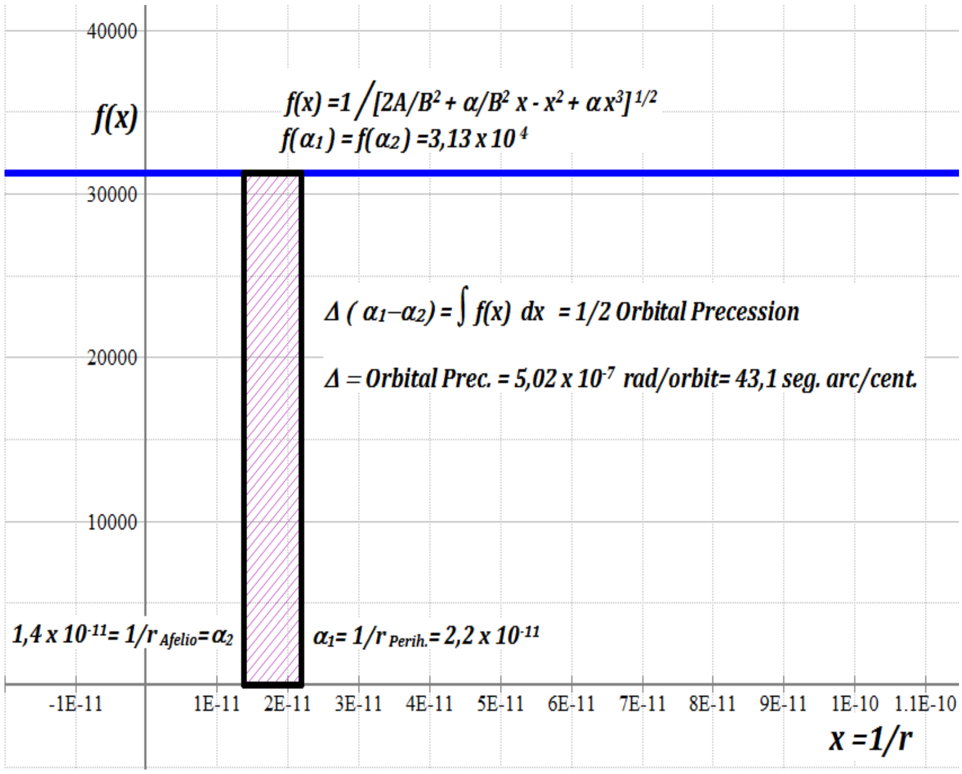

Fig-2b: *Graphic focused on Mercury.* α_1, α_2.

We can remark that *f(x),* has virtually the same values both in the aphelion as in the perihelion and also through the rest of the orbit. This means that this solution, involves a constant angular precession -$\delta(\phi)$- along the whole orbit and also a linear accumulation of the orbital precession -$\Delta(\phi)$-, with a K proportion relative to the true anomaly ϕ. (*Fig-1*).

2.- G.R. angular instantaneous precession. Periodic oscillations .

General Relativity admits also small periodic oscillations that are insignificant contributions and their only effect is to change slightly the position of the perihelion and the interpretation of r_{min} and *e*.[5] The most extended and accepted formulation of G.R. orbit fluctuations about the average orbital precession, based also in the Schwarzschild solution is : [1],[2],[8],[9]

$$\alpha(\phi) = \frac{3GM}{c^2 p}\left[\; 1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\phi\right) + e\phi\,\sin\phi\right] = \frac{3GM}{c^2 p}\left[\; 1 + \frac{e^2}{2} + j(\phi) + e\phi\sin\phi\;\right]$$

We will analyse the magnitude of the periodic oscillations that produces function $j(\phi)$ related with the mean value of a constant angular precession that involves the last term $e\ \phi\sin\phi$. It must be underlined that the cumulative effect produce by $[K\ e\ \phi]$, has also a periodic origin and implication; it really represents the magnitude of $[\sin(K\ e\ \phi)]$ that makes $\alpha(\phi)$ consistent in equation (1) and, as a result of it, the effect of the perturbation is shaped definitely as an angle, a real angular precession.

