The recent vision about preferred extremals and solutions of the modified Dirac equation

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Abstract

During years several approaches to what preferred extremals of Kähler action and solutions of the modified Dirac equation could be have been proposed and the challenge is to see whether at least some of these approaches are consistent with each other. It is good to list various approaches first.

1. For preferred extremals generalization of conformal invariance to 4-D situation is very attractive approach and leads to concrete conditions formally similar to those encountered in string model. The approach based on basic heuristics for massless equations, on effective 3-dimensionality, and weak form of electric magnetic duality is also promising. An alternative approach is inspired by number theoretical considerations and identifies space-time surfaces as associative or co-associative sub-manifolds of octonionic imbedding space.
2. There are also several approaches for solving the modified Dirac equation. The most promising approach is assumes that the solutions are restricted on 2-D stringy world sheets and/or partonic 2-surfaces. This strange looking view is a rather natural consequence of number theoretic vision. The conditions stating that electric charge is conserved for preferred extremals is an alternative very promising approach.

The question whether these various approaches are mutually consistent is discussed. It indeed turns out that the approach based on the conservation of electric charge leads under rather general assumptions to the proposal that solutions of the modified Dirac equation are localized on 2-dimensional string world sheets and/or partonic 2-surfaces. Einstein's equations are satisfied for the preferred extremals and this implies that the earlier proposal for the realization of Equivalence Principle is not needed. This leads to a considerable progress in the understanding of super Virasoro representations for super-symplectic and super-Kac-Moody algebra. In particular, the proposal is that super-Kac-Moody currents assignable to string world sheets define duals of gauge potentials and their generalization for gravitons: in the approximation that gauge group is Abelian - motivated by the notion of finite measurement resolution - the exponents for the sum of KM charges would define non-integrable phase factors. One can also identify Yangian as the algebra generated by these charges. The approach allows also to understand the special role of the right handed neutrino in SUSY according to TGD.

1 Introduction

During years several approaches to what preferred extremals of Kähler action and solutions of the modified Dirac equation could be have been proposed and the challenge is to see whether at least some of these approaches are consistent with each other. It is good to list various approaches first.

1. For preferred extremals generalization of conformal invariance to 4-D situation is very attractive approach and leads to concrete conditions formally similar to those encountered in string model [K1]. In particular, Einstein's equations with cosmological constant follow as consistency conditions and field equations reduce to a purely algebraic statements analogous to those appearing in equations for minimal surfaces if one assumes that space-time surface has Hermitian structure or its Minkowskian variant Hamilton-Jacobi structure(Appendix). The older approach based on basic heuristics for massless equations, on effective 3-dimensionality, and weak form of electric magnetic duality, and Beltrami flows is also promising. An alternative approach is inspired by number theoretical considerations and identifies space-time surfaces as associative or co-associative sub-manifolds of octonionic imbedding space [K9].

   The basic step of progress was the realization that the known extremals of Kähler action - certainly limiting cases of more general extremals - can be deformed to more general extremals having interpretation as preferred extremals.

   (a) The generalization boils down to the condition that field equations reduce to the condition that the traces $Tr(TH^k)$ for the product of energy momentum tensor and second fundamental form vanish. In string models energy momentum tensor corresponds to metric and one obtains minimal surface equations. The equations reduce to purely algebraic conditions stating that T and H^k have no common components. Complex structure of string world sheet makes this possible.

   Stringy conditions for metric stating $g_{zz} = g_{\overline{z}\overline{z}} = 0$ generalize. The condition that field equations reduce to $Tr(TH^k) = 0$ requires that the terms involving Kähler gauge current in field equations vanish. This is achieved if Einstein's equations hold true. The conditions guaranteeing the vanishing of the trace in turn boil down to the existence of Hermitian structure in the case of Euclidian signature and to the existence of its analog - Hamilton-Jacobi structure - for Minkowskian signature (Appendix). These conditions state that certain components of the induced metric vanish in complex coordinates or Hamilton-Jacobi coordinates.

   In string model the replacement of the imbedding space coordinate variables with quantized ones allows to interpret the conditions on metric as Virasoro conditions. In the recent case generalization of classical Virasoro conditions to four-dimensional ones would be in question. An interesting question is whether quantization of these conditions could make sense also in TGD framework at least as a useful trick to deduce information about quantum states in WCW degrees of freedom.

   The interpretation of the extended algebra as Yangian suggested previously [K12] to act as a generalization of conformal algebra in TGD Universe is attractive. There is also the conjecture that preferred extremals could be interpreted as quaternionic of co-quaternionic 4-surface of the octonionic imbedding space with octonionic representation of the gamma matrices defining the notion of tangent space quanternionicity.

2. There are also several approaches for solving the modified Dirac equation. The most promising approach is assumes that the solutions are restricted on 2-D stringy world sheets and/or partonic 2-surfaces. This strange looking view is a rather natural consequence of both strong form of holography and of number theoretic vision, and also follows from the notion of finite measurement resolution having discretization at partonic 2-surfaces as a geometric correlate. The conditions stating that electric charge is conserved for preferred extremals is an alternative very promising approach. One expects that stringy approach based on 4-D generalization of conformal invariance or its 2-D variant at 2-D preferred surfaces should also allow to understand the modified Dirac equation. In accordance with the earlier conjecture, all modes of the modified Dirac operator generate badly broken super-symmetries. Right-handed neutrino allows also holomorphic modes delocalized at entire space-time surface and the delocalization inside

Euclidian region defining the line of generalized Feynman diagram is a good candidate for the right-handed neutrino generating the least broken super-symmetry. This super-symmetry seems however to differ from the ordinary one in that ν_R is expected to behave like a passive spectator in the scattering.

In the following the question whether these various approaches are mutually consistent is discussed. It indeed turns out that the approach based on the conservation of electric charge leads under rather general assumptions to the proposal that solutions of the modified Dirac equation are localized on 2-dimensional string world sheets and/or partonic 2-surfaces. Einstein's equations are satisfied for the preferred extremals and this implies that the earlier proposal for the realization of Equivalence Principle is not needed. This leads to a considerable progress in the understanding of super Virasoro representations for super-symplectic and super-Kac-Moody algebra. In particular, the proposal is that super-Kac-Moody currents assignable to string world sheets define duals of gauge potentials and their generalization for gravitons: in the approximation that gauge group is Abelian - motivated by the notion of finite measurement resolution - the exponents for the sum of KM charges would define non-integrable phase factors. One can also identify Yangian as the algebra generated by these charges. The approach allows also to understand the special role of the right handed neutrino in SUSY according to TGD.

2 About deformations of known extremals of Kähler action

I have done a considerable amount of speculative guesswork to identify what I have used to call preferred extremals of Kähler action. The problem is that the mathematical problem at hand is extremely non-linear and that there is no existing mathematical literature. One must proceed by trying to guess the general constraints on the preferred extremals which look physically and mathematically plausible. The hope is that this net of constraints could eventually chrystallize to Eureka! Certainly the recent speculative picture involves also wrong guesses. The need to find explicit ansatz for the deformations of known extremals based on some common principles has become pressing. The following considerations represent an attempt to combine the existing information to achieve this.

2.1 What might be the common features of the deformations of known extremals

The dream is to discover the deformations of all known extremals by guessing what is common to all of them. One might hope that the following list summarizes at least some common features.

2.1.1 Effective three-dimensionality at the level of action

1. Holography realized as effective 3-dimensionality also at the level of action requires that it reduces to 3-dimensional effective boundary terms. This is achieved if the contraction $j^\alpha A_\alpha$ vanishes. This is true if j^α vanishes or is light-like, or if it is proportional to instanton current in which case current conservation requires that CP_2 projection of the space-time surface is 3-dimensional. The first two options for j have a realization for known extremals. The status of the third option - proportionality to instanton current - has remained unclear.

2. As I started to work again with the problem, I realized that instanton current could be replaced with a more general current $j = *B \wedge J$ or concretely: $j^\alpha = \epsilon^{\alpha\beta\gamma\delta} B_\beta J_{\gamma\delta}$, where B is vector field and CP_2 projection is 3-dimensional, which it must be in any case. The contractions of j appearing in field equations vanish automatically with this ansatz.

3. Almost topological QFT property in turn requires the reduction of effective boundary terms to Chern-Simons terms: this is achieved by boundary conditions expressing weak form of electric magnetic duality. If one generalizes the weak form of electric magnetic duality to $J = \Phi * J$ one has $B = d\Phi$ and j has a vanishing divergence for 3-D CP_2 projection. This is clearly a more general solution ansatz than the one based on proportionality of j with instanton current and would reduce the field equations in concise notation to $Tr(TH^k) = 0$.

4. Any of the alternative properties of the Kähler current implies that the field equations reduce to $Tr(TH^k) = 0$, where T and H^k are shorthands for Maxwellian energy momentum tensor and second fundamental form and the product of tensors is obvious generalization of matrix product involving index contraction.

2.1.2 Could Einstein's equations emerge dynamically?

For j^α satisfying one of the three conditions, the field equations have the same form as the equations for minimal surfaces except that the metric g is replaced with Maxwell energy momentum tensor T.

1. This raises the question about dynamical generation of small cosmological constant Λ: $T = \Lambda g$ would reduce equations to those for minimal surfaces. For $T = \Lambda g$ modified gamma matrices would reduce to induced gamma matrices and the modified Dirac operator would be proportional to ordinary Dirac operator defined by the induced gamma matrices. One can also consider weak form for $T = \Lambda g$ obtained by restricting the consideration to sub-space of tangent space so that space-time surface is only "partially" minimal surface but this option is not so elegant although necessary for other than CP_2 type vacuum extremals.

2. What is remarkable is that $T = \Lambda g$ implies that the divergence of T which in the general case equals to $j^\beta J^\alpha_\beta$ vanishes. This is guaranteed by one of the conditions for the Kähler current. Since also Einstein tensor has a vanishing divergence, one can ask whether the condition to $T = \kappa G + \Lambda g$ could the general condition. This would give Einstein's equations with cosmological term besides the generalization of the minimal surface equations. GRT would emerge dynamically from the non-linear Maxwell's theory although in slightly different sense as conjectured [K11]! Note that the expression for G involves also second derivatives of the imbedding space coordinates so that actually a partial differential equation is in question. If field equations reduce to purely algebraic ones, as the basic conjecture states, it is possible to have $Tr(GH^k) = 0$ and $Tr(gH^k) = 0$ separately so that also minimal surface equations would hold true.

   What is amusing that the first guess for the action of TGD was curvature scalar. It gave analogs of Einstein's equations as a definition of conserved four-momentum currents. The recent proposal would give the analog of ordinary Einstein equations as a dynamical constraint relating Maxwellian energy momentum tensor to Einstein tensor and metric.

3. Minimal surface property is physically extremely nice since field equations can be interpreted as a non-linear generalization of massless wave equation: something very natural for non-linear variant of Maxwell action. The theory would be also very "stringy" although the fundamental action would not be space-time volume. This can however hold true only for Euclidian signature. Note that for CP_2 type vacuum extremals Einstein tensor is proportional to metric so that for them the two options are equivalent. For their small deformations situation changes and it might happen that the presence of G is necessary. The GRT limit of TGD discussed in [K11] [L1] indeed suggests that CP_2 type solutions satisfy Einstein's equations with large cosmological constant and that the small observed value of the cosmological constant is due to averaging and small volume fraction of regions of Euclidian signature (lines of generalized Feynman diagrams).

4. For massless extremals and their deformations $T = \Lambda g$ cannot hold true. The reason is that for massless extremals energy momentum tensor has component T^{vv} which actually quite essential for field equations since one has $H^k_{vv} = 0$. Hence for massless extremals and their deformations $T = \Lambda g$ cannot hold true if the induced metric has Hamilton-Jacobi structure meaning that g^{uu} and g^{vv} vanish. A more general relationship of form $T = \kappa G + \Lambda G$ can however be consistent with non-vanishing T^{vv} but require that deformation has at most 3-D CP_2 projection (CP_2 coordinates do not depend on v).

5. The non-determinism of vacuum extremals suggest for their non-vacuum deformations a conflict with the conservation laws. In, also massless extremals are characterized by a non-determinism with respect to the light-like coordinate but like-likeness saves the situation. This suggests that the transformation of a properly chosen time coordinate of vacuum extremal to a light-like coordinate in the induced metric combined with Einstein's equations in the induced metric of the deformation could allow to handle the non-determinism.

2.1.3 Are complex structure of CP_2 and Hamilton-Jacobi structure of M^4 respected by the deformations?

The complex structure of CP_2 and Hamilton-Jacobi structure of M^4 could be central for the understanding of the preferred extremal property algebraically.

1. There are reasons to believe that the Hermitian structure of the induced metric ((1,1) structure in complex coordinates) for the deformations of CP_2 type vacuum extremals could be crucial property of the preferred extremals. Also the presence of light-like direction is also an essential elements and 3-dimensionality of M^4 projection could be essential. Hence a good guess is that allowed deformations of CP_2 type vacuum extremals are such that (2,0) and (0,2) components the induced metric and/or of the energy momentum tensor vanish. This gives rise to the conditions implying Virasoro conditions in string models in quantization:

$$g_{\xi^i\xi^j} = 0 \ , \quad g_{\overline{\xi}^i\overline{\xi}^j} = 0 \ , \quad i,j = 1,2 \ . \tag{2.1}$$

   Holomorphisms of CP_2 preserve the complex structure and Virasoro conditions are expected to generalize to 4-dimensional conditions involving two complex coordinates. This means that the generators have two integer valued indices but otherwise obey an algebra very similar to the Virasoro algebra. Also the super-conformal variant of this algebra is expected to make sense.

   These Virasoro conditions apply in the coordinate space for CP_2 type vacuum extremals. One expects similar conditions hold true also in field space, that is for M^4 coordinates.

2. The integrable decomposition $M^4(m) = M^2(m)+E^2(m)$ of M^4 tangent space to longitudinal and transversal parts (non-physical and physical polarizations) - Hamilton-Jacobi structure- could be a very general property of preferred extremals and very natural since non-linear Maxwellian electrodynamics is in question. This decomposition led rather early to the introduction of the analog of complex structure in terms of what I called Hamilton-Jacobi coordinates $(u, v, w, \overline{w})$ for M^4. (u, v) defines a pair of light-like coordinates for the local longitudinal space $M^2(m)$ and $(w, \overline{w})$ complex coordinates for $E^2(m)$. The Hamilton-Jacobi conditions on induced metric would be obtained by replacing imaginary unit in the definition of Hermitian metric for some complex coordinates with e, $e^2 = 1$ and defining hyper-complex conjugation as $u \to v$ for light-like-coordinate (Appendix).

   A good guess is that the deformations of massless extremals respect this structure. This condition gives rise to the analog of the constraints leading to Virasoro conditions stating the vanishing of the non-allowed components of the induced metric plus the analogs of hermiticity conditions. Again the generators of the algebra would involve two integers and the structure is that of Virasoro algebra and also generalization to super algebra is expected to make sense. The moduli space of Hamilton-Jacobi structures would be part of the moduli space of the preferred extremals and analogous to the space of all possible choices of complex coordinates. The analogs of infinitesimal holomorphic transformations would preserve the modular parameters and give rise to a 4-dimensional Minkowskian analog of Virasoro algebra. The conformal algebra acting on CP_2 coordinates acts in field degrees of freedom for Minkowskian signature.

2.1.4 Field equations as purely algebraic conditions

If the proposed picture is correct, field equations would reduce basically to purely algebraically conditions stating that the Maxwellian energy momentum tensor has no common index pairs with the second fundamental form. For the deformations of CP_2 type vacuum extremals T is a complex tensor of type (1,1) and second fundamental form H^k a tensor of type (2,0) and (0,2) so that $Tr(TH^k) =$ is true. This requires that second light-like coordinate of M^4 is constant so that the M^4 projection is 3-dimensional. For Minkowskian signature of the induced metric Hamilton-Jacobi structure replaces conformal structure. Here the dependence of CP_2 coordinates on second light-like coordinate of $M^2(m)$ only plays a fundamental role. Note that now T^{vv} is non-vanishing (and light-like). This picture generalizes to the deformations of cosmic strings and even to the case of vacuum extremals.

2.2 What small deformations of CP_2 type vacuum extremals could be?

I was led to these arguments when I tried find preferred extremals of Kähler action, which would have 4-D CP_2 and M^4 projections - the Maxwell phase analogous to the solutions of Maxwell's equations that I conjectured long time ago. It however turned out that the dimensions of the projections can be ($D_{M^4} \leq 3, D_{CP_2} = 4$) or ($D_{M^4} = 4, D_{CP_2} \leq 3$). What happens is essentially breakdown of linear superposition so that locally one can have superposition of modes which have 4-D wave vectors in the same direction. This is actually very much like quantization of radiation field to photons now represented as separate space-time sheets and one can say that Maxwellian superposition corresponds to union of separate photonic space-time sheets in TGD. In the following I shall restrict the consideeration to the deformations of CP_2 type vacuum extremals.

2.2.1 Solution ansatz

I proceed by the following arguments to the ansatz.

1. Effective 3-dimensionality for action (holography) requires that action decomposes to vanishing $j^\alpha A_\alpha$ term + total divergence giving 3-D "boundary" terms. The first term certainly vanishes (giving effective 3-dimensionality) for

$$D_\beta J^{\alpha\beta} = j^\alpha = 0 \ .$$

   Empty space Maxwell equations, something extremely natural. Also for the proposed GRT limit these equations are true.

2. How to obtain empty space Maxwell equations $j^\alpha = 0$? The answer is simple: assume self duality or its slight modification:

$$J = *J$$

   holding for CP_2 type vacuum extremals or a more general condition

$$J = k * J \ ,$$

   In the simplest situation k is some constant not far from unity. * is Hodge dual involving 4-D permutation symbol. $k = constant$ requires that the determinant of the induced metric is apart from constant equal to that of CP_2 metric. It does not require that the induced metric is proportional to the CP_2 metric, which is not possible since M^4 contribution to metric has Minkowskian signature and cannot be therefore proportional to CP_2 metric.

   One can consider also a more general situation in which k is scalar function as a generalization of the weak electric-magnetic duality. In this case the Kähler current is non-vanishing but divergenceless. This also guarantees the reduction to $Tr(TH^k) = 0$. In this case however the proportionality of the metric determinant to that for CP_2 metric is not needed. This solution ansatz becomes therefore more general.

3. Field equations reduce with these assumptions to equations differing from minimal surfaces equations only in that metric g is replaced by Maxwellian energy momentum tensor T. Schematically:

$$Tr(TH^k) = 0 \ ,$$

   where T is the Maxwellian energy momentum tensor and H^k is the second fundamental form - asymmetric 2-tensor defined by covariant derivative of gradients of imbedding space coordinates.

2.2.2 How to satisfy the condition $Tr(TH^k) = 0$?

It would be nice to have minimal surface equations since they are the non-linear generalization of massless wave equations. It would be also nice to have the vanishing of the terms involving Kähler current in field equations as a consequence of this condition. Indeed, $T = \kappa G + \Lambda g$ implies this. In the case of CP_2 vacuum extremals one cannot distinguish between these options since CP_2 itself is constant curvature space with $G \propto g$. Furthermore, if G and g have similar tensor structure the algebraic field equations for G and g are satisfied separately so that one obtains minimal surface property also now. In the following minimal surface option is considered.

1. The first opton is achieved if one has

   $$T = \Lambda g \ .$$

   Maxwell energy momentum tensor would be proportional to the metric! One would have dynamically generated cosmological constant! This begins to look really interesting since it appeared also at the proposed GRT limit of TGD [L1]. Note that here also non-constant value of Λ can be considered and would correspond to a situation in which k is scalar function: in this case the the determinant condition can be dropped and one obtains just the minimal surface equations.

2. Very schematically and forgetting indices and being sloppy with signs, the expression for T reads as

   $$T = JJ - g/4Tr(JJ) \ .$$

   Note that the product of tensors is obtained by generalizing matrix product. This should be proportional to metric.

   Self duality implies that $Tr(JJ)$ is just the instanton density and does not depend on metric and is constant.

   For CP_2 type vacuum extremals one obtains

   $$T = -g + g = 0 \ .$$

   Cosmological constant would vanish in this case.

