Differential Representation of Exact Value of the nth Partial Sum $\sum_{i=1}^{n} \frac{1}{a+(i-1)d}$ of General Harmonic Series

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Abstract

In order to find a differential representation of the nth partial sum $\sum_{i=1}^{n} \frac{1}{a+(i-1)d}$ of the general harmonic series $\sum_{i=1}^{\infty} \frac{1}{a+(i-1)d}$, a theoretical study has been performed analytically. Moreover, some special cases of it such as harmonic number have been discussed.

Keywords: General Harmonic Progression, Harmonic Progression, Harmonic Series

1 Introduction

Problems concerning harmonic series find many applications in various branches of number theory. Moreover; harmonic numbers, sometimes loosely termed harmonic series, are closely associated with the Riemann zeta function and appear in various expressions for various special functions. The form of the general harmonic series can be represented as

$$\sum_{n=1}^{\infty} \frac{1}{a+(n-1)d}, \tag{1}$$

where a and d are real numbers such that $a+(n-1)d \neq 0$ for all values of n. The general harmonic series (1) and its particular cases such as harmonic series are known as divergent

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series. The nth partial sum $S_n = \sum_{i=1}^{n} \frac{1}{a+(i-1)d}$ in exact and convenient form of the general harmonic series (1) and its special cases has been proposed in my recent communications [1, 2, 3]. In the present paper it is devoted to find a differential representation of the nth partial sum $\sum_{i=1}^{n} \frac{1}{a+(i-1)d}$ of the general harmonic series (1).

2 Differential form

From the representation of the nth partial sum of the general harmonic series (1) and its special cases which given in [1, 2, 3], the nth partial sum of the general harmonic series (1) can be expressed as

$$S_n = \sum_{i=1}^{n} \frac{1}{a+(i-1)d} \tag{2}$$

$$= \frac{d}{da}\left\{\log\left(a^n + a^{n-1}d\sum_{i=1}^{n-1} a_i + a^{n-2}d^2 \sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + a^{n-3}d^3 \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + ad^{n-1}a_1 a_2 \cdots a_{n-1}\right)\right\} \tag{3}$$

$$= \frac{\begin{array}{c} na^{n-1} + (n-1)a^{n-2}d\sum_{i=1}^{n-1} a_i + (n-2)a^{n-3}d^2 \sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j \\ +(n-3)a^{n-4}d^3 \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + d^{n-1}a_1 a_2 \cdots a_{n-1} \end{array}}{a^n + a^{n-1}d\sum_{i=1}^{n-1} a_i + a^{n-2}d^2 \sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + a^{n-3}d^3 \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + ad^{n-1}a_1 a_2 \cdots a_{n-1}}, \tag{4}$$

where $a_i = i$ for $i = 1,\ 2,\ \cdots,\ n-1$.

Let $\mathrm{f(a)} = \mathrm{a^n + a^{n-1}d}\sum_{\mathrm{i=1}}^{\mathrm{n-1}} \mathrm{a_i} + \mathrm{a^{n-2}d^2} \sum_{\substack{i,j=1\\(i<j)}}^{\mathrm{n-1}} \mathrm{a_i a_j} + \mathrm{a^{n-3}d^3} \sum_{\substack{i,j,k=1\\(i<j<k)}}^{\mathrm{n-1}} \mathrm{a_i a_j a_k} + \cdots + \mathrm{ad^{n-1}a_1 a_2 \cdots a_{n-1}},$ (5)

where $a_i = i$ for $i = 1,\ 2,\ \cdots,\ n-1$. The special cases of the equations (3) and (4) are given below.

2.1 Case I: harmonic number ($a = 1,\ d = 1$)

When $a = 1,\ d = 1$, the nth partial sum S_n known as a harmonic number which is generally denoted by H_n can be found in differential form

$$H_n = \frac{d}{da}\{\log(f(a))\}\Big|_{a=1,\ d=1} \tag{6}$$

$$= \frac{n + (n-1)\sum_{i=1}^{n-1} a_i + (n-2)\sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + (n-3)\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_{n-1}}{1 + \sum_{i=1}^{n-1} a_i + \sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_{n-1}} \tag{7}$$

$$= \frac{n + (n-1)\sum_{i=1}^{n-1} a_i + (n-2)\sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + (n-3)\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_{n-1}}{n!}, \tag{8}$$

in which $a_i = i$ for $i = 1,\ 2,\ \cdots,\ n-1$; $f(a)$ is given by the equation (5) and the special representation of $n!$ is provided in [4]. Derivation of equations (7) and (8) is given in [1].

2.2 Case II: $d = 1$

Here $d = 1$ is considered, but no restriction is imposed on a. In this case the partial sum S_n in differential form can be evaluated as

$$H_n = \frac{d}{da}\{\log(f(a))\}\Big|_{d=1} \tag{9}$$

$$= \frac{\begin{array}{c} na^{m-1} + (n-1)a^{m-2}\sum_{i=1}^{n-1} a_i + (n-2)a^{n-3}\sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j \\ +(n-3)a^{n-4}\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_{n-1} \end{array}}{a^n + a^{n-1}\sum_{i=1}^{n-1} a_i + a^{n-2}\sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + a^{n-3}\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + a a_1 a_2 \cdots a_{n-1}}, \tag{10}$$

where $a_i = i$ for $i = 1,\ 2,\ \cdots,\ n-1$ and $f(a)$ is given by (5). Derivation of equation (10) has been shown in [2].

2.3 Case III: $a = 1$

This situation is restricted to $a = 1$, but has no restriction on d. Then the partial sum S_n in differential form can be found as

$$S_n = \frac{d}{da}\left\{\log\left(f(a)\right)\right\}\Bigg|_{a=1} \tag{11}$$

$$= \frac{n + (n-1)d\sum_{i=1}^{n-1} a_i + (n-2)d^2 \sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + (n-3)d^3 \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + d^{n-1} a_1 a_2 \cdots a_{n-1}}{1 + d\sum_{i=1}^{n-1} a_i + d^2 \sum_{\substack{i,j=1\\(i<j)}}^{n-1} a_i a_j + d^3 \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} a_i a_j a_k + \cdots + d^{n-1} a_1 a_2 \cdots a_{n-1}}, \tag{12}$$

where $a_i = i$ for $i = 1,\ 2,\ \cdots,\ n-1$ and $f(a)$ is given by (5). The derivation of equation (12) can be done easily in a similar way as the equations (7), (8) and (10) have been derived in [1, 2, 3].

Note :

It is to be noted that the equations (3) and (4) in differential form of the nth partial sum of the general harmonic series (1) are valid when a and d are complex numbers.

References

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