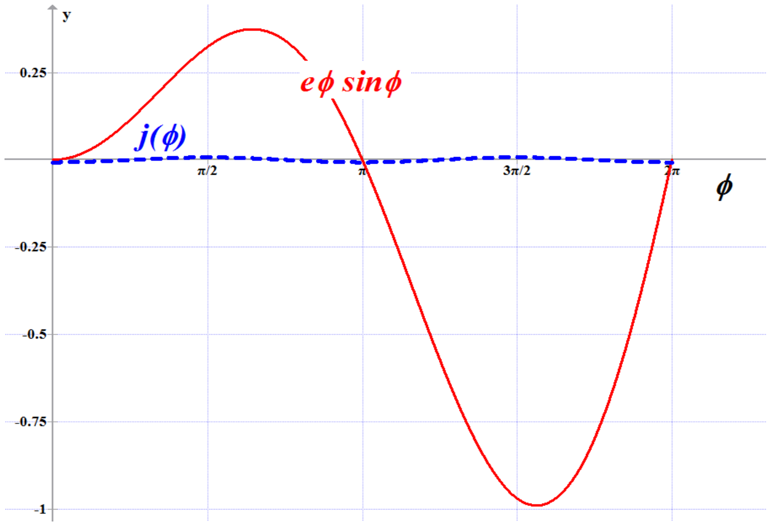

Fig-3 *: Angular precession oscillations : j(φ)*

As we can conclude, the function $j(\phi)$ involves very small variations. Its amplitude is about 3/100 of the mean constant value.

Professor M.Berry [5], presents another $\alpha(\phi)$ function with larger amplitude of oscillations, keeping however the same final orbital precession :

$$\alpha(\phi) = \frac{GM}{c^2 p}\left[\; (3 + 2e^2) + \left(\frac{1 + 3e^2}{e}\right)\cos\phi - e^2\cos^2\phi + 3e\phi\,\sin\phi\right]$$

$$\alpha(\phi) = \frac{GM}{c^2 p}\left[(3 + 2e^2) + j_B(\phi) + 3e\phi\,\sin\phi\right]$$

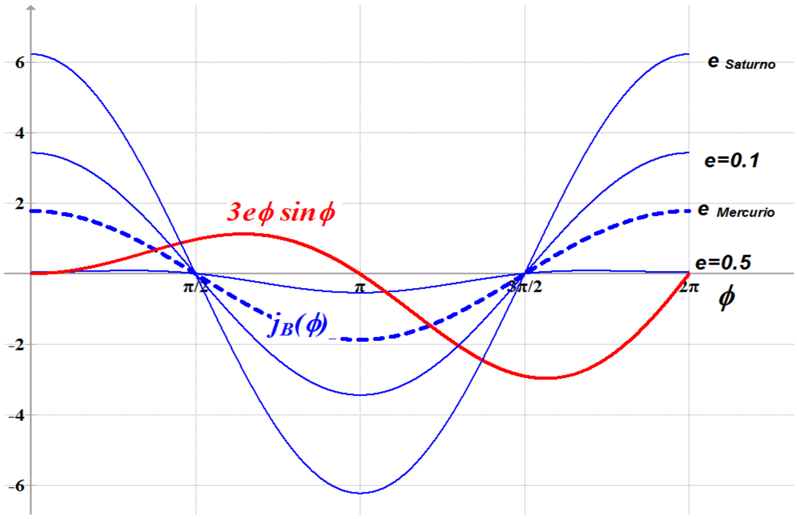

Fig-4 *: Angular precession oscillations;* $j_B(\phi)$

Standing out from *Fig-4*, there are significant oscillations, but with the same final orbital precession as in equation (1). The range of oscillations is equivalent to the magnitude of the theoretical constant precession. The eccentricity of the orbit has clear effects on the angular precession, increasing the amplitude as the eccentricity decreases.

3.- G.R. perturbing gravitational potential / force.

Trying to analyse the oscillations of the angular precession, we can also study the effects of a perturbing potential or force upon the newtonian field. This procedure should allow even more accurate results than those obtained solving the second order differential equation of motion.