3. Could it happen that for deformations a small value of cosmological constant is generated? The condition would reduce to

   $$JJ = (\Lambda - 1)g \ .$$

   Λ must relate to the value of parameter k appearing in the generalized self-duality condition. For the most general ansatz Λ would not be constant anymore.

   This would generalize the defining condition for Kähler form

   $$JJ = -g \ \ (i^2 = -1 \ \ \textit{geometrically})$$

   stating that the square of Kähler form is the negative of metric. The only modification would be that index raising is carried out by using the induced metric containing also M^4 contribution rather than CP_2 metric.

4. Explicitly:

   $$J_{\alpha\mu} J^{\mu}_{\ \beta} = (\Lambda - 1) g_{\alpha\beta} \ .$$

   Cosmological constant would measure the breaking of Kähler structure. By writing $g = s+m$ and defining index raising of tensors using CP_2 metric and their product accordingly, this condition can be also written as

$$Jm = (\Lambda - 1)mJ \ .$$

If the parameter k is constant, the determinant of the induced metric must be proportional to the CP_2 metric. If k is scalar function, this condition can be dropped. Cosmological constant would not be constant anymore but the dependence on k would drop out from the field equations and one would hope of obtaining minimal surface equations also now. It however seems that the dimension of M^4 projection cannot be four. For 4-D M^4 projection the contribution of the M^2 part of the M^4 metric gives a non-holomorphic contribution to CP_2 metric and this spoils the field equations.

For $T = \kappa G + \Lambda g$ option the value of the cosmological constant is large - just as it is for the proposed GRT limit of TGD [K11] [L1]. The interpretation in this case is that the average value of cosmological constant is small since the portion of space-time volume containing generalized Feynman diagrams is very small.

2.2.3 More detailed ansatz for the deformations of CP_2 type vacuum extremals

One can develop the ansatz to a more detailed form. The most obvious guess is that the induced metric is apart from constant conformal factor the metric of CP_2. This would guarantee self-duality apart from constant factor and $j^\alpha = 0$. Metric would be in complex CP_2 coordinates tensor of type (1,1) whereas CP_2 Riemann connection would have only purely holomorphic or anti-holomorphic indices. Therefore CP_2 contributions in $Tr(TH^k)$ would vanish identically. M^4 degrees of freedom however bring in difficulty. The M^4 contribution to the induced metric should be proportional to CP_2 metric and this is impossible due to the different signatures. The M^4 contribution to the induced metric breaks its Kähler property but would preserve Hermitian structure.

A more realistic guess based on the attempt to construct deformations of CP_2 type vacuum extremals is following.

1. Physical intuition suggests that M^4 coordinates can be chosen so that one has integrable decomposition to longitudinal degrees of freedom parametrized by two light-like coordinates u and v and to transversal polarization degrees of freedom parametrized by complex coordinate w and its conjugate. M^4 metric would reduce in these coordinates to a direct sum of longitudinal and transverse parts. I have called these coordinates Hamilton Jacobi coordinates.

2. w would be holomorphic function of CP_2 coordinates and therefore satisfy massless wave equation. This would give hopes about rather general solution ansatz. u and v cannot be holomorphic functions of CP_2 coordinates. Unless wither u or v is constant, the induced metric would receive contributions of type (2,0) and (0,2) coming from u and v which would break Kähler structure and complex structure. These contributions would give no-vanishing contribution to all minimal surface equations. Therefore either u or v is constant: the coordinate line for non-constant coordinate -say u- would be analogous to the M^4 projection of CP_2 type vacuum extremal.

3. With these assumptions the induced metric would remain $(1,1)$ tensor and one might hope that $Tr(TH^k)$ contractions vanishes for all variables except u because the there are no common index pairs (this if non-vanishing Christoffel symbols for H involve only holomorphic or anti-holomorphic indices in CP_2 coordinates). For u one would obtain massless wave equation expressing the minimal surface property.

4. If the value of k is constant the determinant of the induced metric must be proportional to the determinant of CP_2 metric. The induced metric would contain only the contribution from the transversal degrees of freedom besides CP_2 contribution. Minkowski contribution has however rank 2 as CP_2 tensor and cannot be proportional to CP_2 metric. It is however enough that its determinant is proportional to the determinant of CP_2 metric with constant proportionality coefficient. This condition gives an additional non-linear condition to the solution. One would have wave equation for u (also w and its conjugate satisfy massless wave equation) and determinant condition as an additional condition.

   The determinant condition reduces by the linearity of determinant with respect to its rows to sum of conditions involved 0,1,2 rows replaced by the transversal M^4 contribution to metric given if M^4 metric decomposes to direct sum of longitudinal and transversal parts. Derivatives with respect to derivative with respect to particular CP_2 complex coordinate appear linearly in this expression they can depend on u via the dependence of transversal metric components on u. The challenge is to show that this equation has (or does not have) non-trivial solutions.

5. If the value of k is scalar function the situation changes and one has only the minimal surface equations and Virasoro conditions.

What makes the ansatz attractive is that special solutions of Maxwell empty space equations are in question, equations reduces to non-linear generalizations of Euclidian massless wave equations, and possibly space-time dependent cosmological constant pops up dynamically. These properties are true also for the GRT limit of TGD [L1].

2.3 Hamilton-Jacobi conditions in Minkowskian signature

The maximally optimistic guess is that the basic properties of the deformations of CP_2 type vacuum extremals generalize to the deformations of other known extremals such as massless extremals, vacuum extremals with 2-D CP_2 projection which is Lagrangian manifold, and cosmic strings characterized by Minkowskian signature of the induced metric. These properties would be following.

1. The recomposition of M^4 tangent space to longitudinal and transversal parts giving Hamilton-Jacobi structure. The longitudinal part has hypercomplex structure but the second light-like coordinate is constant: this plays a crucial role in guaranteeing the vanishing of contractions in $Tr(TH^k)$. It is the algebraic properties of g and T which are crucial. T can however have light-like component T^{vv}. For the deformations of CP_2 type vacuum extremals $(1,1)$ structure is enough and is guaranteed if second light-like coordinate of M^4 is constant whereas w is holomorphic function of CP_2 coordinates.

2. What could happen in the case of massless extremals? Now one has 2-D CP_2 projection in the initial situation and CP_2 coordinates depend on light-like coordinate u and single real transversal coordinate. The generalization would be obvious: dependence on single light-like coordinate u and holomorphic dependence on w for complex CP_2 coordinates. The constraint is $T = \Lambda g$ cannot hold true since T^{vv} is non-vanishing (and light-like). This property restricted to transversal degrees of freedom could reduce the field equations to minimal surface equations in transversal degrees of freedom. The transversal part of energy momentum tensor would be proportional to metric and hence covariantly constant. Gauge current would remain light-like but would not be given by $j = *d\phi \wedge J$. $T = \kappa G + \Lambda g$ seems to define the attractive option.

It therefore seems that the essential ingredient could be the condition

$$T = \kappa G + \lambda g \ ,$$

which has structure (1,1) in both $M^2(m)$ and $E^2(m)$ degrees of freedom apart from the presence of T^{vv} component with deformations having no dependence on v. If the second fundamental form has (2,0)+(0,2) structure, the minimal surface equations are satisfied provided Kähler current satisfies on of the proposed three conditions and if G and g have similar tensor structure.

One can actually pose the conditions of metric as complete analogs of stringy constraints leading to Virasoro conditions in quantization to give

$$g_{uu} = 0 \ , \quad g_{vv} = 0 \ , \quad g_{ww} = 0 \ , \quad g_{\overline{w}\overline{w}} = 0 \ . \tag{2.2}$$

This brings in mind the generalization of Virasoro algebra to four-dimensional algebra for which an identification in terms of non-local Yangian symmetry has been proposed [K12]. The number of conditions is four and the same as the number of independent field equations. One can consider similar conditions also for the energy momentum tensor T but allowing non-vanishing component T^{vv} if deformations has no v-dependence. This would solve the field equations if the gauge current vanishes or is light-like. On this case the number of equations is 8. First order differential equations are in question and they can be also interpreted as conditions fixing the coordinates used since there is infinite number of manners to choose the Hamilton-Jacobi coordinates.

One can can try to apply the physical intuition about general solutions of field equations in the linear case by writing the solution as a superposition of left and right propagating solutions:

$$\xi^k = f_+^k(u,w) + f_+^k(v,w) \ . \tag{2.3}$$

This could guarantee that second fundamental form is of form (2,0)+(0,2) in both M^2 and E^2 part of the tangent space and these terms if $Tr(TH^k)$ vanish identically. The remaining terms involve contractions of T^{uw}, $T^{u\overline{w}}$ and T^{vw}, $T^{v\overline{w}}$ with second fundamental form. Also these terms should sum up to zero or vanish separately. Second fundamental form has components coming from f_+^k and f_-^k

Second fundamental form H^k has as basic building bricks terms $\hat{H}^k$ given by

$$\hat{H}^k_{\alpha\beta} \ = \ \partial_\alpha\partial_\beta h^k + \left({k \atop l\ m}\right) \partial_\alpha h^l \partial_\beta h^m \ . \tag{2.4}$$

For the proposed ansatz the first terms give vanishing contribution to H^k_{uv}. The terms containing Christoffel symbols however give a non-vanishing contribution and one can allow only f_+^k or f_-^k as in the case of massless extremals. This reduces the dimension of CP_2 projection to $D = 3$.

What about the condition for Kähler current? Kähler form has components of type $J_{w\overline{w}}$ whose contravariant counterpart gives rise to space-like current component. J_{uw} and $J_{u\overline{w}}$ give rise to light-like currents components. The condition would state that the $J^{w\overline{w}}$ is covariantly constant. Solutions would be characterized by a constant Kähler magnetic field. Also electric field is represent. The interpretation both radiation and magnetic flux tube makes sense.

2.4 Deformations of cosmic strings

In the physical applications it has been assumed that the thickening of cosmic strings to Kähler magnetic flux tubes takes place. One indeed expects that the proposed construction generalizes also to the case of cosmic strings having the decomposition $X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$, where X^2 is minimal surface and Y^2 a complex homologically non-trivial sub-manifold of CP_2. Now the starting point structure is Hamilton-Jacobi structure for $M^2_m \times Y^2$ defining the coordinate space.

1. The deformation should increase the dimension of either CP_2 or M^4 projection or both. How this thickening could take place? What comes in mind that the string orbits X^2 can be interpreted as a distribution of longitudinal spaces $M^2(x)$ so that for the deformation w coordinate becomes a holomorphic function of the natural Y^2 complex coordinate so that M^4 projection becomes 4-D but CP_2 projection remains 2-D. The new contribution to the X^2 part of the induced metric is vanishing and the contribution to the Y^2 part is of type $(1,1)$ and the the ansatz $T = \kappa G + \Lambda g$ might be needed as a generalization of the minimal surface equations The ratio of κ and G would be determined from the form of the Maxwellian energy momentum tensor and be fixed at the limit of undeformed cosmic strong to $T = (ag(Y^2) - bg(Y^2)$. The value of cosmological constant is now large, and overall consistency suggests that $T = \kappa G + \Lambda g$ is the correct option also for the CP_2 type vacuum extremals.

2. One could also imagine that remaining CP_2 coordinates could depend on the complex coordinate of Y^2 so that also CP_2 projection would become 4-dimensional. The induced metric would receive holomorphic contributions in Y^2 part. As a matter fact, this option is already implied by the assumption that Y^2 is a complex surface of CP_2.

2.5 Deformations of vacuum extremals?

What about the deformations of vacuum extremals representable as maps from M^4 to CP_2?

1. The basic challenge is the non-determinism of the vacuum extremals. One should perform the deformation so that conservation laws are satisfied. For massless extremals there is also non-determinism but it is associated with the light-like coordinate so that there are no problems with the conservation laws. This would suggest that a properly chosen time coordinate consistent with Hamilton-Jacobi decomposition becomes light-like coordinate in the induced metric. This poses a conditions on the induced metric.

2. Physical intuition suggests that one cannot require $T = \Lambda g$ since this would mean that the rank of T is maximal whereas the original situation corresponds to the vanishing of T. For small deformations rank two for T looks more natural and one could think that T is proportional to a projection of metric to a 2-D subspace. The vision about the long length scale limit of TGD is that Einstein's equations are satisfied and this would suggest $T = kG$ or $T = \kappa G + \Lambda g$. The rank of T could be smaller than four for this ansatz and this conditions binds together the values of κ and G.

3. These extremals have CP_2 projection which in the generic case is 2-D Lagrangian sub-manifold Y^2. Again one could assume Hamilton-Jacobi coordinates for X^4. For CP_2 one could assume Darboux coordinates (P_i, Q_i), $i = 1, 2$, in which one has $A = P_i dQ^i$, and that $Y^2 \subset CP_2$ corresponds to $Q_i = constant$. In principle P_i would depend on arbitrary manner on M^4 coordinates. It might be more convenient to use as coordinates (u, v) for M^2 and (P_1, P_2) for Y^2. This covers also the situation when M^4 projection is not 4-D. By its 2-dimensionality Y^2 allows always a complex structure defined by its induced metric: this complex structure is not consistent with the complex structure of CP_2 (Y^2 is not complex sub-manifold).

   Using Hamilton-Jacobi coordinates the pre-image of a given point of Y^2 is a 2-dimensional sub-manifold X^2 of X^4 and defines also 2-D sub-manifold of M^4. The following picture suggests itself. The projection of X^2 to M^4 can be seen for a suitable choice of Hamilton-Jacobi coordinates as an analog of Lagrangian sub-manifold in M^4 that is as surface for which v and $Im(w)$ vary and u and $Re(w)$ are constant. X^2 would be obtained by allowing u and $Re(w)$ to vary: as a matter fact, (P_1, P_2) and $(u, Re(w))$ would be related to each other. The induced metric should be consistent with this picture. This would requires $g_{uRe(w)} = 0$.

   For the deformations Q_1 and Q_2 would become non-constant and they should depend on the second light-like coordinate v only so that only g_{uu} and g_{uw} and $gu\overline{w}$ $g_{w,w}$ and $g_{\overline{w},\overline{w}}$ receive contributions which vanish. This would give rise to the analogs of Virasoro conditions guaranteeing that T is a tensor of form $(1, 1)$ in both M^2 and E^2 indices and that there are no cross components in the induced metric. A more general formulation states that energy momentum tensor satisfies these conditions. The conditions on T might be equivalent with the conditions for g and G separately.

4. Einstein's equations provide an attractive manner to achieve the vanishing of effective 3-dimensionality of the action. Einstein equations would be second order differential equations and the idea that a deformation of vacuum extremal is in question suggests that the dynamics associated with them is in directions transversal to Y^2 so that only the deformation is dictated partially by Einstein's equations.

5. Lagrangian manifolds do not involve complex structure in any obvious manner. One could however ask whether the deformations could involve complex structure in a natural manner in CP_2 degrees of freedom so that the vanishing of g_{ww} would be guaranteed by holomorphy of CP_2 complex coordinate as function of w.

   One should get the complex structure in some natural manner: in other words, the complex structure should relate to the geometry of CP_2 somehow. The complex coordinate defined by say $z = P_1 + iQ^1$ for the deformation suggests itself. This would suggest that at the limit when one puts $Q_1 = 0$ one obtains $P_1 = P_1(Re(w))$ for the vacuum extremals and the deformation could be seen as an analytic continuation of real function to region of complex plane. This is in spirit with the algebraic approach. The vanishing of Kähler current requires that the Kähler magnetic field is covariantly constant: $D_z J^{z\overline{z}} = 0$ and $D_{\overline{z}} J^{z\overline{z}} = 0$.

6. One could consider the possibility that the resulting 3-D sub-manifold of CP_2 can be regarded as contact manifold with induced Kähler form non-vanishing in 2-D section with natural complex coordinates. The third coordinate variable- call it s- of the contact manifold and second coordinate of its transversal section would depend on time space-time coordinates for vacuum extremals. The coordinate associated with the transversal section would be continued to a complex coordinate which is holomorphic function of w and u.

7. The resulting thickened magnetic flux tubes could be seen as another representation of Kähler magnetic flux tubes: at this time as deformations of vacuum flux tubes rather than cosmic

strings. For this ansatz it is however difficult to imagine deformations carrying Kähler electric field.

2.6 About the interpretation of the generalized conformal algebras

The long-standing challenge has been finding of the direct connection between the super-conformal symmetries assumed in the construction of the geometry of the "world of classical worlds" (WCW) and possible conformal symmetries of field equations. 4-dimensionality and Minkowskian signature have been the basic problems. The recent construction provides new insights to this problem.

1. In the case of string models the quantization of the Fourier coefficients of coordinate variables of the target space gives rise to Kac-Moody type algebra and Virasoro algebra generators are quadratic in these. Also now Kac-Moody type algebra is expected. If one were to perform a quantization of the coefficients in Laurents series for complex CP_2 coordinates, one would obtain interpretation in terms of $su(3) = u(2) + t$ decomposition, where t corresponds to CP_3: the oscillator operators would correspond to generators in t and their commutator would give generators in $u(2)$. SU(3)/SU(2) coset representation for Kac-Moody algebra would be in question. Kac-Moody algebra would be associated with the generators in both M^4 and CP_2 degrees of freedom. This kind of Kac-Moody algebra appears in quantum TGD.

2. The constraints on induced metric imply a very close resemblance with string models and a generalization of Virasoro algebra emerges. An interesting question is how the two algebras acting on coordinate and field degrees of freedom relate to the super-conformal algebras defined by the symplectic group of $\delta M^4_+ \times CP_2$ acting on space-like 3-surfaces at boundaries of CD and to the Kac-Moody algebras acting on light-like 3-surfaces. It has been conjectured that these algebras allow a continuation to the interior of space-time surface made possible by its slicing by 2-surfaces parametrized by 2-surfaces. The proposed construction indeed provides this kind of slicings in both M^4 and CP_2 factor.

3. In the recent case, the algebras defined by the Fourier coefficients of field variables would be Kac-Moody algebras. Virasoro algebra acting on preferred coordinates would be expressed in terms of the Kac-Moody algebra in the standard Sugawara construction applied in string models. The algebra acting on field space would be analogous to the conformal algebra assignable to the symplectic algebra so that also symplectic algebra is present. Stringy pragmatist could imagine quantization of symplectic algebra by replacing CP_2 coordinates in the expressions of Hamiltonians with oscillator operators. This description would be counterpart for the construction of spinor harmonics in WCW and might provide some useful insights.

4. For given type of space-time surface either CP_2 or M^4 corresponds to Kac-Moody algebra but not both. From the point of view of quantum TGD it looks as that something were missing. An analogous problem was encountered at GRT limit of TGD [L1]. When Euclidian space-time regions are allowed Einstein-Maxwell action is able to mimic standard model with a surprising accuracy but there is a problem: one obtains either color charges or M^4 charges but not both. Perhaps it is not enough to consider either CP_2 type vacuum extremal or its exterior but both to describe particle: this would give the direct product of the Minkowskian and Euclidian algebras acting on tensor product. This does not however seem to be consistent with the idea that the two descriptions are duality related (the analog of T-duality).

3 Under what conditions electric charge is conserved for the modified Dirac equation?

One might think that talking about the conservation of electric charge at 21st century is a waste of time. In TGD framework this is certainly not the case and the following arguments suggests that the conservation of electric charge is the Golden Road to the understanding of the spinorial dynamics.

1. In quantum field theories there are two manners to define em charge: as electric flux over 2-D surface sufficiently far from the source region or in the case of spinor field quantum mechani-

cally as combination of fermion number and vectorial isospin. The latter definition is quantum mechanically more appropriate.

2. There is however a problem. In standard approach to gauge theory Dirac equation in the presence of charged classical gauge fields does not conserve the electric charge: electron is transformed to neutrino and vice versa. Quantization solves the problem since the non-conservation can be interpreted in terms of emission of gauge bosons. In TGD framework this does not work since one does not have path integral quantization anymore. Preferred extremals carry classical gauge fields and the question whether em charge is conserved arises. Heuristic picture suggests that em charge must be conserved. This condition might be actually one of the conditions defining what it is to be a preferred extremal. It is not however trivial whether this kind of additional condition can be posed.

3.1 Conditions guaranteing the conservation of em charge

What does the conservation of em charge imply in the case of the modified Dirac equation? The obvious guess that the em charged part of the modified Dirac operator must annihilate the solutions, turns out to be correct as the following argument demonstrates.

1. Em charge as coupling matrix can be defined as a linear combination $Q = aI + bI_3$, $I_3 = J_{kl}\Sigma^{kl}$, where I is unit matrix and I_3 vectorial isospin matrix, J_{kl} is the Kähler form of CP_2, Σ^{kl} denotes sigma matrices, and a and b are numerical constants different for quarks and leptons. Q is covariantly constant in $M^4 \times CP_2$ and its covariant derivatives at space-time surface are also well-defined and vanish.

2. The modes of the modified Dirac equation should be eigen modes of Q. This is the case if the modified Dirac operator D commutes with Q. The covariant constancy of Q can be used to derive the condition

$$\begin{aligned} [D,Q]\,\Psi &= D_1\Psi = 0 \ , \\ D &= \hat{\Gamma}^\mu D_\mu \ , \ \ D_1 = [D,Q] = \hat{\Gamma}^\mu_1 D_\mu \ , \ \ \hat{\Gamma}^\mu_1 = \left[\hat{\Gamma}^\mu, Q\right] \ . \end{aligned} \tag{3.1}$$

Covariant constancy of J is absolutely essential: without it the resulting conditions would not be so simple.

It is easy to find that also $[D_1,Q]\Psi = 0$ and its higher iterates $[D_n,Q]\Psi = 0$, $D_n = [D_{n-1},Q]$ must be true. The solutions of the modified Dirac equation would have an additional symmetry.

3. The commutator $D_1 = [D,Q]$ reduces to a sum of terms involving the commutators of the vectorial isospin $I_3 = J_{kl}\Sigma^{kl}$ with the CP_2 part of the gamma matrices:

$$D_1 = [Q,D] = [I_3,\Gamma_r]\partial_\mu s^r T^{\alpha\mu} D_\alpha \ . \tag{3.2}$$

In standard complex coordinates in which $U(2)$ acts linearly the complexified gamma matrices can be chosen to be eigenstates of vectorial isospin. Only the charged flat space complexified gamma matrices Γ^A denoted by Γ^+ and Γ^- possessing charges +1 and -1 contribute to the right hand side. Therefore the additional Dirac equation $D_1\Psi = 0$ states

$$\begin{aligned} D_1\Psi &= [Q,D]\Psi = I_3(A) e_{Ar}\Gamma^A \partial_\mu s^r T^{\alpha\mu} D_\alpha \Psi \\ &= (e_{+r}\Gamma^+ - e_{-r}\Gamma^-)\partial_\mu s^r T^{\alpha\mu} D_\alpha\Psi = 0 \ . \end{aligned} \tag{3.3}$$

The next condition is

$$D_2\Psi = [Q,D]\Psi = (e_{+r}\Gamma^+ + e_{-r}\Gamma^-)\partial_\mu s^r T^{\alpha\mu} D_\alpha\Psi = 0 \ . \tag{3.4}$$

Only the relative sign of the two terms has changed. The remaining conditions give nothing new.

4. These equations imply two separate equations for the two charged gamma matrices

$$\begin{aligned} D_+\Psi &= T_+^{\alpha}\Gamma^+ D_\alpha\Psi = 0 \ , \\ D_-\Psi &= T_-^{\alpha}\Gamma^- D_\alpha\Psi = 0 \ , \\ T_\pm^{\alpha} &= e_{\pm r}\partial_\mu s^r T^{\alpha\mu} \ . \end{aligned} \tag{3.5}$$

These conditions state what one might have expected: the charged part of the modified Dirac operator annihilates separately the solutions. The reason is that the classical W fields are proportional to $e_{r\pm}$.

The above equations can be generalized to define a decomposition of the energy momentum tensor to charged and neutral components in terms of vierbein projections. The equations state that the analogs of the modified Dirac equation defined by charged components of the energy momentum tensor are satisfied separately.

5. In complex coordinates one expects that the two equations are complex conjugates of each other for Euclidian signature. For the Minkowskian signature an analogous condition should hold true. The dynamics enters the game in an essential manner: whether the equations can be satisfied depends on the coefficients a and b in the expression $T = aG + bg$ implied by Einstein's equations in turn guaranteeing that the solution ansatz generalizing minimal surface solutions holds true [K1].

6. As a result one obtains three separate Dirac equations corresponding to the the neutral part $D_0\Psi = 0$ and charged parts $D_\pm\Psi = 0$ of the modified Dirac equation. By acting on the equations with these Dirac operators one obtains also that the commutators $[D_+, D_-]$, $[D_0, D_\pm]$ and also higher commutators obtained from these annihilate the induced spinor field mode. Therefore possibly infinite-dimensional algebra would annihilate the induced spinor fields unless the charged parts of the energy momentum tensor vanish identically.

3.2 Dirac equation in CP_2 as a test bench

What could the conservation of electric charge mean from the point of view of the solutions of the modified Dirac equation? The field equations for the preferred extremals of Kähler action reduce to purely algebraic conditions in the same manner as the field equations for the minimal surfaces in string model. Could something similar happen also for the modified Dirac equation and could the condition on charged part of the Dirac operator help to achieve this?

1. For CP_2 type vacuum extremals the modified Dirac operator vanishes identically for the Kähler action. For volume action it reduces to the ordinary Dirac operator in CP_2 and one can ask whether ordinary Dirac operator could in this case allow solutions with a well-defined em charge. Since also spinor harmonics of the imbedding space are expected to be important and associated with the representations of conformal symmetries assignable to the boundary of light cone involving symplectic group of $\delta M^4_\pm \times CP_2$, it would be nice if this construction would work for CP_2.

2. One can construct the solutions of the ordinary Dirac equation from covariantly constant right-handed neutrino spinor playing the role of fermionic vacuum annihilated by the second half of complexified gamma matrices. Dirac equation reduces to Laplace equation for a scalar function and solution can be constructed from this "vacuum" by multiplying with the spherical harmonics of CP_2 and applying Dirac operator [K4]. Similar construction works quite generally thanks to the existence of covariantly constant right handed neutrino spinor. Spherical harmonics of CP_2 are only replaced with those of space-time surface possessing either hermitian structure of Hamilton-Jacobi structure (corresponding to Euclidian and Minkowskian signatures of the induced metric (Appendix)).

3. A good guess is that holomorphy codes the statement that only spinor harmonics of form $(n,0)$ and $(0,n)$ should be allowed. The first problem is that these modes are not actually holomorphic since color triplet partial waves are proportional to $1/\sqrt{1+|\xi^1|^2+|\xi^2|^2}$: The are good hopes that covariant derivative containing also a term proportional to Kähler gauge potential and coupling to leptons and quarks differently takes care of this.

   Dirac equation in CP_2 allows only modes with $(m,m+3)$, $(m+3,m)$ for leptons and anti-leptons and modes $(m+1,m)$ and $(m,m+1)$ for quarks. More general solutions could be possible but would not be global solutions. If this picture is correct, the dynamics in fermionic degrees of freedom would be extremely restricted and only only very few color partial waves would survive.

4. Most holomorphic and anti-holomorphic modes for leptons and quarks would represent gauge degrees of freedom. The remaining three modes for quarks could be interpreted in terms of color of ground state. At first this looks good since only color neutral leptons and color triplet quarks would be allowed. This result just what has been observed and the experimental absence has indeed a challenge for quantum TGD. The experimental absence of higher modes would be due to Kac-Moody gauge invariance.

   Lepto-pion hypothesis in its original form and postulating color octet leptons would be however wrong. This might not be a catastrophe: a variant of this hypothesis identifies lepto-pion as quark antiquark pair associated with scaled down variant of hadron physics [K10].

5. A further work described in the sequel however shows that the correct identification of partial waves of imbedding space spinors is in terms of cm degrees of freedom of the partonic 2-surface and can be assigned to the super-symplectic conformal invariance dictating the ground states of Super-Kac-Moody representations. There is therefore no need to modify the earlier well-tested picture.

3.3 How to satisfy the conditions guaranteeing the conservation of em charge?

There are two manners to satisfy the conditions guaranteeing the conservation of em charge leading to three separate Dirac equations. The first option is inspired by string model and solutions are annihilated either by second charged gamma matrix or holomorphic covariant derivative or by the conjugates of these. For the second option the charged modified gamma matrices vanish identically.

3.3.1 Holomorphy of the solutions

In string model holomorphy/antiholomorphy for the modes of the induced spinor field is essential from the point of view of Super-Virasoro conditions. For preferred extremals the holomorphy seems to be in a key role and it would not be surprising if this were the case also in the fermionic sector. Could the additional Dirac equations associated with charged parts of the modified Dirac operator be solved by a generalization of holomorphy or anti-holomorphy? For the second charged Dirac operator - say D_+ - complexified gamma matrices would annihilate spinor mode and for the second one - say D_- - holomorphic covariant derivative would annihilate the spinor. Note that the gamma matrices Γ_+ and Γ_- are hermitian conjugates of each other.

The condition that either charged gamma matrix annihilates the spinor mode for the space-time sheet in question requires super-symmetry. For CP_2 one has this kind of supersymmetry but covariant constancy allows only right handed neutrino spinor: in this case however both holomorphic gamma matrices annihilate the spinor. In super-string models in flat target space, one has maximal supersymmetry allowing maximal number of covariantly constant spinor modes. For modified gamma matrices this kind of situation might be realized. If one allows the restriction of the induced spinor fields to string world sheets or partonic 2-surfaces, the situation simplifies further, and it might be possible to assume holomorphy in the complex coordinate parametrizing the 2-surface.

Either Γ_+ or Γ_- should annihilate the spinor mode. For right-handed neutrino second half of complexified gamma matrices annihilate it. This condition is analogous to the condition that fermionic annihilation operators annihilate the state. In the recent case the condition is weaker since only single complexified gamma matrix annihilates the state. For CP_2 Dirac operator one obtains four basic solutions corresponding to ν_R, $\Gamma_+\nu_R$, $\Gamma_0\nu_R$, $\Gamma_0\Gamma_+\nu_R$. Γ_0 is the second holomorphic complexitifed gamma matrix. Therefore it seems that one might be able to obtain at least two charged states for both quarks and lepton as required by the standard model plus possible higher color partial waves.

A stronger condition is that both Γ_+ and Γ_- vanish and Dirac operator reduces to neutral Dirac operator acting on complex coordinate and its conjugate. If the vanishing of Γ_+ or Γ_- takes place everywhere the energy momentum tensor must be effectively 2-dimensional. This option has been proposed earlier as a solution ansatz. If the vanishing occurs only at 2-D surface, effective 2-dimensionality holds true only at this surface. This option looks more plausible one.

3.3.2 Option for which charged parts of energy momentum tensor vanish

The vanishing of the charged parts of the modified Dirac operator D - or equivalently, those of the energy momentum tensor - would reduce D to its neutral part and the conditions would trivialize. There would be no need for the full holomorphy, which could be an un-physical condition for CP_2 and not favored if one assumes that all color partial waves in CP_2 correspond to physical states in the construction of representations of the symplectic conformal algebra. On the other hand, the reduction of allowed modes to just the observed one (singlet for leptons and triplet for quarks) is an attractive property. In string models one would speak about super-symmetry breaking. Note that holomorphy in the remaining neutral complex or hyper-complex coordinate could provide elegant solution of the modified Dirac equation exactly in the same manner as it does in string models.

It is however not at all clear whether the charged part of the energy momentum tensor can vanish everywhere. Note however that since imvbedding space projections of the energy momentum tensor are in question, the conditions do not mean reduction of the rank of energy momentum tensor to at most two. The possibility that the vanishing occurs only at 2-D surfaces analogous to string world sheets and that induced spinor fields must be restricted to these, is consistent with the existent vision

One can study the option allowing non-vanishing charged Dirac operators and requiring holomorphy, covariant constancy, and extended supersymmetry in more detail. To get some perspective one can the situation in the case of CP_2 type vacuum extremals. In this case the energy momentum tensor and therefore also modified gamma matrices vanish identically. Therefore the situation trivializes and one might hope that for the deformations of CP_2 type vacuum extremals the charged parts of the modified Dirac operator vanish or that holomorphy makes sense.

3.4 Could the solutions of the modified Dirac equation be restricted to 2-D surfaces?

The condition that the charged Dirac operators (and also neutral one) annihilate physical states is rather strong. If the charged parts $T^\alpha_\pm$ of the energy momentum tensor vanish, these conditions apply only to the space-time surface. These conditions are however quite strong and the question is whether one could require them for 2-D sub-manifolds of space-time surface- the analogs of string world sheets- only, and assume that the modes of Dirac equation are restricted on these.

There are several other manners to end up with this view.

1. The vision inspired by the finite measurement resolution is that the solutions of the modified Dirac equation are singular and restricted to 2-dimensional surfaces identifiable as string world sheets in Minkowskian signature and as partonic 2-surfaces in Euclidian signature. For 3-D light-like surfaces solutions would be restricted at word lines defining strands of braid defining discretization as space-time correlate for the finite measurement resolution. The interesting self-referential aspect would be that physical system itself would define the finite measurement resolution.

2. Another interpretation is in terms of the number theoretical vision [K9]. Space-time surfaces define associative or co-associative 4-surfaces with tangent space allowing quaternionic of co-quaternionic structure or its Minkowskian variant. This is a formulation for the idea that classical dynamics is determined by associativity condition. One can however go further and require also commutativity or co-commutativity and this leads to string world sheets or partonic 2-surfaces. The vanishing of the charged components of the energy momentum tensor could be indeed seen as a condition stating that the surface is complex or co-complex sub-manifold (or hyper-complex or co-hyper-complex one).

3. Also strong form of holography leads to the idea that 2-D partonic surfaces or string world sheets and the 4-D tangent space data at them should be enough for the formulation of quantum theory and 4-D space-time surfaces are necessary only for the realization of quantum classical correspondence. One could say that space-time surface is analogous to phase space and that in quantum theory only 2-D slice of it analogous to Lagrangian sub-manifold can be used.

The general vision about preferred extremals involves a non-trivial aspect not yet mentioned [K1] and this allows to developed an argument in favor of reductionof spinorial dynamics to that at 2-D surfaces.

1. Various conserved currents are suggested to define integrable flows meaning that one can identify a global coordinate varying along the flow lines. Could the charged parts of the energy momentum tensor defined as currents define Beltrami flow? If so, these currents have expression of form $J_\pm = \Phi_\pm \nabla \Psi_\pm$, where $\Phi_\pm$ and $\Psi_\pm$ are complex scalar functions such that the latter ones define the global coordinate. If this is the case, then the surface at which $J_\pm$ vanishes corresponds to the surface $\Phi_+ = \overline{\Phi}_- = 0$ and by complex valuedness of $\Phi_\pm = 0$ is 2-dimensional rather than 0-dimensional as for a generic vector field. The charged parts of energy momentum tensor vanish identically as do the corresponding modified gamma matrices.

2. Vanishing of $\Phi_\pm$ would reduce the 4-D conformal algebra to 2-dimensional conformal algebra associated with the string world sheet or partonic 2-surface, and this is just what is expected on basis of physical intuition. One could say that locally space-time surface reduces to effectively 2-D surface. Charge conservation would select 2-dimensional string world sheets and/or partonic 2-surfaces and reproduce the earlier picture inspired by the notion of finite measurement resolution, by number theoretical considerations, and by strong form of holography.

A couple of further comments about are in order.

1. A natural consistency condition is that the modified gamma matrices in the modified Dirac operator are parallel to the 2-D surface. Otherwise one obtains covariant derivatives in transversal direction giving delta functions. This requires that modified gamma matrices generate a 2-D subspace tangential to X^2 at X^2. This condition need not hold true elsewhere. This would mean that with respect to the effective metric defined by the anticommutators of the modified gamma matrices space-time surface becomes effectively 2-dimensional locally. Effective 2-dimensionality of the effective metric was conjectured already earlier but now it is restricted to string world sheets and partonic 2-surfaces thus appearing as singularities of the preferred extremals. String world sheets and partonic 2-surfaces must obey some dynamics and minimal surface equation in the effective metric is a good guess since it automatically would reduce the situation to 2-D one.

2. Weak form of electric magnetic duality states that at partonic 2-surface X^2 the Kähler electric field strength $J^{n\beta}$ in 2-dimensional tangent plane of X^4 transversal to X^2 is proportional to the 4-D dual of the Kähler magnetic field strength $J_{\alpha\beta}$ at X^2 : $J^{n\beta} = k\epsilon^{n\beta\gamma\delta}J_{\gamma\delta}$, $k = constant$. The transversal plane is not unique without some additional condition and the natural condition is that it defines tangent plane to the string world sheet.

3.5 The algebra spanned by the modified Dirac operators

The conservation of em charge for the modified Dirac equation implies that the electromagnetic charge determined defined as $Q = aI + bI_3$ is conserved in the classical electro-weak gauge fields identified as induced gauge fields. This condition is highly non-trivial and as has been found could hold true only at 2-D surfaces implying a stringy localization of fermions.

For the modified Dirac equation additional consistency conditions analogous to Super-Virasoro conditions follow from the conditions that electromagnetic charge is constant for the modes of the modified Dirac equation. The conditions state that the possibly infinite-dimensional algebra or super-algebra generated by the neutral part and two charged parts with charges ± 1 of the modified Dirac operator annihilates the preferred solutions of the modified Dirac equation.

For super-algebra option one would start with anti-commutators of the Dirac operators and consider commutators when either generators has even value of em charge. For algebra option one would consider only commutators which are formally Lie commutators of gamma matrix valued vector fields $\Gamma_i^\alpha D_\alpha$ which ordinary derivatives replaced with covariant derivatives and components of the vector fields replaced with the modified gamma matrices obtained by contracting the neutral or charged part of the energy momentum tensor with flat space gamma matrices.

The commutator is the more feasible option as following arguments show.

1. Ordinary modified Dirac equation gives rise to a conserved fermionic current and its conservation could be seen as a consequence of the modified Dirac equation. The statement that the modified Dirac operators annihilate the induced spinors could thus be equivalent with the statement that SU(2) triplet of fermionic currents is conserved. In old fashioned hadron physics this corresponds to conserved vector current hypothesis.

2. The algebra defined by the commutators has the structure of vectorial SU(2) algebra and the natural guess is that this algebra relates closely to $\mathcal{N}=2$ super-conformal algebra for which super-generators G form SU(2) triplets and which allows conserved $U(1)$ currents besides energy momentum tensor.

   In the recent case the U(1) charge would be em charge. As a matter fact, $\mathcal{N}=2$ algebra is accompanied also by SU(2) algebra of conserved currents and the attractive interpretation is that the Dirac operators generate this algebra.

3. The algebra of Dirac operators annihilating the induced spinor field would define algebra of divergences of fermionic currents of form $J_i^\alpha = \overline{\Psi}\Gamma_i^\alpha\Psi$. These currents are conserved if the modified Dirac equations are satisfied. The algebra generated by the commutators of these fermionic currents assuming anti-commutation relations for the induced spinor fields should be equivalent with the algebra of Dirac operators. This should fix the anti-commutation relations for the induced spinor fields.

4. The outcome is a bosonic algebra of vector currents. By replacing $\overline{\Psi}$ or Ψ with a mode of the induced spinor field one would obtain super-algebra generators of extended super-algebra. The divergences of fermionic and bosonic generators would generate algebra which vanishes identically for the solutions of the modified Dirac equation. The currents themselves would be non-vanishing.

5. If the charged currents vanish at 2-surface then the commutator of charged Dirac operator vanishes identically. The commutators of neutral and charged Dirac operators need not vanish identically and it might be necessary to pose this as an additional conditions. The vanishing conditions reduce to the vanishing of the ordinary commutator $[T_0, T_\pm]$ of vector fields T_0 and $T_\pm$.

What about the super algebra part of the Super-Virasoro algebra. Is it also present?

1. One must notice that it is the "gamma matrix fields" defined by neutral and charged parts of the modified gamma matrices Γ_i^A forming an SU(2) triplet and these anti-commute classically to parts of the modified metric for which certain parts should vanish. These "certain parts" should vanish also for the induced metric resulting as anti-commutators of the induced gamma matrices. The conditions are not expected to be independent and should correspond to Virasoro conditions for the induced metric.