The effective G.R. potential is displayed in equation (4), where the last term, is the perturbation potential added to the classic newtonian one. [1],[6],[10], [12]

$$V_{eff} = \frac{GM}{r} + \frac{h^2}{2r^2} + \frac{GM\,h^2}{c^2 r^3} \qquad (4)$$

$V(r) = \dfrac{GM\,h^2}{c^2 r^3} = (G.R.\ \ Perturbing\ \ pot.)$; and then, perturbing force is: $F(r) = \dfrac{\partial V(r)}{\partial r} = -\dfrac{3GMh^2}{c^2 r^4}$

We will now analyse some approaches and methods that explore the orbital precession produced by any potential or perturbation force.

a) Published in 1982, B. Davies [11] presented a solution to the orbital precession, based on the Laplace-Runge-Lenz vector, located in the same plane as the orbit and pointing in the direction of the perihelion. Its angular velocity, measures the precession if there is any external disturbance.
The magnitude of the total force would be equal to the usual newtonian, added with a function -*g(r)*- as a perturbing factor.

$$F(r) = -\frac{GM\,m}{r^2}\left[1 + g(r)\right]$$

The solution to the orbital precession is then :

$$\Delta = \frac{2}{e}\int_0^{\pi} g(r)\ \cos\phi\ d\phi \qquad rad.$$

Considering G.R. perturbing force and also an elliptic orbit:

$g(r) = \dfrac{3h^2}{c^2 r^2}$; $h^2 = GM\ p$; $r = \dfrac{p}{1 + e\ \cos\phi}$; and then, in agreement with the orbit´s symmetry

$$\Delta = \frac{3GM}{c^2\, p}\, 2\int_0^{\pi} \frac{1}{e}\left(1+e\ \cos\phi\right)^2 \cos\phi\ d\phi = \frac{3GM}{c^2\, p}\int_0^{2\pi} \frac{1}{e}\left(1+e\ \cos\phi\right)^2 \cos\phi\ d\phi = K\int_0^{2\pi} \delta_D(\phi)\, d\phi \quad rad.$$

Then, the angular precession is:

$$\delta_D(\phi) = K\frac{1}{e}\left(1+e\ \cos\phi\right)^2 \cos\phi \quad rad.$$

Davies also remarks that the factor $\cos\phi$, brings a positive sign to the precession in the part of the orbit when the planet is closer to the focus than the average distance; the rest is negative. Therefore, that statement supports that one half of the relativistic precession, is in the opposite direction to the advance of the planet in its orbit.

b) Published in 2005, M.G. Stewart [12] also starts his approach from the Laplace-Runge-Lenz vector, but providing the following alternative formulation :

$$\omega_{GR} = -\frac{r^2\, F(r)\, h^2\, \dot{\phi}}{GM\ e}\cos\phi \quad rad./sec.$$

with ω_{GR} , angular velocity of L-R-L vector.

$$\Delta = \int_0^{\tau} \omega_{GR}\ dt = \int_0^{2\pi} \omega_{GR}(1/\dot{\phi})\ d\phi = \frac{3GM}{c^2\, p}\int_0^{2\pi} \frac{1}{e}\left(1+e\ \cos\phi\right)^2 \cos\phi\, d\phi = K\int_0^{2\pi} \delta_D(\phi)\, d\phi$$

The result is just the same to the previous one.

c) Publish in 2007, G. Adkins [10], studies the precession by solving the equation of motion, adding a perturbing potential $V(u)$ in the following expression:

$$\Delta = \int \frac{du}{\sqrt{\dfrac{2E}{mh^2} + \dfrac{2GM}{h^2}u - u^2 + \dfrac{2V(u)}{mh^2}}} \quad rad.$$

By the change of variables $u = (1+e\,z)/p$, he obtains the following formulation of the orbital precession:

$$\Delta = \frac{-2p}{GMme^2}\int_{-1}^{1} \frac{z\,dz}{\sqrt{1-z^2}}\ \frac{V(z)}{dz}$$

and for a power-law potential, he concludes:

$$V(r) = \alpha_{-(n+1)}\ r^{-(n+1)} \quad ; \qquad \Delta_{-(n+1)} = \frac{-2(n+1)\alpha_{-(n+1)}}{GMmp^n\, e}\int_{-1}^{1} \frac{z\,dz}{\sqrt{1-z^2}}\,(1+ez)^n$$