2. The SU(2) Super Virasoro algebra discussed above would naturally relate to $\mathcal{N}=2$ variant of the ordinary Super-Virasoro algebra and electromagnetic charge would take the role of conserved $U(1)$ current accompanying $\mathcal{N}=2$ algebra. This algebra indeed involves SU(2) as an additional symmetry algebra. Modified Dirac equations would correspond to conservation of the SU(2) currents and the vanishing conditions on the induced metric and/or its analog defined by the anti-commutators of the modified gamma matrices would correspond to Virasoro conditions. It is however not clear whether the modified gamma matrices - or rather, their second quantized variants - should annihilate the physical states. This condition would correspond for the induced spinor fields a condition stating that second half of complexified modified gamma matrices annihilates the right handed neutrino spinor serving as the analog of fermionic Fock vacuum.

3.6 Connection with the number theoretical vision about field equations

The recent progress in the understanding of preferred extremals of Kähler action suggests also an interesting connection to the number theoretic vision about field equations [K9]. In particular, it might be possible to understand how one can have Hermitian/Harmilton-Jacobi structure simultaneously with quaternionic structure and how quaternionic structure is possible for the Minkowskian signature of the induced metric.

One can imagine two manners of introducing octonionic and quaternionic structures. The first one is based on the introduction of octonionic representation of gamma matrices and second on the notion of octonion real-analycity.

1. If quaternionic structure is defined in terms of the octonionic representation of the imbedding space gamma matrices, there seems to be no obvious problems since one considers automatically complexification of quaternions represented in terms of gamma matrices. For the approach based on the notion of quaternion real analyticity, one is forced to use Wick rotation to define the quaternionic structure in Minkowskian regions or to introduce what I have called hyper-quaternionic structure by imbedding the space-time surface to a sub-space M^8 of complexified octonions. This is admittedly artificial.

2. The octonionic representation effectively replaces $SO(7,1)$ as tangent space group with G_2 and means selection of preferred $M^2 \subset M^4$ having interpretation complex plane of octonionic space. A more general condition is that the tangent space of space-time surface at each point contains preferred sub-space $M^2(x) \subset M^4$ forming an integrable distribution. The same condition is involved with the definition of Hamilton-Jacobi structure. What puts bells ringing is that the modified Dirac equation for the octonionic representation of gamma matrices allows the conservation of electromagnetic charge in the proposed sense a observed for years ago. One can ask whether the conditions on the charged part of energy momentum tensor could relate to the reduction of $SO(7,1)$ to G_2.

3. Octonionic gamma matrices appear also in the proposal stating that space-time surfaces are quaternionic in the sense that tangent space of the space-time surface is quaternionic in the sense that induced octonionic gamma matrices generate a quaternionic sub-space at a given point of space-time time. Besides this the already mentioned additional condition stating that the tangent space contains preferred sub-space $M^2 \subset M^4$ or integrable distribution of this kind of sub-spaces is required. It must be emphasized that induced rather than modified gamma matrices are natural in these conditions.

3.6.1 Definition of quaternionicity based on gamma matrices

The definition of quaternionicity in terms of gamma matrices looks more promising. This however raises two questions.

1. Can the quaternionicity of the space-time surface together with a preferred distribution of tangent planes $M^2(x) \subset M^4$ or $E^2(x) \subset CP_2$ be equivalent with the reduction of the field equations to the analogs of minimal surface equations stating that certain components of the induced metric in complex/Hamilton-Jacobi coordinates vanish in turn guaranteeing that field equations reduce to algebraic identifies following from the fact that energy momentum tensor and second fundamental form have no common components? This should be the case if one requires that the two solution ansätze are equivalent.

2. Can the conditions for the modified Dirac equation select complex of co-complex 2-sub-manifold of space-time surface identified as quaternionic or co-quaternionic 4-surface? Could the conditions stating the vanishing of charged energy momentum currents state that the spinor fields are localized to complex or co-complex (hyper-complex or co-hypercomplex) 2-surfaces?

One should assign to the space-time sheets both quaternionic and Hermitian or Hamilton-Jacobi structure. There are two structures involved. Euclidian metric is an essential aspect of what it is to be quaternionic or octonions. It however seems that one can assign to the induced metric only Hermitian or Hamilton-Jacobi structure. This leads to a serious of innocent questions.

1. Could these two structures be associated with energy momentum tensor and metric respectively? Anti-commutators of the modified gamma matrices define an effective metric as

$$G_{\alpha\beta} = T_{\alpha\mu} T^{\mu}_{\ \beta} \ .$$

This effective metric should have a deep physical and mathematical meaning but this meaning has remained a mystery. Note that iT takes the role of Kähler form for Kähler metric in this expression with imaginary unit plus symmetry replacing antisymmetry. The neutral and charged parts of T would be analogous to quaternionic imaginary units and real part but one would have sum over the squares of all of them rather than only single square as for Kähler metric.

2. Could G be assigned with the quaternionic structure and induced metric to the Hermitian/Hamilton-Jacobi structures? Or perhaps vice versa? Could the neutral and charged components of the energy momentum tensor somehow correspond to quaternionic units?

The basic potential problem with the assignment of quaternionic structure to the induced gamma matrices is the signature of the metric in Minkowskian regions.

1. If quaternionic structrures is defined in terms of the octonionic representation of the imbedding space gamma matrices, there seems to be no obvious problems since one considers automatically complexification of quaternions.

2. For the approach based on the notion of quaternion real analyticity, one is forced to use Wick rotation to define the quaternionic structure or to introduce hyper-quaternionic structure by imbedding the space-time surface to a sub-space M¡sup¿8¡/sup¿ of complexified octonions. This is admittedly artificial.

Could one pose the additional requirement that the signature of the effective metric G defined by the modified gamma marices (and to be distinguished from Einstein tensor) is Euclidian in the sense that all four eigenvalues of this tensor would have same sign.

1. For the induced metric the projections of gamma matrices are given by

$$\Gamma_\alpha = \Gamma^a e_{a\alpha} \ , \ \ e_{a\alpha} = e_{ak}\partial_\alpha h^k \ .$$

For the modified gamma matrices their analogs would be given by

$$\Gamma_\alpha = \Gamma^a E_{a\alpha} \ , \ \ E_{a\alpha} = e_{a\mu} T^{\mu}_{\alpha} \ .$$

Projection would be followed by a multiplication with energy momentum tensor. One cannot induce G from any metric defined in the imbedding space but the notion of tangent space quaternionicity is well-defined.

2. What quaternionic structure for G could mean? One can imagine several options.

   (a) For the ordinary complex structure metric has vanishing diagonal components and the inner product for infinitesimal vectors is just $g_{z\bar{z}}(dz_1 d\overline{z}_2 + dz_2 d\overline{z}_1)$. Could this formula generalize to $g_{Q\bar{Q}}(dQ_1 d\overline{Q}_2 + dQ_2 d\overline{Q}_1)$? The generalization would be a direct generalization of conformal invariance to 4-D context stating that 4-metric is quaternion-conformally equivalent to flat metric. This would give additional strong condition on energy momentum tensor:

$$G = T_{\alpha\mu} T^{\mu}_{\ \beta} = T^2 \delta_{\alpha\beta} \ .$$

   The proportionality to Euclidian metric means in Minkowskian realm that the G is of form $G = T^2(2u_\alpha u_\beta - g_{\alpha\beta}$. Here u is time-like vector field satisfying $u^\alpha u_\alpha = 1$ and having interpretation as a local four-velocity (in Robertson-Walker cosmology similar situation is encountered). The eigen value problem in the form $G^{\alpha}_{\beta} x^\beta = \lambda x^\alpha$ makes sense and eigenvectors would be u^α with eigenvalue $\lambda = T^2$ and three vectors orthogonal to it with eigenvalue $-T^2$. This requires integrable flow defined by u and defining a preferred time coordinate. In number theoretic vision this kind of time coordinate is introduced and corresponds to the direction assignable to the octonionic real unit. Note that the vanishing of charged projections of the energy momentum tensor does not imply a reduction of the rank of T so that this options might work.

(b) Quaternionicity could mean also the structure of hyper-Kähler manifold. Metric and Kähler form for Kähler manifold are generalized to metric representing quaternion real unit and three covariantly constant Kähler forms I_i obeying the multiplication rules for quaternions. The necessary condition is that the holonomy group equals to SU(2) identifiable as automorphism group of quaternions. One can also define quaternionic structure: there would exist three antisymmetric tensors, whose squares give the negative of the metric. CP_2 allows quaternionic structure in this sense and only one of these forms is covariantly constant.

Could space-time surface allow Hyper-Kähler or quaternionic structure somehow induced from that of CP_2? This does not work for G. G is quadratic in energy momentum tensor and therefore involves four power of J rather than being square of projection of J or two other quaternionic imaginary units of CP_2. One can of course ask whether the induced quaternionic units could obey the multiplication of quaternionic units and have same square given by the projection of CP_2 metric. In this case CP_2 metric would define the effective metric and would be indeed Euclidian. For the ansatz for preferred extremals with Minkowskian signature CP_2 projection is at most 3-dimensional but also in this case the imaginary units might allow a realization as projections.

3.6.2 Definition of quaternionicity based on octonion real-analyticity

Second definition of quaternionicity is on more shaky basis and motivated by the solutions of 2-D Laplace equation: quaternionic space-time surfaces would be obtained as zero loci of octonion real–analytic functions. Unfortunately octonion real–analyticity does not make sense in Minkowskian signature.

One could understand octonion real-analyticity in Minkowskian signature if one could understand the deeper meaning of Wick rotation. Octonion real analyticity formulated as a condition for the vanishing of the imaginary part of octonion real-analytic function makes sense for in octonionic coordinates for $E^4 \times CP_2$ with Euclidian signature of metric. $M^4 \times CP_2$ is however only a subspace of complexified octonions and not closed with respect to multiplication so that octonion real-analytic functions do not make sense in $M^4 \times CP_2$. Wick rotation should transform the solution candidate defined by an octonion real-analytic function to that defined in $M^4 \times CP_2$. A natural additional condition is that Wick rotation should reduce to that taking $M^2 \subset M^4$ to $E^2 \subset E^4$.

The following trivial observation made in the construction of Hamilton-Jacobi structure in M^4 with Minkowskian signature of the induced metric (see the Appendix) as a Wick rotation of Hermitian structure in E^4 might help here.

1. The components of the metric of E^2 in complex coordinates $(z, \overline{z})$ for E^2 are given by $g_{w\overline{w}} = -1$ whereas the metric of M^2 in light-like coordinates $(u = x+t, v = x-t)$ is given by $g_{uv} = -1$. The metric is same and M^2 and E^2 correspond only to different interpretations for the coordinates! One could say that $M^4 \times CP_2$ and $E^4 \times CP_2$ have same metric tensor, Kähler structure, and spinor structure. Since only these appear in field equations, one could hope that the solutions of field equations in $M^4 \times CP_2$ and $E^4 \times CP_2$ are obtained by Wick rotation. This for preferred extremals at least and if the field equations reduce to purely algebraic ones.

2. If one accepts the proposed construction of preferred extremals of Kähler action discussed in [K13], the field equations indeed reduce to purely algebraic conditions satisfied if space-time surface possesses Hermitian structure in the case of Euclidian signature of the induced metric and Hamilton-Jacobi structure in the case of Minkowskian signature. Just as in the case of minimal surfaces, energy momentum tensor and second fundamental form have no common non-vanishing components. The algebraization requires as a consistency condition Einstein's equations with a cosmological term. Gravitational constant and cosmological constant follow as predictions.

3. If Wick rotation in the replacement of E^2 coordinates $(z, \overline{z})$ with M^2 coordinates (u, v) makes sense, one can hope that field equations for the preferred extremals hold true also for a Wick rotated surfaces obtained by mapping $M^2 \subset M^4$ to $E^2 \subset E^4$. Also Einstein's equations should be satisfied by the Wick rotated metric with Euclidian signature.

4. Wick rotation makes sense also for the surfaces defined by the vanishing of the imaginary part (complementary to quaternionic part) of octonion real-analytic function. Therefore one can hope that this ansatz could work. Wick rotation is non-trivial geometrically. For instance, light-like lines $v = 0$ of hyper-complex plane M^2 are taken to $\overline{z} = 0$ defining a point of complex plane E^2. Note that non-invertible hyper-complex numbers correspond to the two light-like lines $u = 0$ and $v = 0$ whereas non-invertible complex numbers correspond to the origin of E^2.

5. If the conjecture holds true, one can apply to both factors in $E^4 = E^2 \times E^2$ and to get preferred extremals in $M^{2,2} \times CP_2$. Minkowski space $M^{2,2}$ is essential in twistor approach and the possibility to carry out Wick rotation for preferred extremals could justify Wick rotation in quantum theory.

4 Preferred extremals and solutions of the modified Dirac equation and super-conformal symmetries

The new vision about preferred exrtremals and modified Dirac equations is bound to check the existing vision about super-conformal symmetries. One important discovery is that Einstein's equations follow from the vanishing of terms proportional to Kähler current in field equations for preferred extremals and Equivalence Principle at the classical level is realized automatically in all scales in contrast to the earlier belief. This obviously must have implications to the general vision about Super-Virasoro representations and one must be ready to modify the existing picture based on the assumption that quantum version of Equivalence Principle is realized in terms coset representations.

The very special role of right handed neutrino is also bound to have profound implications. A further important outcome is the identification of gauge potentials as duals of Kac-Moody currents at the boundaries of string world sheets: quantum gauge potentials are defined only where they are needed that is the curves defining the non-integrable phase factors. This gives also rise to the realization of the conjecture Yangian in terms of the Kac-Moody charges and commutators in accordance with the earlier conjecture.

4.1 Super-conformal symmetries

It is good to summarize first the basic ideas about Super-Virasoro representations. TGD allows two kinds of super-conformal symmetries.

1. The first super-conformal symmetry is associated with $\delta M^4_\pm \times CP_2$ and corresponds to symplectic symmetries of $\delta M^4_\pm \times CP_2$. The reason for extension of conformal symmetries is metric 2-dimensionality of the light-like boundary $\delta M^4_\pm$ defining upper/lower boundary of causal diamond (CD). This super-conformal symmetry is something new and corresponds to replacing finite-dimensional Lie-group G for Kac-Moody symmetry with infinite-dimensional symplectic group. The light-like radial coordinate of $\delta M^4_\pm$ takes the role of the real part of complex coordinate z for ordinary conformal symmetry. Together with complex coordinate of S^2 it defines 3-D restriction of Hamilton-Jacobi variant of 4-D super-conformal symmetries. One can continue the conformal symmetries from light-cone boundary to CD by forming a slicing by parallel copies of $\delta M^4_\pm$. There are two possible slicings corresponding to the choices δM^4_+ and δM^4_- assignable to the upper and lower boundaries of CD. These two choices correspond to two arrows of geometric time for the basis of zero energy states in ZEO.

2. Super-symplectic degrees of freedom determine the electroweak and color quantum numbers of elementary particles. Bosonic emergence implies that ground states assignable to partonic 2-surfaces correspond to partial waves in $\delta M^4_\pm$ and one obtains color partial waves in particular. These partial waves correspond to the solutions for the Dirac equation in imbedding space and

the correlation between color and electroweak quantum numbers is not quite correct. Super-Kac-Moody generators give the compensating color for massless states obtained from tachyonic ground states guaranteeing that standard correlation is obtained. Super-symplectic degrees are therefore directly visible in particle spectrum. One can say that at the pointlike limit the WCW spinors reduce to tensor products of imbedding space spinors assignable to the center of mass degrees of freedom for the partonic 2-surfaces defining wormhole throats.

I have proposed a physical interpretation of super-symplectic vibrational degrees of freedom in terms of degrees of freedom assignable to non-perturbative QCD. These degrees of freedom would be responsible for most of the baryon masses but their theoretical understanding is lacking in QCD framework.

3. The second super-conformal symmetry is assigned light-like 3-surfaces and to the isometries and holonomies of the imbedding space and is analogous to the super-Kac-Moody symmetry of string models. Kac-Moody symmetries could be assigned to the light-like deformations of light-like 3-surfaces. Isometries give tensor factor $E^2 \times SU(3)$ and holonomies factor $SU(2)_L \times U(1)$. Altogether one has 5 tensor factors to super-conformal algebra. That the number is just five is essential for the success p-adic mass calculations [K6, K4].

   The construction of solutions of the modified Dirac equation suggests strongly that the fermionic representation of the Super-Kac-Moody algebra can be assigned as conserved charges associated with the space-like braid strands at both the 3-D space-like ends of space-time surfaces and with the light-like (or space-like with a small deformation) associated with the light-like 3-surfaces. The extension to Yangian algebra involving higher multilinears of super-Kac Moody generators is also highly suggestive. These charges would be non-local and assignable to several wormhole contacts simultaneously. The ends of braids would correspond points of partonic 2-surfaces defining a discretization of the partonic 2-surface having interpretation in terms of finite measurement resolution.

   These symmetries would correspond to electroweak and strong gauge fields and to gravitation. The duals of the currents giving rise to Kac-Moody charges would define the counterparts of gauge potentials and the conserved Kac-Moody charges would define the counterparts of non-integrable phase factors in gauge theories. The higher Yangian charges would define generalization of non-integrable phase factors. This would suggest a rather direct connection with the twistorial program for calculating the scattering amplitudes implies also by zero energy ontology.

Quantization recipes have worked in the case of super-string models and one can ask whether the application of quantization to the coefficients of powers of complex coordinates or Hamilton-Jacobi coordinates could lead to the understanding of the 4-D variants of the conformal symmetries and give detailed information about the representations of the Kac-Moody algebra too.

4.2 What is the role of the right-handed neutrino?

A highly interesting aspect of Super-Kac-Moody symmetry is the special role of right handed neutrino.

1. Only right handed neutrino allows besides the modes restricted to 2-D surfaces also the 4D modes delocalized to the entire space-time surface. The first ones are holomorphic functions of single coordinate and the latter ones holomorphic functions of two complex/Hamilton-Jacobi coordinates. Only ν_R has the full $D = 4$ counterpart of the conformal symmetry and the localization to 2-surfaces has interpretation as super-conformal symmetry breaking halving the number of super-conformal generators.

2. This forces to ask for the meaning of super-partners. Are super-partners obtained by adding ν_R neutrino localized at partonic 2-surface or delocalized to entire space-time surface or its Euclidian or Minkowskian region accompanying particle identified as wormhole throat? Only the Euclidian option allows to assign right handed neutrino to a unique partonic 2-surface. For the Minkowskian regions the assignment is to many particle state defined by the partonic 2-surfaces associated with the 3-surface. Hence for spartners the 4-D right-handed neutrino must be associated with the 4-D Euclidian line of the generalized Feynman diagram.

3. The orthogonality of the localized and de-localized right handed neutrino modes requires that 2-D modes correspond to higher color partial waves at the level of imbedding space. If color octet is in question, the 2-D right handed neutrino as the candidate for the generator of standard SUSY would combine with the left handed neutrino to form a massive neutrino. If 2-D massive neutrino acts as a generator of super-symmetries, it is is in the same role as badly broken supersymmeries generated by other 2-D modes of the induced spinor field (SUSY with rather large value of $\mathcal{N}$) and one can argue that the counterpart of standard SUSY cannot correspond to this kind of super-symmetries. The right-handed neutrinos delocalized inside the lines of generalized Feynman diagrams, could generate $\mathcal{N}=2$ variant of the standard SUSY.

4.2.1 How particle and right handed neutrino are bound together?

Ordinary SUSY means that apart from kinematical spin factors sparticles and particles behave identically with respect to standard model interactions. These spin factors would allow to distinguish between particles and sparticles. But is this the case now?

1. One can argue that 2-D particle and 4-D right-handed neutrino behave like independent entities, and because ν_R has no standard model couplings this entire structure behaves like a particle rather than sparticle with respect to standard model interactions: the kinematical spin dependent factors would be absent.

2. The question is also about the internal structure of the sparticle. How the four-momentum is divided between the ν_R and and 2-D fermion. If ν_R carries a negligible portion of four-momentum, the four-momentum carried by the particle part of sparticle is same as that carried by particle for given four-momentum so that the distinctions are only kinematical for the ordinary view about sparticle and trivial for the view suggested by the 4-D character of ν_R.

Could sparticle character become manifest in the ordinary scattering of sparticle?