Inserting G.R. perturbing potential and changing $z = \cos\phi$:

$$\Delta = \frac{3GM}{c^2\, p}\int_0^{2\pi} \frac{1}{e}\left(1+e\ \cos\phi\right)^2 \cos\phi\, d\phi = K\int_0^{2\pi} \delta_D(\phi)\, d\phi \quad rad.$$

Identical result to the one obtained previously, but from different arguments.

d) Published in 2008, O. I. Chashchina[13] proposed the calculation of the orbital precession based on the Hamilton vector with this definition:

$$\vec{u} = \vec{v} - \frac{\alpha}{h}\,\vec{e}_{\varphi} \quad ; \quad \alpha = GM$$

Then, the orbital precession is :

$$\Delta = -\frac{p^2}{\alpha e}\int_0^{2\pi} \frac{\cos\phi}{(1+e\cos\phi)^2}\ \frac{dV(r)}{dr}\, d\phi$$

V(r) is the perturbing potential :

$$\frac{dV(r)}{dr} = -\frac{3GM\,h^2}{c^2 r^4}\text{ ; and then,}$$

$$\Delta = \frac{3GM}{c^2\,p}\int_0^{2\pi}\frac{1}{e}(1+e\cos\phi)^2\cos\phi\,d\phi = K\int_0^{2\pi}\delta_D(\phi)\,d\phi \quad rad.$$

Again it is exactly the same result, however starting from a different hypothesis.

e) We will check these results but now with another test, based on a new approach. This is the Landau & Lifshitz formulation [7], which defines the precession produced by a perturbing potential. This formula is valid as a theorem, suitable for any small perturbation whatever could be its physical origin and returning the exact value. Integration is performed over an unperturbed orbit [14]:

$$\Delta(\phi) = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^{\pi} r^2\,\delta U\;d\phi\right)$$

where $M = m\,h$ = angular momentum, δU= perturbing potential energy = $m\,V(r)$
then, the angular precession is :

$$\delta_L(\phi) = \frac{\partial}{\partial h}\left[\frac{1}{h}\;r^2\;V(r)\;\;d\phi\right]$$

derivates referred to h are:

$$\frac{\partial p}{\partial h} = 2\frac{p}{h} \quad ; \quad \frac{\partial e}{\partial h} = -\frac{1}{h}\frac{1-e^2}{e} \quad [10];$$

and then :

$$\delta_L(\phi) = K\left[\;(1+e\cos\phi) + \frac{1-e^2}{e}\cos\phi\;\right] \quad rad.$$

The solution is different, however similar to the previous ones, with identical value of the final orbital precession (*Fig-5*) :

- The maximum values are somewhat lower at 0 and 2π but higher at π.
- The angular precession is null at ϕ = 1.77 *rad.* and ϕ = 4.50 *rad* while previously was null at $\pi/2$ y $3\pi/2$.