1. If ν_R behaves as an independent unit not bound to the particle, it would continue in the original direction as particle scatters: sparticle would decay to particle and right-handed neutrino. If ν_R carries a non-negligible energy the scattering could be detected via a missing energy. If not, then the decay could be detected by the interactions revealing the presence of ν_R. ν_R can have only gravitational interactions. What these gravitational interactions are is not however quite clear since the proposed identification of gravitational gauge potentials is as duals of Kac-Moody currents analogous to gauge potentials located at the boundaries of string world sheets. Does this mean that 4-D right-handed neutrino has no quantal gravitational interactions? Does internal consistency require ν_R to have a vanishing gravitational and inertial masses and does this mean that this particle carries only spin?

2. The cautious conclusion would be following: if delocalized ν_R and parton are un-correlated particle and sparticle cannot be distinguished experimentally and one might perhaps understand the failure to detect standard SUSY at LHC. Note however that the 2-D fermionic oscillator algebra defines badly broken large $\mathcal{N}$ SUSY containing also massive (longitudinal momentum square is non-vanishing) neutrino modes as generators.

4.2.2 Taking a closer look on sparticles

It is good to take a closer look at the delocalized right handed neutrino modes.

1. At imbedding space level that is in cm mass degrees of freedom they correspond to covariantly constant CP_2 spinors carrying light-like momentum which for causal diamond could be discretized. For non-vanishing momentum one can speak about helicity having opposite sign for ν_R and $\overline{\nu}_R$. For vanishing four-momentum the situation is delicate since only spin remains and Majorana like behavior is suggestive. Unless one has momentum continuum, this mode might be important and generate additional SUSY resembling standard $\mathcal{N}=1$ SUSY.

2. At space-time level the solutions of modified Dirac equation are holomorphic or anti-holomorphic.

(a) For non-constant holomorphic modes these characteristics correlate naturally with fermion number and helicity of ν_R . One can assign creation/annihilation operator to these two kinds of modes and the sign of fermion number correlates with the sign of helicity.

(b) The covariantly constant mode is naturally assignable to the covariantly constant neutrino spinor of imbedding space. To the two helicities one can assign also oscillator operators $\{a_\pm, a_\pm^\dagger\}$. The effective Majorana property is expressed in terms of non-orthogonality of ν_R and and $\overline{\nu}_R$ translated to the the non-vanishing of the anti-commutator $\{a_+^\dagger, a_-\} = \{a_-^\dagger, a_+\} = 1$. The reduction of the rank of the 4×4 matrix defined by anti-commutators to two expresses the fact that the number of degrees of freedom has halved. $a_+^\dagger = a_-$ realizes the conditions and implies that one has only $\mathcal{N} = 1$ SUSY multiplet since the state containing both ν_R and $\overline{\nu}_R$ is same as that containing no right handed neutrinos.

(c) One can wonder whether this SUSY is masked totally by the fact that sparticles with all possible conformal weights n for induced spinor field are possible and the branching ratio to $n = 0$ channel is small. If momentum continuum is present, the zero momentum mode might be equivalent to nothing.

What can happen in spin degrees of freedom in super-symmetric interaction vertices if one accepts this interpretation? As already noticed, this depends solely on what one assumes about the correlation of the four-momenta of particle and ν_R.

1. For SUSY generated by covariantly constant ν_R and $\overline{\nu}_R$ there is no neutrino four-momentum involved so that only spin matters. One cannot speak about the change of direction for ν_R. In the scattering of sparticle the direction of particle changes and introduces different spin quantization axes. ν_R retains its spin and in new system it is superposition of two spin projections. The presence of both helicities requires that the transformation $\nu_R \rightarrow \overline{\nu}_R$ happens with an amplitude determined purely kinematically by spin rotation matrices. This is consistent with fermion number conservation modulo 2. $\mathcal{N} = 1$ SUSY based on Majorana spinors is highly suggestive.

2. For SUSY generated by non-constant holomorphic and anti-holomorphic modes carrying fermion number the behavior in the scattering is different. Suppose that the sparticle does not split to particle moving in the new direction and ν_R moving in the original direction so that also ν_R or $\overline{\nu}_R$ carrying some massless fraction of four-momentum changes its direction of motion. One can form the spin projections with respect to the new spin axis but must drop the projection which does not conserve fermion number. Therefore the kinematics at the vertices is different. Hence $\mathcal{N} = 2$ SUSY with fermion number conservation is suggestive when the momentum directions of particle and ν_R are completely correlated.

3. Since right-handed neutrino has no standard model couplings, p-adic thermodynamics for 4-D right-handed neutrino must correspond to a very low p-adic temperature $T = 1/n$. This implies that the excitations with non-vanishing conformal weights are effectively absent and one would have $\mathcal{N} = 1$ SUSY effectively.

   The simplest assumption is that particle and sparticle correspond to the same p-adic mass scale and have degenerate masses: it is difficult to imagine any good reason for why the p-adic mass scales should differ. This should have been observed -say in decay widths of weak bosons - unless the spartners correspond to large hbar phase and therefore to dark matter. Note that for the badly broken 2-D N=2 SUSY in fermionic sector this kind of almost degeneracy cannot be excluded and I have considered an explanation for the mysterious X and Y mesons in terms of this degeneracy [K5].

4.2.3 Why space-time SUSY is not possible in TGD framework?

LHC suggests that one does not have $\mathcal{N} = 1$ SUSY in standard sense. Why one cannot have standard space-time SUSY in TGD framework. Let us begin by listing all arguments popping in mind.

1. Could covariantly constant ν_R represents a gauge degree of freedom? This is plausible since the corresponding fermion current is non-vanishing.

2. The original argument for absence of space-time SUSY years ago was indirect: $M^4 \times CP_2$ does not allow Majorana spinors so that $\mathcal{N}=1$ SUSY is excluded.

3. One can however consider $\mathcal{N}=2$ SUSY by including both helicities possible for covariantly constant ν_R. For ν_R the four-momentum vanishes so that one cannot distinguish the modes assigned to the creation operator and its conjugate via complex conjugation of the spinor. Rather, one oscillator operator and its conjugate correspond to the two different helicities of right-handed neutrino with respect to the direction determined by the momentum of the particle. The spinors can be chosen to be real in this basis. This indeed gives rise to an irreducible representation of spin 1/2 SUSY algebra with right-handed neutrino creation operator acting as a ladder operator. This is however $\mathcal{N}=1$ algebra and right-handed neutrino in this particular basis behaves effectively like Majorana spinor. One can argue that the system is mathematically inconsistent. By choosing the spin projection axis differently the spinor basis becomes complex. In the new basis one would have $\mathcal{N}=2$, which however reduces to $\mathcal{N}=1$ in the real basis.

4. Or could it be that fermion and sfermion do exist but cannot be related by SUSY? In standard SUSY fermions and sfermions forming irreducible representations of super Poincare algebra are combined to components of superfield very much like finite-dimensional representations of Lorentz group are combined to those of Poincare. In TGD framework ν_R generates in space-time interior generalization of 2-D super-conformal symmetry but covarianlty constant ν_R cannot give rise to space-time SUSY.

   This would be very natural since right-handed neutrinos do not have any electroweak interactions and are are delocalized into the interior of the space-time surface unlike other particles localized at 2-surfaces. It is difficult to imagine how fermion and ν_R could behave as a single coherent unit reflecting itself in the characteristic spin and momentum dependence of vertices implied by SUSY. Rather, it would seem that fermion and sfermion should behave identically with respect to electroweak interactions.

The third argument looks rather convincing and can be developed to a precise argument.

1. If sfermion is to represent elementary bosons, the products of fermionic oscillator operators with the oscillator operators assignable to the covariantly constant right handed neutrinos must define might-be bosonic oscillator operators as $b_n = a_n a$ and $b_n^\dagger = a_n^\dagger a^\dagger$ One can calculate the commutator for the product of operators. If fermionic oscillator operators commute, so do the corresponding bosonic operators. The commutator $[b_n, b_n^\dagger]$ is however proportional to occupation number for ν_R in $\mathcal{N}=1$ SUSY representation and vanishes for the second state of the representation. Therefore $\mathcal{N}=1$ SUSY is a pure gauge symmetry.

2. One can however have both irreducible representations of SUSY: for them either fermion or sfermion has a non-vanishing norm. One would have both fermions and sfermions but they would not belong to the same SUSY multiplet, and one cannot expect SUSY symmetries of 3-particle vertices.

3. For instance, γFF vertex is closely related to $\gamma \tilde{F}\tilde{\overline{F}}$ in standard SUSY. Now one expects this vertex to decompose to a product of $\gamma F\overline{F}$ vertex and amplitude for the creation of $\nu_R \tilde{\nu}_R$ from vacuum so that the characteristic momentum and spin dependent factors distinguishing between the couplings of photon to scalar and and fermion are absent. Both states behave like fermions. The amplitude for the creation of $\nu_R \tilde{\nu}_R$ from vacuum is naturally equal to unity as an occupation number operator by crossing symmetry. The presence of right-handed neutrinos would be invisible if this picture is correct. Whether this invisible label can have some consequences is not quite clear: one could argue that the decay rates of weak bosons to fermion pairs are doubled unless one introduces $1/\sqrt{2}$ factors to couplings.

   Where the sfermions might make themselves visible are loops. What loops are? Consider boson line first. Boson line is replaced with a sum of two contributions corresponding to ordinary contribution with fermion and antifermion at opposite throats and second contribution with fermion and antifermion accompanied by right-handed neutrino ν_R and its antiparticle which now has opposite helicity to ν_R. The loop for ν_R decomposes to four pieces since also the propagation from wormhole throat to the opposite wormhole throat must be taken into account.

Each of the four propagators equals to $a^\dagger_{1/2}a^\dagger_{-1/2}$ or its hermitian conjugate. The product of these is slashed between vacuum states and anti-commutations give imaginary unit per propagator giving $i^4 = 1$. The two contributions are therefore identical and the scaling $g \rightarrow g/\sqrt{2}$ for coupling constants guarantees that sfermions do not affect the scattering amplitudes at all. The argument is identical for the internal fermion lines.

4.3 WCW geometry and super-conformal symmetries

The vision about the geometry of WCW has been roughly the following and the recent steps of progress induce to it only small modifications if any.

1. Kähler geometry is forced by the condition that hermitian conjugation allows geometrization. Kähler function is given by the Kähler action coming from space-time regions with Euclidian signature of the induced metric identifiable as lines of generalized Feynman diagrams. Minkowskian regions give imaginary contribution identifiable as the analog of Morse function and implying interference effects and stationary phase approximation. The vision about quantum TGD as almost topological QFT inspires the proposal that Kähler action reduces to 3-D terms reducing to Chern-Simons terms by the weak form of electric-magnetic duality. The recent proposal for preferred extremals is consistent with this property realizing also holography implied by general coordinate invariance. Strong form of general coordinate invariance implying effective 2-dimensionality in turn suggests that Kähler action is expressible in terms of areas of partonic 2-surfaces and string world sheets.

2. The complexified gamma matrices of WCW come as hermitian conjugate pairs and anti-commute to the Kähler metric of WCW. Also bosonic generators of symplectic transformations of $\delta M^4_\pm \times CP_2$ a assumed to act as isometries of WCW geometry can be complexified and appear as similar pairs. The action of isometry generators co-incides with that of symplectic generators at partonic 2-surfaces and string world sheets but elsewhere inside the space-time surface it is expected to be deformed from the symplectic action. The super-conformal transformations of $\delta M^4_\pm \times CP_2$ acting on the light-like radial coordinate of $\delta M^4_\pm$ act as gauge symmetries of the geometry meaning that the corresponding WCW vector fields have zero norm.

3. WCW geometry has also zero modes which by definition do not contribute to WCW metric expect possibly by the dependence of the elements of WCW metric on zero modes through a conformal factor. In particular, induced CP_2 Kähler form and its analog for sphere $r_M = constant$ of light cone boundary are symplectic invariants, and one can define an infinite number of zero modes as invariants defined by Kähler fluxes over partonic 2-surfaces and string world sheets. This requires however the slicing of CD parallel copies of δM^4_+ or δM^4_-. The physical interpretation of these non-quantum fluctuating degrees of freedom is as classical variables necessary for the interpretation of quantum measurement theory. Classical variable would metaphorically correspond the position of the pointer of the measurement instrument.

4. The construction receives a strong philosophical inspiration from the geometry of loop spaces. Loop spaces allow a unique Kähler geometry with maximal isometry group identifiable as Kac-Moody group. The reason is that otherwise Riemann connection does not exist. The only problem is that curvature scalar diverges since the Riemann tensor is by constant curvature property proportional to the metric. In 3-D case one would have union of constant curvature spaces labelled by zero modes and the situation is expected to be even more restrictive. The conjecture indeed is that WCW geometry exists only for $H = M^4 \times CP_2$: infinite-D Kähler geometric existence and therefore physics would be unique. One can also hope that Ricci scalar is finite and therefore zero by the constant curvature property so that Einstein's equations are satisfied.

5. WCW Hamiltonians determined the isometry currents and WCW metric is given in terms of the anti-commutators of the Killing vector fields associated with symplectic isometry currents. The WCW Hamiltonians generating symplectic isometries correspond to the Hamiltonians spanning the symplectic group of $\delta M^4_\pm \times CP_2$. One can say that the space of quantum fluctuating degrees of freedom is this symplectic group of $\delta M^4_\pm \times CP_2$ or its subgroup or coset space: this must have very deep implications for the structure of the quantum TGD.

6. Zero energy ontology brings in additional delicacies. Basic objects are now unions of partonic 2-surfaces at the ends of CD. Also string world sheets would naturally contribute. One can generalize the expressions for the isometry generators in a straightforward manner by requiring that given isometry restricts to a symplectic transformation at partonic 2-surfaces and string world sheets.

7. One could criticize the effective metric 2-dimensionality forced by general consistency arguments as something non-physical. The Hamiltonians are expressed using only the data at partonic 2-surfaces: this includes also 4-D tangent space data via the weak form of electric-magnetic duality so that one has only effective 2-dimensionality. Obviously WCW geometry must have large gauge symmetries besides zero modes. The super-conformal symmetries indeed represent gauge symmetries of this kind. Effective 2-dimensionality realizing strong form of holography in turn is induced by the strong form of general coordinate invariance. Light-like 3-surfaces at which the signature of the induced metric changes must be equivalent with the 3-D space-like ends of space-time surfaces at the light-boundaries of space-time surfaces as far as WCW geometry is considered. This requires that the data from their 2-D intersections defining partonic 2-surfaces should dictate the WCW geometry. Note however that Super-Kac-Moody charges giving information about the interiors of 3-surfaces appear in the construction of the physical states.

What is the role of the right handed neutrino in this construction?

1. In the construction of components of WCW metric as anti-commutators of super-generators only the covariantly constant right-handed neutrino appears in the super-generators analogous to super-Kac-Moody generators. All holomorphic modes of right handed neutrino characterized by two integers could in principle contribute to the WCW gamma matrices identified as fermionic super-symplectic generators anti-commuting to the metric. At the space-like ends of space-time surface the holomorphic generators would restrict to symplectic generators since the radial light-like coordinate r_M identified and complex coordinate of CP_2 allowing identification as restrictions of two complex coordinates or Hamilton-Jacobi coordinates to light-like boundary.

2. The non-covariantly constant modes could also correspond to purely super-conformal gauge degrees of freedom. Originally the restriction to right-handed neutrino looked somewhat unsatisfactory but the recent view about Super-Kac-Moody symmetries makes its special role rather natural. One could say that WCW geometry possesses the maximal $D = 4$ supersymmetry.

3. One can of course ask whether the Super-Kac-Moody generators assignable to the isometries of H and expressible as conserved charges associated with the boundaries of string world sheets could contribute to the WCW geometry via the anti-commutators. This option cannot be excluded but in this case the interpretation in terms of Hamiltonians is not obvious.

4.4 Equivalence Principle

An important physical input has been the condition that a generalization of Equivalence Principle is obtained.

1. The proposal has been that inertial and gravitational masses can be assigned with the super-symplectic and super-Kac-Moody representations via the condition that the scaling generator L_0 defined as a difference of the corresponding generators for the two representations annihilates physical states. This requires that super-Kac-Moody algebra can be regarded in some sense as a sub-algebra of super-symplectic algebra. For isometries this would be natural but in the case of holonomies the situation is problematic. The idea has been that the ordinary realization of Equivalence Principle follows as Einstein's equations for fluctuations around vacuum extremals expressing the average energy momentum tensor for the fluctuations.

2. The emergence of Einstein's equations for preferred extremals as additional conditions [K1, K11] allowing the algebraization of the equations to analogs of minimal surface equations changes the situation completely. Is there anymore need to realize Equivalence Principle at quantum level? If one drops this condition one can imagine very simple option obtained as tensor product of

the super-symplectic and super-Kac-Moody representations. Of course, coset representations for the symplectic group and its suitable subgroup - say subgroup defining measurement resolution - can be present but would not nothing to do with Equivalence Principle.

3. One can of course argue that one has very naturally to different mass squared operators and therefore inertial and gravitational masses. Inertial mass squared would be naturally assignable to the representations of the super-symplectic algebra imbedding space d'Alembertian and gravitational mass squared with the spinor d'Alembertian at string world sheets at space-time surfaces. Quantum level realization for Equivalence Principle could mean that these two mass squared operators are identical or something analogous to this. One can however criticize this idea as un-necessary and also because the signature of the effective metric defined by the modified Dirac gamma matrices is speculated to be Euclidian.

4.5 Constraints from p-adic mass calculations and ZEO

A further important physical input comes from p-adic thermodynamics forming a core element of p-adic mass calculations.

1. The first thing that one can get worried about relates to the extension of conformal symmetries. If the conformal symmetries generalize to $D = 4$, how can one take seriously the results of p-adic mass calculations based on 2-D conformal invariance? There is no reason to worry. The reduction of the conformal invariance to 2-D one for the preferred extremals takes care of this problem. This however requires that the fermionic contributions assignable to string world sheets and/or partonic 2-surfaces - Super- Kac-Moody contributions - should dictate the elementary particle masses. For hadrons also symplectic contributions should be present. This is a valuable hint in attempts to identify the mathematical structure in more detail.

2. ZEO suggests that all particles, even virtual ones correspond to massless wormhole throats carrying fermions. As a consequenc, twistor approach would work and the kinematical constraints to vertices would allow th cancellation of divergences. This would suggest that the p-adic thermal expectation value is for the longitudinal M^2 momentum squared (the definition of CD selects $M^1 \subset M^2 \subset M^4$ as also does number theoretic vision). Also propagator would be determined by M^2 momentum. Lorentz invariance would be obtained by integration of the moduli for CD including also Lorentz boosts of CD.

3. In the original approach one allows states with arbitrary large values of L_0 as physical states. Usually one would require that L_0 annihilates the states. In the calculations however mass squared was assumed to be proportional L_0 apart from vacuum contribution. This is a questionable assumption. ZEO suggests that total mass squared vanishes and that one can decompose mass squared to a sum of longitudinal and transversal parts. If one can do the same decomposition to longitudinal and transverse parts also for the Super Virasoro algebra then one can calculate longitudinal mass squared as a p-adic thermal expectation in the transversal super-Virasoro algebra and only states with $L_0 = 0$ would contribute and one would have conformal invariance in the standard sense.

4. In the original approach the assumption motivated by Lorentz invariance has been that mass squared is replaced with conformal weight in thermodynamics, and that one first calculates the thermal average of the conformal weight and then equates it with mass squared. This assumption is somewhat ad hoc. ZEO however suggests an alternative interpretation in which one has zero energy states for which longitudinal mass squared of positive energy state derive from p-adic thermodynamics. Thermodynamics - or rather, its square root - would become part of quantum theory in ZEO. M-matrix is indeed product of hermitian square root of density matrix multiplied by unitary S-matrix and defines the entanglement coefficients between positive and negative energy parts of zero energy state.