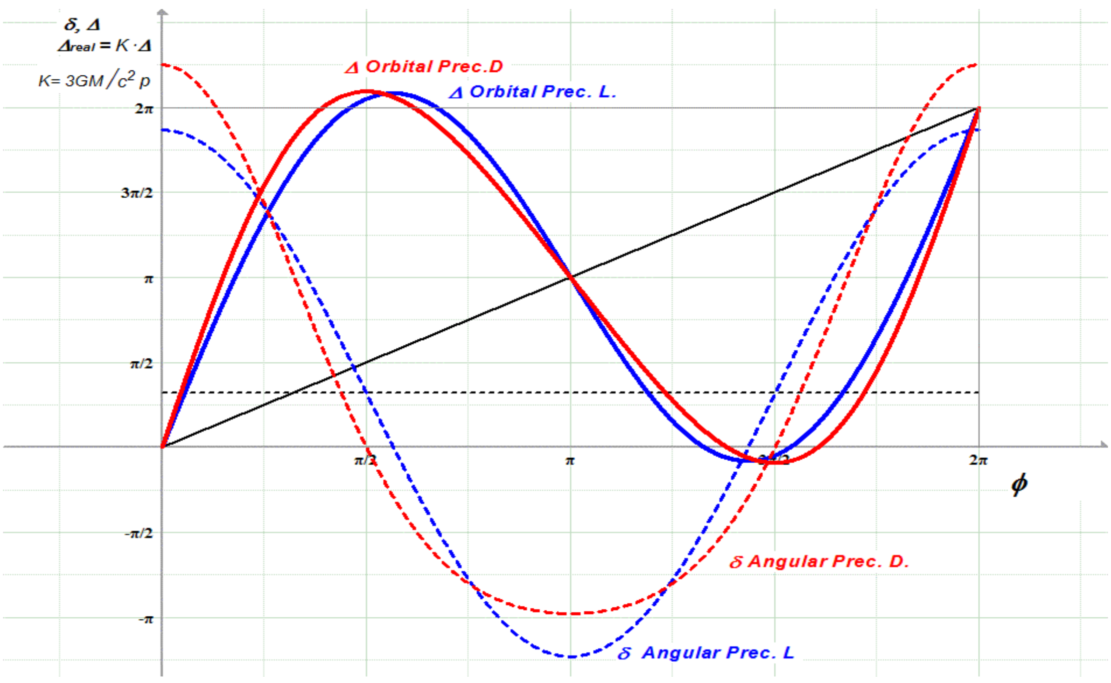

Fig-5. Theoretic approaches to Angular(δ) and Orbital (Δ) precession.

The results obtained, shows that the oscillation is such that between $\pi/2$ and $3\pi/2$, the orbital precession turns back, opposite to Mercury's own progress in its orbit. At these points (maximum and minimum) there is an equivalent lead/lag of 1,9 sec.arc./century related with the magnitude of the orbital precession at the final/ initial point of the orbit. (*Fig-5*)

Another issue is the clear influence that has the eccentricity in the magnitude of oscillations. The lower is the eccentricity, the greater the fluctuation of the angular precession because they are inversely proportional. (*Fig-6*)

In case of Mars ($e = 0.093$) , there would be a lead/lag of 1.3 sec.arc./century equivalent itself to the magnitude of the relativistic precession at the final/ initial point of the orbit. The Earth ($e = 0.017$) should have a lead/lag of 37.1 sec.arc./century, (*Fig-7*) nearly ten times the relativistic precession and Venus ($e = 0.0068$) should have 203.4 sec.arc./century, 24 times the final precession.

If this theoretical formulation is correct, these results should have significant observational data records, in the registered orbital precession of these planets,

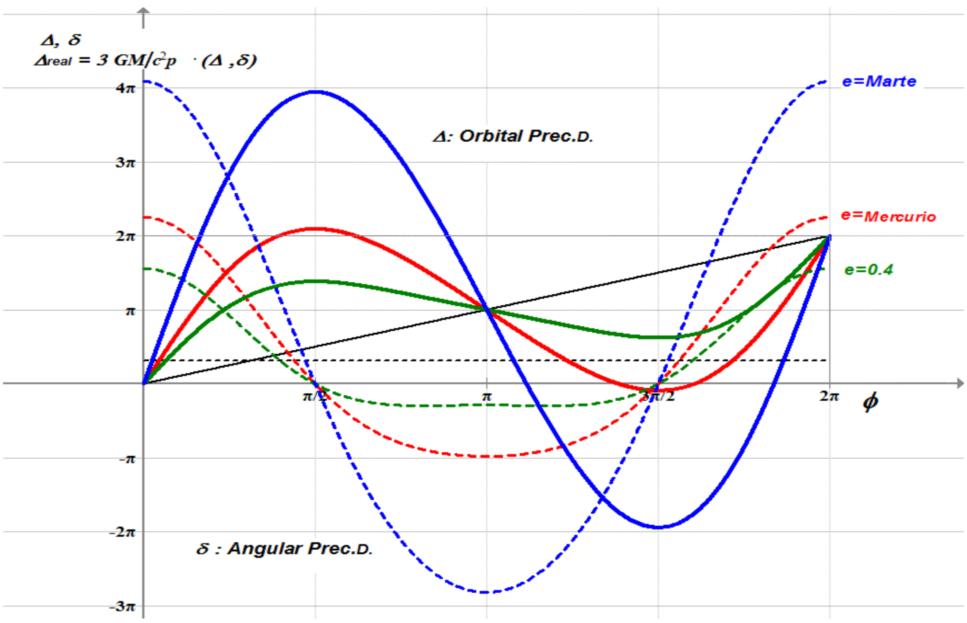

Fig-6. Eccentricity, Angular(δ) and Orbital (Δ) precession.