5. The crucial constraint is that the number of super-conformal tensor factors is $N = 5$: this suggests that thermodynamics applied in Super-Kac-Moody degrees of freedom assignable to string world sheets is enough, when one is interested in the masses of fermions and gauge bosons. Super-symplectic degrees of freedom can also contribute and determine the dominant

contribution to baryon masses. Should also this contribution obey p-adic thermodynamics in the case when it is present? Or does the very fact that this contribution need not be present mean that it is not thermal? The symplectic contribution should correspond to hadronic p-adic length prime rather the one assignable to (say) u quark. Hadronic p-adic mass squared and partonic p-adic mass squared cannot be summed since primes are different. If one accepts the basic rules [K7], longitudinal energy and momentum are additive as indeed assumed in perturbative QCD.

6. Calculations work if the vacuum expectation value of the mass squared must be assumed to be tachyonic. There are two options depending on whether one whether p-adic thermodynamics gives total mass squared or longitudinal mass squared.

   (a) One could argue that the total mass squared has naturally tachyonic ground state expectation since for massless extremals longitudinal momentum is light-like and transversal momentum squared is necessary present and non-vanishing by the localization to topological light ray of finite thickness of order p-adic length scale. Transversal degrees of freedom would be modeled with a particle in a box.

   (b) If longitudinal mass squared is what is calculated, the condition would require that transversal momentum squared is negative so that instead of plane wave like behavior exponential damping would be required. This would conform with the localization in transversal degrees of freedom.

7. What about Equivalence Principle in this framework? A possible quantum counterpart of Equivalence Principle could be that the longitudinal parts of the imbedding space mass squared operator for a given massless state equals to that for d'Alembert operator assignable to the modified Dirac action. The attempts to formulate this in more precise manner however seem to produce only additional troubles.

4.6 The emergence of Yangian symmetry and gauge potentials as duals of Kac-Moody currents

Yangian symmetry plays a key role in $\mathcal{N} = 4$ super-symmetric gauge theories. What is special in Yangian symmetry is that the algebra contains also multi-local generators. In TGD framework multilocality would naturally correspond to that with respect to partonic 2-surfaces and string world sheets and the proposal has been that the Super-Kac-Moody algebras assignable to string worlds sheets could generalize to Yangian.

Witten has written a beautiful exposition of Yangian algebras [B2]. Yangian is generated by two kinds of generators J^A and Q^A by a repeated formation of commutators. The number of commutations tells the integer characterizing the multi-locality and provides the Yangian algebra with grading by natural numbers. Witten describes a 2-dimensional QFT like situation in which one has 2-D situation and Kac-Moody currents assignable to real axis define the Kac-Moody charges as integrals in the usual manner. It is also assumed that the gauge potentials defined by the 1-form associated with the Kac-Moody current define a flat connection:

$$\partial_\mu j^A_\nu - \partial_\nu j^A_\nu + [j^A_\mu, j^A_\nu] = 0 \ . \tag{4.1}$$

This condition guarantees that the generators of Yangian are conserved charges. One can however consider alternative manners to obtain the conservation.

1. The generators of first kind - call them J^A - are just the conserved Kac-Moody charges. The formula is given by

$$J_A = \int_{-\infty}^{\infty} dx j^{A0}(x,t) \ . \tag{4.2}$$

2. The generators of second kind contain bi-local part. They are convolutions of generators of first kind associated with different points of string described as real axis. In the basic formula one has integration over the point of real axis.

$$Q^A = f^A_{BC}\int_{-\infty}^{\infty} dx \int_x^{\infty} dy j^{B0}(x,t)j^{C0}(y,t) - 2\int_{-\infty}^{\infty} j^A_x dx \quad . \tag{4.3}$$

   These charges are indeed conserved if the curvature form is vanishing as a little calculation shows.

How to generalize this to the recent context?

1. The Kac-Moody charges would be associated with the braid strands connecting two partonic 2-surfaces - Strands would be located either at the space-like 3-surfaces at the ends of the space-time surface or at light-like 3-surfaces connecting the ends. Modified Dirac equation would define Super-Kac-Moody charges as standard Noether charges. Super charges would be obtained by replacing the second quantized spinor field or its conjugate in the fermionic bilinear by particular mode of the spinor field. By replacing both spinor field and its conjugate by its mode one would obtain a conserved c-number charge corresponding to an anti-commutator of two fermionic super-charges. The convolution involving double integral is however not number theoretically attactive whereas single 1-D integrals might make sense.

2. An encouraging observation is that the Hodge dual of the Kac-Moody current defines the analog of gauge potential and exponents of the conserved Kac-Moody charges could be identified as analogs for the non-integrable phase factors for the components of this gauge potential. This identification is precise only in the approximation that generators commute since only in this case the ordered integral $P(exp(i\int Adx))$ reduces to $P(exp(i\int Adx))$.Partonic 2-surfaces connected by braid strand would be analogous to nearby points of space-time in its discretization implying that Abelian approximation works. This conforms with the vision about finite measurement resolution as discretization in terms partonic 2-surfaces and braids.

   This would make possible a direct identification of Kac-Moody symmetries in terms of gauge symmetries. For isometries one would obtain color gauge potentials and the analogs of gauge potentials for graviton field (in TGD framework the contraction with M^4 vierbein would transform tensor field to 4 vector fields). For Kac-Moody generators corresponding to holonomies one would obtain electroweak gauge potentials. Note that super-charges would give rise to a collection of spartners of gauge potentials automatically. One would obtain a badly broken SUSY with very large value of $\mathcal{N}$ defined by the number of spinor modes as indeed speculated earlier [K3].

3. The condition that the gauge field defined by 1-forms associated with the Kac-Moody currents are trivial looks unphysical since it would give rise to the analog of topological QFT with gauge potentials defined by the Kac-Moody charges. For the duals of Kac-Moody currents defining gauge potentials only covariant divergence vanishes implying that curvature form is

$$F_{\alpha\beta} = \epsilon_{\alpha\beta}[j_\mu, j^\mu] \quad , \tag{4.4}$$

   so that the situation does not reduce to topological QFT unless the induced metric is diagonal. This is not the case in general for string world sheets.

4. It seems however that there is no need to assume that j_μ defines a flat connection. Witten mentions that although the discretization in the definition of J^A does not seem to be possible, it makes sense for Q^A in the case of $G = SU(N)$ for any representation of G. For general G and its general representation there exists no satisfactory definition of Q. For certain representations, such as the fundamental representation of $SU(N)$, the definition of Q^A is especially simple. One just takes the bi-local part of the previous formula:

$$Q^A = f^A_{BC}\sum_{i<j} J^B_i J^C_j \ . \tag{4.5}$$

What is remarkable that in this formula the summation need not refer to a discretized point of braid but to braid strands ordered by the label i by requiring that they form a connected polygon. Therefore the definition of J^A could be just as above.

5. This brings strongly in mind the interpretation in terms of twistor diagrams. Yangian would be identified as the algebra generated by the logarithms of non-integrable phase factors in Abelian approximation assigned with pairs of partonic 2-surfaces defined in terms of Kac-Moody currents assigned with the modified Dirac action. Partonic 2-surfaces connected by braid strand would be analogous to nearby points of space-time in its discretization. This would fit nicely with the vision about finite measurement resolution as discretization in terms partonic 2-surfaces and braids.

The resulting algebra satisfies the basic commutation relations

$$\left[J^A, J^B\right] = f^{AB}_C J^C \ , \ \left[J^A, Q^B\right] = f^{AB}_C Q^C \ . \tag{4.6}$$

plus the rather complex Serre relations described in [B2].

4.7 Quantum criticality and electro-weak gauge symmetries

Quantum criticality is one of the basic guiding principles of Quantum TGD. What it means mathematically is however far from clear.

1. What is obvious is that quantum criticality implies quantization of Kähler coupling strength as a mathematical analog of critical temperature so that the theory becomes mathematically unique if only single critical temperature is possible. Physically this means the presence of long range fluctuations characteristic for criticality and perhaps assignable to the effective hierarchy of Planck constants having explanation in terms of effective covering spaces of the imbedding space. This hierarchy follows from the vacuum degeneracy of Kähler action, which in turn implies 4-D spin-glass degeneracy. It is easy to interpret the degeneracy in terms of criticality.

2. At more technical level one would expect criticality to corresponds deformations of a given preferred extremal defining a vanishing second variation of Kähler action. This is analogous to the vanishing of also second derivatives of potential function at extremum in certain directions so that the matrix defined by second derivatives does not have maximum rank. Entire hierarchy of criticalities is expected and a good finite-dimensional model is provided by the catastrophe theory of Thom [A5]. Cusp catastrophe [A1] is the simplest catastrophe one can think of, and here the folds of cusp where discontinuous jump occurs correspond to criticality with respect to one control variable and the tip to criticality with respect to both control variables.

3. I have discussed what criticality could mean for modified Dirac action [K2] and claimed that it leads to the existence of additional conserved currents defined by the variations which do not affect the value of Kähler action. These arguments are far from being mathematically rigorous and the recent view about the solutions of the modified Dirac equation predicting that the spinor modes are restricted to 2-D string world sheets requires a modification of these arguments.

In the following these arguments are updated. The unexpected result is that critical deformations induce conformal scalings of the modified metric and electro-weak gauge transformations of the induced spinor connection at X^2. Therefore holomorphy brings in the Kac-Moody symmetries associated with isometries of H (gravitation and color gauge group) and quantum criticality those associated with the holonomies of H (electro-weak-gauge group) as additional symmetries.

4.7.1 The variation of modes of the induced spinor field in a variation of space-time surface respecting the preferred extremal property

Consider first the variation of the induced spinor field in a variation of space-time surface respecting the preferred extremal property. The deformation must be such that the deformed modified Dirac operator D annihilates the modified mode. By writing explicitly the variation of the modified Dirac action (the action vanishes by modified Dirac equation) one obtains deformations and requiring its vanishing one obtains

$$\delta\Psi = D^{-1}(\delta D)\Psi \ . \tag{4.7}$$

D^{-1} is the inverse of the modified Dirac operator defining the analog of Dirac propagator and δD defines vertex completely analogous to $\gamma^k\delta A_k$ in gauge theory context. The functional integral over preferred extremals can be carried out perturbatively by expressing δD in terms of δh^k and one obtains stringy perturbation theory around X^2 associated with the preferred extremal defining maximum of Kähler function in Euclidian region and extremum of Kähler action in Minkowskian region (stationary phase approximation).

What one obtains is stringy perturbation theory for calculating n-points functions for fermions at the ends of braid strands located at partonic 2-surfaces and representing intersections of string world sheets and partonic 2-surfaces at the light-like boundaries of CDs. δD- or more precisely, its partial derivatives with respect to functional integration variables - appear atthe vertices located anywhere in the interior of X^2 with outcoming fermions at braid ends. Bosonic propagators are replaced with correlation functions for δh^k. Fermionic propagator is defined by D^{-1}.

After 35 years or hard work this provides for the first time a reasonably explicit formula for the N-point functions of fermions. This is enough since by bosonic emergence [K8] these N-point functions define the basic building blocks of the scattering amplitudes. Note that bosonic emergence states that bosons corresponds to wormhole contacts with fermion and antifermion at the opposite wormhole throats.

4.7.2 What critical modes could mean for the induced spinor fields?

What critical modes could mean for the induced spinor fields at string world sheets and partonic 2-surfaces. The problematic part seems to be the variation of the modified Dirac operator since it involves gradient. One cannot require that covariant derivative remains invariant since this would require that the components of the induced spinor connection remain invariant and this is quite too restrictive condition. Right handed neutrino solutions delocalized into entire X^2 are however an exception since they have no electro-weak gauge couplings and in this case the condition is obvious: modified gamma matrices suffer a local scaling for critical deformations:

$$\delta\Gamma^\mu = \Lambda(x)\Gamma^\mu \ . \tag{4.8}$$

This guarantees that the modified Dirac operator D is mapped to ΛD and still annihilates the modes of ν_R labelled by conformal weight, which thus remain unchanged.

What is the situation for the 2-D modes located at string world sheets? The condition is obvious. Ψ suffers an electro-weak gauge transformation as does also the induced spinor connection so that D_μ is not affected at all. Criticality condition states that the deformation of the space-time surfaces induces a conformal scaling of Γ^μ at X^2. It might be possible to continue this conformal scaling of the entire space-time sheet but this might be not necessary and this would mean that all critical deformations induced conformal transformations of the effective metric of the space-time surface defined by $\{\Gamma^\mu,\Gamma^\nu\} = 2G^{\mu\nu}$. Thus it seems that effective metric is indeed central concept (recall that if the conjectured quaternionic structure is associated with the effective metric, it might be possible to avoid problem related to the Minkowskian signature in an elegant manner).

In fact, one can consider even more general action of critical deformation: the modes of the induced spinor field would be mixed together in the infinitesimal deformation besides infinitesimal electroweak gauge transformation, which is same for all modes. This would extend electroweak gauge symmetry. Modified Dirac equation holds true also for these deformations. One might wonder whether the conjectured dynamically generated gauge symmetries assignable to finite measurement resolution could be generated in this manner.

The infinitesimal generator of a critical deformation J_M can be expressed as tensor product of matrix A_M acting in the space of zero modes and of a generator of infinitemal electro-weak gauge transformation $T_M(x)$ acting in the same manner on all modes: $J_M = A_M \otimes T_M(x)$. A_M is a spatially constant matrix and $T_M(x)$ decomposes to a direct sum of left- and right-handed $SU(2) \times U(1)$ Lie-algebra generators. Left-handed Lie-algebra generator can be regarded as a quaternion and right handed as a complex number. One can speak of a direct sum of left-handed local quaternion $q_{M,L}$ and right-handed local complex number $c_{M,R}$. The commutator $[J_M, J_N]$ is given by $[J_M, J_N] = [A_M, A_N] \otimes \{T_M(x), T_N(x)\} + \{A_M, A_N\} \otimes [T_M(x), T_N(x)]$. One has $\{T_M(x), T_N(x)\} = \{q_{M,L}(x), q_{N,L}(x)\} \oplus \{c_{M,R}(x), c_{N,R}(x)\}$ and $[T_M(x), T_N(x)] = [q_{M,L}(x), q_{N,L}(x)]$. The commutators make sense also for more general gauge group but quaternion/complex number property might have some deeper role.

Thus the critical deformations would induce conformal scalings of the effective metric and dynamical electro-weak gauge transformations. Electro-weak gauge symmetry would be a dynamical symmetry restricted to string world sheets and partonic 2-surfaces rather than acting at the entire space-time surface. For 4-D delocalized right-handed neutrino modes the conformal scalings of the effective metric are analogous to the conformal transformations of M^4 for $\mathcal{N} = 4$ SYMs. Also ordinary conformal symmetries of M^4 could be present for string world sheets and could act as symmetries of generalized Feynman graphs since even virtual wormhole throats are massless. An interesting question is whether the conformal invariance associated with the effective metric is the analog of dual conformal invariance in $\mathcal{N} = 4$ theories.

Critical deformations of space-time surface are accompanied by conserved fermionic currents. By using standard Noetherian formulas one can write

$$J_i^{\mu} = \overline{\Psi}\Gamma^{\mu}\delta_i\Psi + \delta_i\overline{\Psi}\Gamma^{\mu}\Psi \ . \tag{4.9}$$

Here $\delta\Psi_i$ denotes derivative of the variation with respect to a group parameter labeled by i. Since $\delta\Psi_i$ reduces to an infinitesimal gauge transformation of Ψ induced by deformation, these currents are the analogs of gauge currents. The integrals of these currents along the braid strands at the ends of string world sheets define the analogs of gauge charges. The interpretation as Kac-Moody charges is also very attractive and I have proposed that the 2-D Hodge duals of gauge potentials could be identified as Kac-Moody currents. If so, the 2-D Hodge duals of J would define the quantum analogs of dynamical electro-weak gauge fields and Kac-Moody charge could be also seen as non-integral phase factor associated with the braid strand in Abelian approximation (the interpretation in terms of finite measurement resolution is discussed earlier).

One can also define super currents by replacing $\overline{\Psi}$ or Ψ by a particular mode of the induced spinor field as well as c-number valued currents by performing the replacement for both $\overline{\Psi}$ or Ψ. As expected, one obtains a super-conformal algebra with all modes of induced spinor fields acting as generators of super-symmetries restricted to 2-D surfaces. The number of the charges which do not annihilate physical states as also the effective number of fermionic modes could be finite and this would suggest that the integer $\mathcal{N}$ for the supersymmetry in question is finite. This would conform with the earlier proposal inspired by the notion of finite measurement resolution implying the replacement of the partonic 2-surfaces with collections of braid ends.

Note that Kac-Moody charges might be associated with "long" braid strands connecting different wormhole throats as well as short braid strands connecting opposite throats of wormhole contacts. Both kinds of charges would appear in the theory.

4.7.3 What is the interpretation of the critical deformations?

Critical deformations bring in an additional gauge symmetry. Certainly not all possible gauge transformations are induced by the deformations of preferred extremals and a good guess is that they correspond to holomorphic gauge group elements as in theories with Kac-Moody symmetry. What is the physical character of this dynamical gauge symmetry?

1. Do the gauge charges vanish? Do they annihilate the physical states? Do only their positive energy parts annihilate the states so that one has a situation characteristic for the representation

of Kac-Moody algebras. Or could some of these charges be analogous to the gauge charges associated with the constant gauge transformations in gauge theories and be therefore non-vanishing in the absence of confinement. Now one has electro-weak gauge charges and these should be non-vanishing. Can one assign them to deformations with a vanishing conformal weight and the remaining deformations to those with non-vanishing conformal weight and acting like Kac-Moody generators on the physical states?

2. The simplest option is that the critical Kac-Moody charges/gauge charges with non-vanishing positive conformal weight annihilate the physical states. Critical degrees of freedom would not disappear but make their presence known via the states labelled by different gauge charges assignable to critical deformations with vanishing conformal weight. Note that constant gauge transformations can be said to break the gauge symmetry also in the ordinary gauge theories unless one has confinement.

3. The hierarchy of quantum criticalities suggests however entire hierarchy of electro-weak Kac-Moody algebras. Does this mean a hierarchy of electro-weak symmetries breakings in which the number of Kac-Moody generators not annihilating the physical states gradually increases as also modes with a higher value of positive conformal weight fail to annihilate the physical state?

   The only manner to have a hierarchy of algebras is by assuming that only the generators satisfying $n \ mod \ N = 0$ define the sub-Kac-Moody algebra annihilating the physical states so that the generators with $n \ mod \ N \neq 0$ would define the analogs of gauge charges. I have suggested for long time ago the relevance of kind of fractal hierarchy of Kac-Moody and Super-Virasoro algebras for TGD but failed to imagine any concrete realization.

   A stronger condition would be that the algebra reduces to a finite dimensional algebra in the sense that the actions of generators Q_n and Q_{n+kN} are identical. This would correspond to periodic boundary conditions in the space of conformal weights. The notion of finite measurement resolution suggests that the number of independent fermionic oscillator operators is proportional to the number of braid ends so that an effective reduction to a finite algebra is expected.

   Whatever the correct interpretation is, this would obviously refine the usual view about electro-weak symmetry breaking.

These arguments suggests the following overall view. The holomorphy of spinor modes gives rise to Kac-Moody algebra defined by isometries and includes besides Minkowskian generators associated with gravitation also SU(3) generators associated with color symmetries. Vanishing second variations in turn define electro-weak Kac-Moody type algebra.

Note that criticality suggests that one must perform functional integral over WCW by decomposing it to an integral over zero modes for which deformations of X^4 induce only an electro-weak gauge transformation of the induced spinor field and to an integral over moduli corresponding to the remaining degrees of freedom.

4.8 The importance of being light-like

The singular geometric objects associated with the space-time surface have become increasingly important in TGD framework. In particular, the recent progress has made clear that these objects might be crucial for the understanding of quantum TGD. The singular objects are associated not only with the induced metric but also with the effective metric defined by the anti-commutators of the modified gamma matrices appearing in the modified Dirac equation and determined by the Kähler action.

4.8.1 The singular objects associated with the induced metric

Consider first the singular objects associated with the induced metric.

1. At light-like 3-surfaces defined by wormhole throats the signature of the induced metric changes from Euclidian to Minkowskian so that 4-metric is degenerate. These surfaces are carriers of elementary particle quantum numbers and the 4-D induced metric degenerates locally to 3-D one at these surfaces.

2. Braid strands at light-like 3-surfaces are most naturally light-like curves: this correspond to the boundary condition for open strings. One can assign fermion number to the braid strands. Braid strands allow an identification as curves along which the Euclidian signature of the string world sheet in Euclidian region transforms to Minkowskian one. Number theoretic interpretation would be as a transformation of complex regions to hyper-complex regions meaning that imaginary unit i satisfying $i^2 = -1$ becomes hyper-complex unit e satisfying $e^2 = 1$. The complex coordinates $(z, \overline{z})$ become hyper-complex coordinates $(u = t + ex, v = t - ex)$ giving the standard light-like coordinates when one puts $e = 1$.

4.8.2 The singular objects associated with the effective metric

There are also singular objects assignable to the effective metric. According to the simple arguments already developed, string world sheets and possibly also partonic 2-surfaces are singular objects with respect to the effective metric defined by the anti-commutators of the modified gamma matrices rather than induced gamma matrices. Therefore the effective metric seems to be much more than a mere formal structure.

1. For instance, quaternionicity of the space-time surface could allow an elegant formulation in terms of the effective metric avoiding the problems due to the Minkowski signature. This is achieved if the effective metric has Euclidian signature $\epsilon \times (1, 1, 1, 1)$, $\epsilon = \pm 1$ or a complex counterpart of the Minkowskian signature $\epsilon(1, 1, -1, -1)$.

2. String word sheets and perhaps also partonic 2-surfaces could be understood as singularities of the effective metric. What happens that the effective metric with Euclidian signature $\epsilon \times (1, 1, 1, 1)$ transforms to the signature $\epsilon(1, 1, -1, -1)$ (say) at string world sheet so that one would have the degenerate signature $\epsilon \times (1, 1, 0, 0)$ at the string world sheet.

   What is amazing is that this works also number theoretically. It came as a total surprise to me that the notion of hyper-quaternions as a closed algebraic structure indeed exists. The hyper-quaternionic units would be given by $(1, I, iJ, iK)$, where i is a commuting imaginary unit satisfying $i^2 = -1$. Hyper-quaternionic numbers defined as combinations of these units with real coefficients do form a closed algebraic structure which however fails to be a number field just like hyper-complex numbers do. Note that the hyper-quaternions obtained with real coefficients from the basis $(1, iI, iJ, iK)$ fail to form an algebra since the product is not hyper-quaternion in this sense but belongs to the algebra of complexified quaternions. The same problem is encountered in the case of hyper-octonions defined in this manner. This has been a stone in my shoe since I feel strong disrelish towards Wick rotation as a trick for moving between different signatures.

3. Could also partonic 2-surfaces correspond to this kind of singular 2-surfaces? In principle, 2-D surfaces of 4-D space intersect at discrete points just as string world sheets and partonic 2-surfaces do so that this might make sense. By complex structure the situation is algebraically equivalent to the analog of plane with non-flat metric allowing all possible signatures (ϵ_1, ϵ_2) in various regions. At light-like curve either ϵ_1 or ϵ_2 changes sign and light-like curves for these two kinds of changes can intersect as one can easily verify by drawing what happens. At the intersection point the metric is completely degenerate and simply vanishes.

4. Replacing real 2-dimensionality with complex 2-dimensionality, one obtains by the universality of algebraic dimension the same result for partonic 2-surfaces and string world sheets. The braid ends at partonic 2-surfaces representing the intersection points of 2-surfaces of this kind would have completely degenerate effective metric so that the modified gamma matrices would vanish implying that energy momentum tensor vanishes as does also the induced Kähler field.

5. The effective metric suffers a local conformal scaling in the critical deformations identified in the proposed manner. Since ordinary conformal group acts on Minkowski space and leaves the boundary of light-cone invariant, one has two conformal groups. It is not however clear whether the M^4 conformal transformations can act as symmetries in TGD, where the presence of the induced metric in Kähler action breaks M^4 conformal symmetry. As found, also in TGD framework the Kac-Moody currents assigned to the braid strands generate Yangian: this is expected to be true also for the Kac-Moody counterparts of the conformal algebra associated with

quantum criticality. On the other hand, in twistor program one encounters also two conformal groups and the space in which the second conformal group acts remains somewhat mysterious object. The Lie algebras for the two conformal groups generate the conformal Yangian and the integrands of the scattering amplitudes are Yangian invariants. Twistor approach should apply in TGD if zero energy ontology is right. Does this mean a deep connection?

What is also intriguing that twistor approach in principle works in strict mathematical sense only at signatures $\epsilon \times (1, 1, -1 - 1)$ and the scattering amplitudes in Minkowski signature are obtained by analytic continuation. Could the effective metric give rise to the desired signature? Note that the notion of massless particle does not make sense in the signature $\epsilon \times (1, 1, 1, 1)$.

These arguments provide genuine a support for the notion of quaternionicity and suggest a connection with the twistor approach.

5 Twistor revolution and TGD

Lubos Motl wrote a nice summary about the talk of Nima Arkani Hamed about twistor revolution in Strings 2012 and gave also a link to the talk [B1]. It seems that Nima and collaborators are ending to a picture about scattering amplitudes which strongly resembles that provided bt generalized Feynman diagrammatics in TGD framework

TGD framework is much more general than $\mathcal{N} = 4$ SYM and is to it same as general relativity for special relativity whereas the latter is completely explicit. Of course, I cannot hope that TGD view could be taken seriously - at least publicly. One might hope that these approaches could be combined some day: both have a lot to give for each other. Below I compare these approaches.

5.1 The origin of twistor diagrammatics

In TGD framework zero energy ontology forces to replace the idea about continuous unitary evolution in Minkowski space with something more general assignable to causal diamonds (CDs), and S-matrix is replaced with a square root of density matrix equal to a hermitian l square root of density matrix multiplied by unitary S-matrix. Also in twistor approach unitarity has ceased to be a star actor. In p-Adic context continuous unitary time evolution fails to make sense also mathematically.

Twistor diagrammatics involves only massless on mass shell particles on both external and internal lines. Zero energy ontology (ZEO) requires same in TGD: wormhole lines carry parallely moving massless fermions and antifermions. The mass shell conditions at vertices are enormously powerful and imply UV finiteness. Also IR finiteness follows if external particles are massive.

What one means with mass is however a delicate matter. What does one mean with mass? I have pondered 35 years this question and the recent view is inspired by p-adic mass calculations and ZEO, and states that observed mass is in a well-defined sense expectation value of longitudinal mass squared for all possible choices of $M^2 \subset M^4$ characterizing the choices of quantization axis for energy and spin at the level of "world of classical worlds" (WCW) assignable with given causal diamond CD.

The choice of quantization axis thus becomes part of the geometry of WCW. All wormhole throats are massless but develop non-vanishing longitudinal mass squared. Gauge bosons correspond to wormhole contacts and thus consist of pairs of massless wormhole throats. Gauge bosons could develop 4-D mass squared but also remain massless in 4-D sense if the throats have parallel massless momenta. Longitudinal mass squared is however non-vanishing andp-adic thermodynamics predicts it.

5.2 The emergence of 2-D sub-dynamics at space-time level

Nima et al introduce ordering of the vertices in 4-D case. Ordering and related braiding are however essentially 2-D notions. Somehow 2-D theory must be a part of the 4-D theory also at space-time level, and I understood that understanding this is the challenge of the twistor approach at this moment.

The twistor amplitude can be represented as sum over the permutations of n external gluons and all diagrams corresponding to the same permutation are equivalent. Permutations are more like braidings since they carry information about how the permutation proceeded as a homotopy. Yang-Baxter equation emerges and states associativity of the braid group. The allowed braidings are minimal braidings in the sense that the repetitions of permutations of two adjacent vertices are not considered to be separate. Minimal braidings reduce to ordinary permutations. Nima also talks about affine braidings which I interpret as analogs of Kac-Moody algebras meaning that one uses projective representations which for Kac-Moody algebra mean non-trivial central extension. Perhaps the condition is that the square of a permutation permuting only two vertices which each other gives only a non-trivial phase factor. Lubos suggests an alternative interpretation which would select only special permutations and cannot be therefore correct.

There are rules of identifying the permutation associated with a given diagram involving only basic 3-gluon vertex with white circle and its conjugate. Lubos explains this "Mickey Mouse in maze" rule in his posting in detail: to determine the image $p(n)$ of vertex n in the permutation put a mouse in the maze defined by the diagram and let it run around obeying single rule: if the vertex is black turn to the right and if the vertex is white turn to the left. The mouse cannot remain in a loop: if it would do so, the rule would force it to run back to n after single full loop and one would have a fixed point: $p(n) = n$. The reduction in the number of diagrams is enormous: the infinity of different diagrams reduces to $n!$ diagrams!

What happens in TGD framework?

1. In TGD framework string world sheets and partonic 2-surfaces (or either or these if they are dual notions as conjectured) at space-time surface would define the sought for 2-D theory, and one obtains indeed perturbative expansion with fermionic propagator defined by the inverse of the modified Dirac operator and bosonic propagator defined by the correlation function for small deformations of the string world sheet. The vertices of twistor diagrams emerge as braid ends defining the intersections of string world sheets and partonic 2-surfaces.

   String model like description becomes part of TGD and the role of string world sheets in X^4 is highly analogous to that of string world sheets connecting branes in $AdS^5 \times S^5$ of $\mathcal{N} = 4$ SYM. In TGD framework 10-D $AdS^5 \times S^5$ is replaced with 4-D space-time surface in $M^4 \times CP_2$. The meaning of the analog of AdS^5 duality in TGD framework should be understood. In particular, it could it be that the descriptions involving string world sheets on one hand and partonic 2-surfaces - or 3-D orbits of wormhole throats defining the generalized Feynman diagram- on the other hand are dual to each other. I have conjectured something like this earlier but it takes some time for this kind of issues to find their natural answer.

2. As described in the article, string world sheets and partonic 2-surfaces emerge directly from the construction of the solutions of the modified Dirac equation by requiring conservation of em charge. This result has been conjectured already earlier but using other less direct arguments. 2-D "string world sheets" as sub-manifolds of the space-time surface make the ordering possible, and guarantee the finiteness of the perturbation theory involving n-point functions of a conformal QFT for fermions at wormhole throats and n-point functions for the deformations of the space-time surface. Conformal invariance should dictate these n-point functions to a high degree. In TGD framework the fundamental 3-vertex corresponds to joining of light-like orbits of three wormhole contacts along their 2-D ends (partonic 2-surfaces).

5.3 The emergence of Yangian symmetry

Yangian symmetry associated with the conformal transformations of M^4 is a key symmetry of Grassmannian approach. Is it possible to derive it in TGD framework?

1. TGD indeed leads to a concrete representation of Yangian algebra as generalization of color and electroweak gauge Kac-Moody algebra using general formula discussed in Witten's article about Yangian algebras (see the article).

2. Article discusses also a conjecture about 2-D Hodge duality of quantized YM gauge potentials assignable to string world sheets with Kac-Moody currents. Quantum gauge potentials are defined only where they are needed - at string world sheets rather than entire 4-D space-time.

3. Conformal scalings of the effective metric defined by the anticommutators of the modified gamma matrices emerges as realization of quantum criticality. They are induced by critical deformations (second variations not changing Kähler action) of the space-time surface. This algebra can be generalized to Yangian using the formulas in Witten's article (see the article).

4. Critical deformations induce also electroweak gauge transformations and even more general symmetries for which infinitesimal generators are products of U(n) generators permuting n modes of the modified Dirac operator and infinitesimal generators of local electro-weak gauge transformations. These symmetries would relate in a natural manner to finite measurement resolution realized in terms of inclusions of hyperfinite factors with included algebra taking the role of gauge group transforming to each other states not distinguishable from each other.

5. How to end up with Grassmannian picture in TGD framework? This has inspired some speculations in the past. From Nima's lecture one however learns that Grassmannian picture emerges as a convenient parametrization. One starts from the basic 3-gluon vertex or its conjugate expressed in terms of twistors. Momentum conservation implies that with the three twistors λ_i or their conjugates are proportional to each other (depending on which is the case one assigns white or black dot with the vertex). This constraint can be expressed as a delta function constraint by introducing additional integration variables and these integration variables lead to the emergence of the Grassmannian $G_{n,k}$ where n is the number of gluons, and k the number of positive helicity gluons.

   Since only momentum conservation is involved, and since twistorial description works because only massless on mass shell virtual particles are involved, one is bound to end up with the Grassmannian description also in TGD.

5.4 The analog of AdS^5 duality in TGD framework

The generalization of AdS^5 duality of $\mathcal{N} = 4$ SYMs to TGD framework is highly suggestive and states that string world sheets and partonic 2-surfaces play a dual role in the construction of M-matrices. Some terminology first.

1. Let us agree that string world sheets and partonic 2-surfaces refer to 2-surfaces in the slicing of space-time region defined by Hermitian structure or Hamilton-Jacobi structure.

2. Let us also agree that *singular* string world sheets and partonic 2-surfaces are surfaces at which the *effective* metric defined by the anticommutators of the modified gamma matrices degenerates to effectively 2-D one.

3. Braid strands at wormhole throats in turn would be loci at which the *induced* metric of the string world sheet transforms from Euclidian to Minkowskian as the signature of induced metric changes from Euclidian to Minkowskian.

AdS^5 duality suggest that string world sheets are in the same role as string world sheets of 10-D space connecting branes in AdS^5 duality for $\mathcal{N} = 4$ SYM. What is important is that there should exist a duality meaning two manners to calculate the amplitudes. What the duality could mean now?

1. Also in TGD framework the first manner would be string model like description using string world sheets. The second one would be a generalization of conformal QFT at light-like 3-surfaces (allowing generalized conformal symmetry) defining the lines of generalized Feynman diagram. The correlation functions to be calculated would have points at the intersections of partonic 2-surfaces and string world sheets and would represent braid ends.

2. General Coordinate Invariance (GCI) implies that physics should be codable by 3-surfaces. Light-like 3-surfaces define 3-surfaces of this kind and same applies to space-like 3-surfaces. There are also preferred 3-surfaces of this kind. The orbits of 2-D wormhole throats at which 4-metric degenerates to 3-dimensional one define preferred light-like 3-surfaces. Also the space-like 3-surfaces at the ends of space-time surface at light-like boundaries of causal diamonds (CDs) define preferred space-like 3-surfaces. Both light-like and space-like 3-surfaces should code for the same physics and therefore their intersections defining partonic 2-surfaces plus the 4-D tangent space data at them should be enough to code for physics. This is strong form of GCI implying effective 2-dimensionality. As a special case one obtains singular string world sheets at which the effective metric reduces to 2-dimensional and singular partonic 2-surfaces defining the wormhole throats. For these 2-surfaces situation could be especially simple mathematically.

3. The guess inspired by strong GCI is that string world sheet -partonic 2-surface duality holds true. The functional integrals over the deformations of 2 kinds of 2-surfaces should give the same result so tthat functional integration over either kinds of 2-surfaces should be enough. Note that the members of a given pair in the slicing intersect at discrete set of points and these points define braid ends carrying fermion number. Discretization and braid picture follow automatically.

4. Scattering amplitudes in the twistorial approach could be thus calculated by using *any* pair in the slicing - or only either member of the pair if the analog of AdS^5 duality holds true as argued. The possibility to choose any pair in the slicing means general coordinate invariance as a symmetry of the Kähler metric of WCW and of the entire theory suggested already early: Kähler functions for difference choices in the slicing would differ by a real part of holomorphic function and give rise to same Kähler metric of "world of classical worlds" (WCW). For a general pair one obtains functional integral over deformations of space-time surface inducing deformations of 2-surfaces with only other kind 2-surface contributing to amplitude. This means the analog of stringy QFT: Minkowskian or Euclidian string theory depending on choice.

5. For singular string world sheets and partonic 2-surfaces an enormous simplification results. The propagators for fermions and correlation functions for deformations reduce to 1-D instead of being 2-D: the propagation takes place only along the light-like lines at which the string world sheets with Euclidian signature (inside CP_2 like regions) change to those with Minkowskian signature of induced metric. The local reduction of space-time dimension would be very real for particles moving along sub-manifolds at which higher dimensional space-time has reduced metric dimension: they cannot get out from lower-D sub-manifold. This is like ending down to 1-D black hole interior and one would obtain the analog of ordinary Feynman diagrammatics. This kind of Feynman diagrammatics involving only braid strands is what I have indeed ended up earlier so that it seems that I can trust good intuition combined with a sloppy mathematics sometimes works;-).

   These singular lines represent orbits of point like particles carrying fermion number at the orbits of wormhole throats. Furthermore, in this representation the expansions coming from string world sheets and partonic 2-surfaces are identical automatically. This follows from the fact that only the light-like lines connecting points common to singular string world sheets and singular partonic 2-surfaces appear as propagator lines!

6. The TGD analog of AdS^5 duality of $\mathcal{N} = 4$ SUSYs would be trivially true as an identity in this special case, and the good guess is that it is true also generally. One could indeed use integral over either string world sheets or partonic 2-sheets to deduce the amplitudes.

What is important to notice that singularities of Feynman diagrams crucial for the Grassmannian approach of Nima and others would correspond at space-time level 2-D singularities of the effective metric defined by the modified gamma matrices defined as contractions of canonical momentum currents for Kähler action with ordinary gamma matrices of the imbedding space and therefore directly reflecting classical dynamics.

5.5 Problems of the twistor approach from TGD point of view

Twistor approach has also its problems and here TGD suggests how to proceed. Signature problem is the first problem.

1. Twistor diagrammatics works in a strict mathematical sense only for $M^{2,2}$ with metric signature (1,1,-1,-1) rather than M^4 with metric signature (1,-1,-1,-1). Metric signature is wrong in the physical case. This is a real problem which must be solved eventually.

2. Effective metric defined by anticommutators of the modified gamma matrices (to be distinguished from the induced gamma matrices) could solve that problem since it would have the correct signature in TGD framework (see the article). String world sheets and partonic 2-surfaces would correspond to the 2-D singularities of this effective metric at which the even-even signature (1,1,1,1) changes to even-even signature (1,1,-1,-1). Space-time at string world sheet would

become locally 2-D with respect to effective metric just as space-time becomes locally 3-D with respect to the induced metric at the light-like orbits of wormhole throats. String world sheets become also locally 1-D at light-like curves at which Euclidian signature of world sheet in induced metric transforms to Minkowskian.

3. Twistor amplitudes are indeed singularities and string world sheets implied in TGD framework by conservation of em charge would represent these singularities at space-time level. At the end of the talk Nima conjectured about lower-dimensional manifolds of space-time as representation of space-time singularities. Note that string world sheets and partonic 2-surfaces have been part of TGD for years. TGD is of course to $\mathcal{N} = 4$ SYM what general relativity is for the special relativity. Space-time surface is dynamical and possesses induced and effective metrics rather than being flat.

Second limitation is that twistor diagrammatics works only for planar diagrams. This is a problem which must be also fixed sooner or later.

1. This perhaps dangerous and blasphemous statement that I will regret it some day but I will make it;-). Nima and others have not yet discovered that $M^2 \subset M^4$ must be there but will discover it when they begin to generalize the results to non-planar diagrams and realize that Feynman diagrams are analogous to knot diagrams in 2-D plane (with crossings allowed) and that this 2-D plane must correspond to $M^2 \subset M^4$. The different choices of causal diamond CD correspond to different choices of M^2 representing choice of quantization axes 4-momentum and spin. The integral over these choices guarantees Lorentz invariance. Gauge conditions are modified: longitudinal M^2 projection of massless four-momentum is orthogonal to polarization so that three polarizations are possible: states are massive in longitudinal sense.

2. In TGD framework one replaces the lines of Feynman diagrams with the light-like 3-surfaces defining orbits of wormhole throats. These lines carry many fermion states defining braid strands at light-like 3-surfaces. There is internal braiding associated with these braid strands. String world sheets connect fermions at different wormhole throats with space-like braid strands. The M^2 projections of generalized Feynman diagrams with 4-D "lines" replaced with genuine lines define the ordinary Feynman diagram as the analog of braid diagram. The conjecture is that one can reduce non-planar diagrams to planar diagrams using a procedure analogous to the construction of knot invariants by un-knotting the knot in Alexandrian manner by allowing it to be cut temporarily.