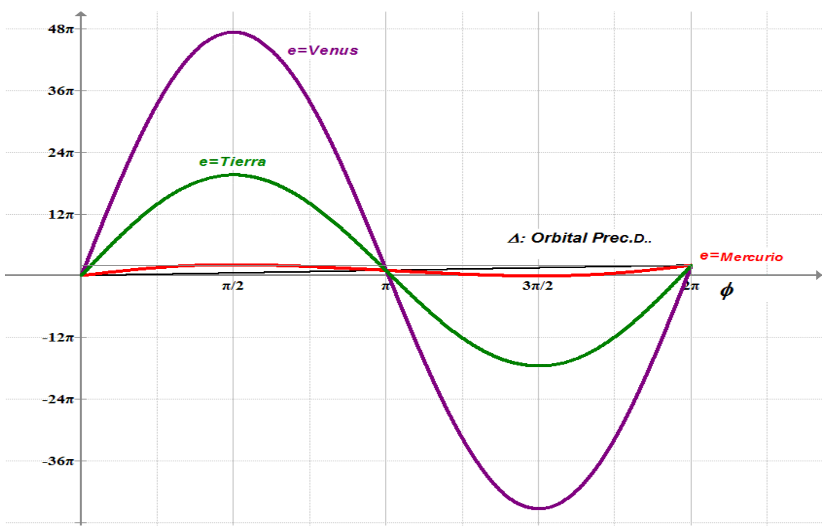

Fig-7. Eccentricity and Orbital (Δ) precession. Earth.Venus

4.- Mercury's orbit as an open free-fall path.

The currently precession of Mercury, is far larger to the one with only a relativistic origin. This is due to the effect produced by the rest of the planets, causing also precessions that must be added.

The largest precession is produced by Venus (277 ar.sec./cent.) followed by Jupiter (154 ar.sec./cent), the Earth-Moon system (91 ar.sec./cent) and the rest of the planets for a total of 532 ar.sec./cent. Relativistic Precession is 43 ar.sec./cent, therefore we can conclude that the real precession detected in astronomical observations is equivalent to 575 ar.sec./cent.

To study the oscillations of the angular precession related to the final magnitude in each orbit, it would be necessary to have for at least one year, data from the position of Mercury with the best possible accuracy. These data should be reduced with the other perturbations of the planets, as well as considering the effect of the equinoxes´s precession. In this way, we could examine Mercury's orbit as an open free-fall path, isolated from other planets gravitational interference. It is certainly a difficult and complex duty but clearly available with the current development of our technology and also not expensive.

MESSENGER spacecraft, now orbiting the planet, should provide an excellent opportunity to perform it, giving precise radiometric data on the day to day real position of Mercury.

To assess the influence of each planet in the orbit of Mercury, is not enough to replace it by the approximation due to a uniform ring of matter. We need to perform a software calculation based on elliptical and inclined orbits, positioning each planet in every moment.

5.- Conclusions and open comments.

a) It is positively difficult to consider a constant and lineal kinematics action for this angular precession. It should be appropriate as a first solution and approximation to the real motion and trajectory. Except angular momentum and total energy, no other feature is constant: perturbing potential, central force, acceleration, velocities, curvatures, radius, etc.

b) Angular precession oscillates about a mean value. The magnitude depends on the theoretical approach we use. In all the proposals, angular precession has a non-zero effect in the perihelion neither the aphelion, nodes where radial velocity is null.

c) The orbital precession produced by the perturbing potential, involves oscillations with a negative advance and turns back, opposite to Mercury's own progress in its orbit. Any elliptic orbit with eccentricity < 0.22 would have the same behaviour. However, the final one orbit precession does not change in any case.

d) Eccentricity should have great influence in the magnitude of oscillations of the angular precession.

e) The astronomical determination of the angular and orbital precession at each single point of the orbit, has not been yet achieved, so it is considered that MESSENGER spacecraft, now orbiting the planet, should provide an opportunity to perform it. That event will clarify some issues, now that we are close to reach the centenary of the formulation and first success of General Relativity.

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