3. The permutations of string vertices emerge naturally as one constructs diagrams by adding to the interior of polygon sub-polygons connected to the external vertices. This corresponds to the addition of internal partonic two-surfaces. There are very many equivalent diagrams of this kind. Only permutations matter and the permutation associated with a given diagram of this kind can be deduced by the Mickey-Mouse rule described explicitly by Lubos. A connection with planar operads is highly suggestive and also conjecture already earlier in TGD framework.

6 $M^8 - H$ duality, preferred extremals, criticality, and Mandelbrot fractals

$M^8 - H$ duality [K9] represents an intriguing connection between number theory and TGD but the mathematics involved is extremely abstract and difficult so that I can only represent conjectures. In the following the basic duality is used to formulate a general conjecture for the construction of preferred extremals by iterative procedure. What is remarkable and extremely surprising is that the iteration gives rise to the analogs of Mandelbrot fractals and space-time surfaces can be seen as fractals defined as fixed sets of iteration. The analogy with Mandelbrot set can be also seen as a geometric correlate for quantum criticality.

6.1 $M^8 - H$ duality briefly

$M^8 - M^4 \times CP_2$ duality [?]tates that certain 4-surfaces of M^8 regarded as a sub-space of complexified octonions can be mapped in a natural manner to 4-surfaces in $M^4 \times CP_2$: this would mean that

$M^4 \times CP_2$ and therefore also the symmetries of standard model would have purely number theoretical meaning.

Consider a distribution of two planes $M^2(x)$ integrating to a 2-surface $\tilde{M}^2$ with the property that a fixed 1-plane M^1 defining time axis globally is contained in each $M^2(x)$ and therefore in $\tilde{M}^2$. M^1 defines real axis of octonionic plane M^8 and M^2(x) a local hyper-complex plane. Quaternionic subspaces with this property can be parameterized by points of CP_2: this leads to $M^8 - H$ duality as can be shown by a simple argument.

1. Hyper-octonionic subspace of complexified octonions is obtained by multiplying octonionic imaginary units by commuting imaginary unit. This does not bring anything new as far as automorphisms are considered so that it is enough to consider octonions (so that M^2 is replaced with C). Octonionic frame consists of orthogonal octonionic units. The space of octonionic frames containing sub-frame spanning fixed C is parameterized by $SU(3)$. The reason is that complexified octonionic units can be decomposed to the representations of $SU(3) \subset G_2$ as $1 + 1 + 3 + \overline{3}$ and the sub-frame 1+1 spans the preferred C.

2. The quaternionic planes H are represented by frames defined by four unit octonions spanning a quaternionic plane. Fixing $C \subset H$ means fixing the 1+1 part in the above decomposition. The sub-group of $SU(3)$ leaving the plane H invariant can perform only a rotation in the plane defined by two quaternionic units in 3. This sub-group is $U(2)$ so that the space of quaternionic planes $H \supset C$ is parameterized by $SU(3)/U(2) = CP_2$.

3. Therefore quaternionic tangent plane $H \supset C$ can be mapped to a point of CP_2. In particular, any quaternionic surface in E^8, whose tangent plane at each point is quaternionic and contains C, can be mapped to $E^4 \times CP_2$ by mapping the point $(e_1, e_2) \in E^4 \times E^4$ to $(e_1, s) \in e^4 \times CP_2$. The generalization from E^8 to M^8 is trivial. This is essentially what $M^8 - H$ duality says.

This can be made more explicit. Define quaternionic surfaces in M^8 as 4-surfaces, whose tangent plane is quaternionic at each point x and contains the local hyper-complex plane $M^2(x)$ and is therefore labelled by a point $s(x) \in CP_2$. One can write these surfaces as union over 2-D surfaces associated with points of $\tilde{M}^2$:

$$X^4 = \cup_{x \in \tilde{M}^2} X^2(x) \subset E^6 \quad .$$

These surfaces can be mapped to surfaces of $M^4 \times CP_2$ via the correspondence $(m(x), e(x)) \to (m, s(T(X^4(x))$. Also the image surface contains at given point x the preferred plane $M^2(x) \supset M^1$. One can also write these surfaces as union over 2-D surfaces associated with points of $\tilde{M}^2$:

$$X^4 = \cup_{x \in \tilde{M}^2} X^2(x) \subset E^2 \times CP_2 \quad .$$

One can also ask what are the conditions under which one can map surfaces $X^4 = \cup_{x \in \tilde{M}^2} X^2 \subset E^2 \times CP_2$ to 4-surfaces in M^8. The map would be given by $(m, s) \to (m, T^4(s)$ and the surface would be of the form as already described. The surface X^4 must be such that the distribution of 4-D tangent planes defined in M^8 is integrable and this gives complicated integrability conditions. One might hope that the conditions might hold true for preferred extremals satisfying some additional conditions.

One must make clear that the conditions discussed above do not allow most general possible surface. The point is that for preferred extremals with Euclidian signature of metric the M^4 projection is 3-dimensional and involves light like projection. Here the fact that light-like line $L \subset M^2$ spans M^2 in the sense that the complement of its orthogonal complement in M^8 is M^2. Therefore one could consider also more general solution ansatz for which one has

$$X^4 = \cup_{x \in L_x \subset \tilde{M}^2} X^3(x) \subset E^2 \times CP_2 \quad .$$

One can also consider co-quaternionic surfaces as surfaces for which tangent space is in the dual of a quaternionic subspace. This says that the normal bundle rather than tangent bundle is quaternionic.

6.2 The integrability conditions

The integrability conditions are associated with the expression of tangent vectors of $T^(X^4)$ as a linear combination of coordinate gradients ∇m^k, where m^k denote the coordinates of M^8. Consider the 4 tangent vectors $e_{i)}$ for the quaternionic tangent plane (containing $M^2(x)$) regarded as vectors of M^8. $e_{i)}$ have components $e^k_{i)}$, $i = 1, .., 4$, $k = 1, ..., 8$. One must be able to express $e_{i)}$ as linear combinations of coordinate gradients ∇m^k:

$$e^k_{i)} = e^\alpha_{i)} \partial_\alpha m^k \quad .$$

Here x^α and e^k denote coordinates for X^4 and M^8. By forming inner products of of $e_{i)}$ one finds that matrix $e^\alpha_{i)}$ represents the components of vierbein at X^4. One can invert this matrix to get $e^{i)}_\alpha$ satisfying $e^{i)}_\alpha e^\beta_{i)} = \delta^\beta_\alpha$ and $e^{i)}_\alpha e^\alpha_{j)} = \delta^i_j$. One can solve the coordinate gradients ∇m^k from above equation to get

$$\partial_\alpha m^k = e^{i)}_\alpha e^k_{i)} \equiv E^k_\alpha \quad .$$

The integrability conditions follow from the gradient property and state

$$D_\alpha E^k_\beta = D_\beta E^k_\alpha \quad .$$

One obtains $8 \times 6 = 48$ conditions in the general case. The slicing to a union of two-surfaces labeled by $M^2(x)$ reduces the number of conditions since the number of coordinates m^k reduces from 8 to 6 and one has 36 integrability conditions but still them is much larger than the number of free variables- essentially the six transversal coordinates m^k.

For co-quaternionic surfaces one can formulate integrability conditions now as conditions for the existence of integrable distribution of orthogonal complements for tangent planes and it seems that the conditions are formally similar.

6.3 How to solve the integrability conditions and field equations for preferred extremals?

The basic idea has been that the integrability condition characterize preferred extremals so that they can be said to be quaternionic in a well-defined sense. Could one imagine solving the integrability conditions by some simple ansatz utilizing the core idea of $M^8 - H$ duality? What comes in mind is that M^8 represents tangent space of $M^4 \times CP_2$ so that one can assign to any point (m, s) of 4-surface $X^4 \subset M^4 \times CP_2$ a tangent plane $T^4(x)$ in its tangent space M^8 identifiable as subspace of complexified octonions in the proposed manner. Assume that $s \in CP_2$ corresponds to a fixed tangent plane containing M^2_x, and that all planes M^2_x are mapped to the same standard fixed hyper-octonionic plane $M^2 \subset M^8$, which does not depend on x. This guarantees that s corresponds to a unique quaternionic tangent plane for given $M^2(x)$.

Consider the map $T \circ s$. The map takes the tangent plane T^4 at point $(m, e) \in M^4 \times E^4$ and maps it to $(m, s_1 = s(T^4)) \in M^4 \times CP_2$. The obvious identification of quaternionic tangent plane at (m, s_1) would be as T^4. One would have $T \circ s = Id$. One could do this for all points of the quaternion surface $X^4 \subset E^4$ and hope of getting smooth 4-surface $X^4 \subset H$ as a result. This is the case if the integrability conditions at various points $(m, s(T^4)(x)) \in H$ are satisfied. One could equally well start from a quaternionic surface of H and end up with integrability conditions in M^8 discussed above. The geometric meaning would be that the quaternionic surface in H is image of quaternionic surface in M^8 under this map.

Could one somehow generalize this construction so that one could iterate the map $T \circ s$ to get $T \circ s = Id$ at the limit? If so, quaternionic space-time surfaces would be obtained as limits of iteration for rather arbitrary space-time surface in either M^8 or H. One can also consider limit cycles, even limiting manifolds with finite-dimension which would give quaternionic surfaces. This would give a connection with chaos theory.

1. One could try to proceed by discretizing the situation in M^8 and H. One does not fix quaternionic surface at either side but just considers for a fixed $m_2 \in M^2(x)$ a discrete collection X $\{(T^4_i\} \supset M^2(x)$ of quaternionic planes in M^8. The points $e_{2,i} \subset E^2 \subset M2 \times E^2 = M^4$ are not fixed. One can also assume that the points $s_i = s(T^4_i)$ of CP_2 defined by the collection of

planes form in a good approximation a cubic lattice in CP_2 but this is not absolutely essential. Complex Eguchi-Hanson coordinates ξ^i are natural choice for the coordinates of CP_2. Assume also that the distances between the nearest CP_2 points are below some upper limit.

2. Consider now the iteration. One can map the collection X to H by mapping it to the set $s(X)$ of pairs $((m_2, s_i)$. Next one must select some candidates for the points $e_{2,i} \in E^2 \subset M^4$ somehow. One can define a piece-wise linear surface in $M^4 \times CP_2$ consisting of 4-planes defined by the nearest neighbors of given point $(m_2, e_{2,i}, s_i)$. The coordinates $e_{2,i}$ for $E^2 \subset M^4$ can be chosen rather freely. The collection $(e_{2,i,i})$ defines a piece-wise linear surface in H consisting of four-cubes in the simplest case. One can hope that for certain choices of $e_{2,i}$ the four-cubes are quaternionic and that there is some further criterion allowing to choose the points $e_{2,i}$ uniquely. The tangent planes contain by construction $M^2(x)$ so that the product of remaining two spanning tangent space vectors (e_3, e_4) must give an element of M^2 in order to achieve quaternionicity. Another natural condition would be that the resulting tangent planes are not only quaternionic but also as near as possible to the planes T_i^4. These conditions allow to find $e_{2,i}$ giving rise to geometrically determined quaternionic tangent planes as near as possible to those determined by s^i.

3. What to do next? Should one replace the quaternionic planes T_i^4 with geometrically determined quaternionic planes as near as possible to them and map them to points s_i slightly different from the original one and repeat the procedure? This would not add new points to the approximation, and this is an unsatisfactory feature.

4. Second possibility is based on the addition of the quaternionic tangent planes obtained in this manner to the original collection of quaternionic planes. Therefore the number of points in discretization increases and the added points of CP_2 are as near as possible to existing ones. One can again determine the points $e_{2,i}$ in such a manner that the resulting geometrically determined quaternionic tangent planes are as near as possible to the original ones. This guarantees that the algorithm converges.

5. The iteration can be stopped when desired accuracy is achieved: in other words the geometrically determined quaternionic tangent planes are near enough to those determined by the points s_i. Also limit cycles are possible and would be assignable to the transversal coordinates e_{2i} varying periodically during iteration. One can quite well allow this kind of cycles, and they would mean that e_2 coordinate as a function of CP_2 coordinates characterizing the tangent plane is many-valued. This is certainly very probable for solutions representable locally as graphs $M^4 \rightarrow CP_2$. In this case the tangent planes associated with distant points in E^2 would be strongly correlated which must have non-trivial physical implications. The iteration makes sense also p-adically and it might be that in some cases only p-adic iteration converges for some value of p.

It is not obvious whether the proposed procedure gives rise to a smooth or even continuous 4-surface. The conditions for this are geometric analogs of the above described algebraic integrability conditions for the map assigning to the surface in $M^4 \times CP_2$ a surface in M^8. Therefore $M^8 - H$ duality could express the integrability conditions and preferred extremals would be 4-surfaces having counterparts also in the tangent space M^8 of H.

One might hope that the self-referentiality condition $s \circ T = Id$ for the CP_2 projection of (m, s) or its fractal generalization could solve the complicated integrability conditions for the map T. The image of the space-time surface in tangent space M^8 in turn could be interpreted as a description of space-time surface using coordinates defined by the local tangent space M^8. Also the analogy for the duality between position and momentum suggests itself.

Is there any hope that this kind of construction could make sense? Or could one demonstrate that it fails? If s would fix completely the tangent plane it would be probably easy to kill the conjecture but this is not the case. Same s corresponds for different planes M_x^2 to different point tangent plane. Presumably they are related by a local G_2 or $SO(7)$ rotation. Note that the construction can be formulated without any reference to the representation of the imbedding space gamma matrices in terms of octonions. Complexified octonions are enough in the tangent space of M^8.

6.4 Connection with Mandelbrot fractal and fractals as fixed sets for iteration

The occurrence of iteration in the construction of preferred extremals suggests a deep connection with the standard construction of 2-D fractals by iteration - about which Mandelbrot fractal [A4, A2] is the canonical example. $X^2(x)$ (or $X^3(x)$ in the case of light-like $L(x) \subset M^2(x)$) could be identified as a union of orbits for the iteration of $s \circ T$. The appearance of the iteration map in the construction of solutions of field equation would answer positively to a long standing question whether the extremely beautiful mathematics of 2-D fractals could have some application at the level of fundamental physics according to TGD.

X^2 (or X^3) would be completely analogous to Mandelbrot set in the sense that it would be boundary separating points in two different basis of attraction. In the case of Mandelbrot set iteration would take points at the other side of boundary to origin on the other side and to infinity. The points of Mandelbrot set are permuted by the iteration. In the recent case $s \circ T$ maps X^2 (or X^3) to itself. This map need not be diffeomorphism or even continuous map. The criticality of X^2 (or X^3) could be seen as a geometric correlate for quantum criticality.

In fact, iteration plays a very general role in the construction of fractals. Very general fractals can be defined as fixed sets of iteration and simple rules for iteration produce impressive representations for fractals appearing in Nature. The book of Michael Barnsley [A3] gives fascinating pictures about fractals appearing in Nature using this method. Therefore it would be highly satisfactory if space-time surfaces would be in a well-defined sense fixed sets of iteration. This would be also numerically beautiful aspect since fixed sets of iteration can be obtained as infinite limit of iteration for almost arbitrary initial set. This construction recipe would also give a concrete content for the notion measurement resolution at the level of construction of preferred extremals.

What is intriguing is that there are several very attractive approaches to the construction of preferred extremals. The challenge of unifying them still remains to be met.

7 Appendix: Hamilton-Jacobi structure

In the following the definition of Hamilton-Jacobi structure is discussed in detail.

7.1 Hermitian and hyper-Hermitian structures

The starting point is the observation that besides the complex numbers forming a number field there are hyper-complex numbers. Imaginary unit i is replaced with e satisfying $e^2 = 1$. One obtains an algebra but not a number field since the norm is Minkowskian norm $x^2 - y^2$, which vanishes at light-cone $x = y$ so that light-like hypercomplex numbers $x \pm e$) do not have inverse. One has "almost" number field.

Hyper-complex numbers appear naturally in 2-D Minkowski space since the solutions of a massless field equation can be written as $f = g(u = t - ex) + h(v = t + ex)$ whith $e^2 = 1$ realized by putting $e = 1$. Therefore Wick rotation relates sums of holomorphic and antiholomorphic functions to sums of hyper-holomorphic and anti-hyper-holomorphic functions. Note that u and v are hyper-complex conjugates of each other.

Complex n-dimensional spaces allow Hermitian structure. This means that the metric has in complex coordinates $(z_1, ..., z_n)$ the form in which the matrix elements of metric are nonvanishing only between z_i and complex conjugate of z_j. In 2-D case one obtains just $ds^2 = g_{z\overline{z}}dzd\overline{z}$. Note that in this case metric is conformally flat since line element is proportional to the line element $ds^2 = dzd\overline{z}$ of plane. This form is always possible locally. For complex n-D case one obtains $ds^2 = g_{i\overline{j}}dz^i d\overline{z}^j$. $g_{i\overline{j}} = \overline{g_{j\overline{i}}}$ guaranteing the reality of ds^2. In 2-D case this condition gives $g_{z\overline{z}} = \overline{g_{z\overline{z}}}$.

How could one generalize this line element to hyper-complex n-dimensional case. In 2-D case Minkowski space M^2 one has $ds^2 = g_{uv}dudv$, $g_{uv} = 1$. The obvious generalization would be the replacement $ds^2 = g_{u_iv_j}du^i dv^j$. Also now the analogs of reality conditions must hold with respect to $u_i \leftrightarrow v_i$.

7.2 Hamilton-Jacobi structure

Consider next the path leading to Hamilton-Jacobi structure.

4-D Minkowski space $M^4 = M^2 \times E^2$ is Cartesian product of hyper-complex M^2 with complex plane E^2, and one has $ds^2 = dudv + dzd\overline{z}$ in standard Minkowski coordinates. One can also consider more general integrable decompositions of M^4 for which the tangent space $TM^4 = M^4$ at each point is decomposed to $M^2(x) \times E^2(x)$. The physical analogy would be a position dependent decomposition of the degrees of freedom of massless particle to longitudinal ones ($M^2(x)$: light-like momentum is in this plane) and transversal ones ($E^2(x)$: polarization vector is in this plane). Cylindrical and spherical variants of Minkowski coordinates define two examples of this kind of coordinates (it is perhaps a good exercize to think what kind of decomposition of tangent space is in question in these examples). An interesting mathematical problem highly relevant for TGD is to identify all possible decompositions of this kind for empty Minkowski space.

The integrability of the decomposition means that the planes $M^2(x)$ are tangent planes for 2-D surfaces of M^4 analogous to Euclidian string world sheet. This gives slicing of M^4 to Minkowskian string world sheets parametrized by euclidian string world sheets. The question is whether the sheets are stringy in a strong sense: that is minimal surfaces. This is not the case: for spherical coordinates the Euclidian string world sheets would be spheres which are not minimal surfaces. For cylindrical and spherical coordinates hower $M^2(x)$ integrate to plane M^2 which is minimal surface.

Integrability means in the case of $M^2(x)$ the existence of light-like vector field J whose flow lines define a global coordinate. Its existence implies also the existence of its conjugate and together these vector fields give rise to $M^2(x)$ at each point. This means that one has $J = \Psi\nabla\Phi$: Φ indeed defines the global coordinate along flow lines. In the case of M^2 either the coordinate u or v would be the coordinate in question. This kind of flows are called Beltrami flows. Obviously the same holds for the transversal planes E^2.

One can generalize this metric to the case of general 4-D space with Minkowski signature of metric. At least the elements g_{uv} and $g_{z\overline{z}}$ are non-vanishing and can depend on both u, v and $z, \overline{z}$. They must satisfy the reality conditions $g_{z\overline{z}} = \overline{g_{z\overline{z}}}$ and $g_{uv} = \overline{g_{vu}}$ where complex conjugation in the argument involves also $u \leftrightarrow v$ besides $z \leftrightarrow \overline{z}$.

The question is whether the components g_{uz}, g_{vz}, and their complex conjugates are non-vanishing if they satisfy some conditions. They can. The direct generalization from complex 2-D space would be that one treats u and v as complex conjugates and therefore requires a direct generalization of the hermiticity condition

$$g_{uz} = \overline{g_{v\overline{z}}} \quad , \quad g_{vz} = \overline{g_{u\overline{z}}} \quad .$$

This would give complete symmetry with the complex 2-D (4-D in real sense) spaces. This would allow the algebraic continuation of hermitian structures to Hamilton-Jacobi structures by just replacing i with e for some complex coordinates.

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