Important Relations over Some Summations $\left(\sum_{\substack{i,j=1\\(i<j)}}^{n} ij,\quad \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk,\quad \sum_{\substack{i,j,k,l=1\\(i<j<k<l)}}^{n} ijkl,\ \cdots\right)$

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Abstract

The paper is focused to find important relations and identities over some summations for natural numbers such as $\sum_{\substack{i,j=1\\(i<j)}}^{n} ij,\ \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk,\ \sum_{\substack{i,j,k,l=1\\(i<j<k<l)}}^{n} ijkl,\ \cdots$. These relations are believed to find applications in the various branches of number theory particularly in the proposed theorems of Maiti [1, 2, 3, 4, 5] which help to represent the factorial $n!$ entirely new way and also help to exhibit the nth partial sum of the general harmonic series $\sum_{n=1}^{\infty} \frac{1}{a+(n-1)b}$ and its particular cases.

Keywords: Harmonic Number, Harmonic Progression, Harmonic Series

1 Introduction

In order to derive the new representation of $n!$ as well as the the nth partial sum of the general harmonic series $\sum_{n=1}^{\infty} \frac{1}{a+(n-1)b}$ and its particular cases, some summations for natural numbers such as $\sum_{i=1}^{n} i,\ \sum_{\substack{i,j=1\\(i<j)}}^{n} ij,\ \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk,\ \sum_{\substack{i,j,k,l=1\\(i<j<k<l)}}^{n} ijkl,\ \cdots$ have been used extensively in my

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recent communications [1, 2, 3, 4, 5]. It is known that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. To the best of the author knowledge and observation, however, calculation procedure/formula of the higher order summation or evaluation of the higher order summation from the previous order summation has not found in the existing literatures. This paper is devoted to calculate the summations $\sum_{\substack{i,j=1\\(i<j)}}^{n} ij$, $\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk$, $\sum_{\substack{i,j,k,l=1\\(i<j<k<l)}}^{n} ijkl$, $\cdots$ successively.

2 Calculations of the summations

2.1 Double summations

From the list $\sum_{\substack{i,j=1\\(i<j)}}^{2} ij = 2$, $\sum_{\substack{i,j=1\\(i<j)}}^{3} ij = 11 = 2+9 = \sum_{\substack{i,j=1\\(i<j)}}^{2} ij + 3\sum_{i=1}^{2} i$, $\sum_{\substack{i,j=1\\(i<j)}}^{4} ij = 35 = 11+24 = \sum_{\substack{i,j=1\\(i<j)}}^{3} ij + 4\sum_{i=1}^{3} i$, $\sum_{\substack{i,j=1\\(i<j)}}^{5} ij = 85 = 35+50 = \sum_{\substack{i,j=1\\(i<j)}}^{4} ij + 5\sum_{i=1}^{4} i$, $\cdots$ a conjecture naturally arises which indicate that $\sum_{\substack{i,j=1\\(i<j)}}^{n} ij = \sum_{\substack{i,j=1\\(i<j)}}^{n-1} ij + n\sum_{i=1}^{n-1} i$ for all natural values of $n > 2$. That can be proved easily by taking the help of mathematical induction.

This implies that

$$\sum_{\substack{i,j=1\\(i<j)}}^{n} ij - \sum_{\substack{i,j=1\\(i<j)}}^{n-1} ij = n\sum_{i=1}^{n-1} i = n \times \frac{(n-1)n}{2} = \frac{n^3-n^2}{2}. \tag{1}$$

Considering the equation (1) for $n = 3, 4, 5, \cdots, n$ and then adding these equations, it is found that

$$\sum_{\substack{i,j=1\\(i<j)}}^{n} ij - \sum_{\substack{i,j=1\\(i<j)}}^{2} ij = \frac{1}{2}\left(\sum_{i=3}^{n}(i^3-i^2)\right) = \frac{1}{2}\left(\sum_{i=1}^{n}(i^3-i^2)\right) - \frac{1}{2}\{(1^3+2^3)-(1^2+2^2)\}. \tag{2}$$

Thus

$$\sum_{\substack{i,j=1\\(i<j)}}^{n} ij = \frac{\sum_{i=1}^{n}(i^3-i^2)}{2} = \frac{n^2(n+1)^2}{8} - \frac{n(n+1)(2n+1)}{6} = \frac{(n-1)n(n+1)(3n+2)}{24} \text{ for } n \geq 2. \tag{3}$$

2.2 Triple summations

We have the list $\sum_{\substack{i,j,k=1\\(i<j<k)}}^{3} ijk = 6$, $\sum_{\substack{i,j,k=1\\(i<j<k)}}^{4} ijk = 50 = 6 + 44 = \sum_{\substack{i,j,k=1\\(i<j<k)}}^{3} ijk + 4\sum_{\substack{i,j=1\\(i<j)}}^{3} ij$, $\sum_{\substack{i,j,k=1\\(i<j<k)}}^{5} ijk = 225 = 50 + 175 = \sum_{\substack{i,j,k=1\\(i<j<k)}}^{4} ijk + 4\sum_{\substack{i,j=1\\(i<j)}}^{4} ij$, $\cdots$. This list indicate that $\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk = \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} ijk + n\sum_{\substack{i,j=1\\(i<j)}}^{n-1} ij$ for all natural values of $n > 3$. One can prove the above relation easily with the help of mathematical induction.

The said relation implies that

$$\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk - \sum_{\substack{i,j=1\\(i<j<k)}}^{n-1} ijk = n\sum_{\substack{i,j=1\\(i<j)}}^{n-1} ij = n \times \frac{(n-2)(n-1)n(3n-1)}{24} = \frac{3n^5 - 10n^4 + 9n^3 - 2n^2}{24}. \quad (4)$$

If we let us take the equation (4) for $n = 4,\ 5,\ 6,\ \cdots,\ n$ and let us permit to add these relations, we can calculate

$$\begin{aligned}
&\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk - \sum_{\substack{i,j,k=1\\(i<j<k)}}^{3} ijk \\
&= \frac{1}{24}\left(\sum_{i=4}^{n}(3i^5 - 10i^4 + 9i^3 - 2i^2)\right) \\
&= \frac{1}{24}\left(\sum_{i=1}^{n}(3i^5 - 10i^4 + 9i^3 - 2i^2)\right) - \frac{1}{24}\{3(1^5 + 2^5 + 3^5) - 10(1^4 + 2^4 + 3^4) + 9(1^3 + 2^3 + 3^3) \\
&\qquad - 2(1^2 + 2^2 + 3^2)\}. \qquad (5)
\end{aligned}$$

Hence

$$\sum_{\substack{i,j,k=1\\(i<j<k)}}^{n} ijk = \frac{\sum_{i=1}^{n}(3i^5 - 10i^4 + 9i^3 - 2i^2)}{24} \quad \text{for n} \geq 3 \qquad (6)$$

$$= \frac{1}{24}\left[3\left\{\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}\right\} - 10\left\{\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right\} + 9\left\{\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right\} - 2\left\{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right\}\right] \quad (7)$$

$$= \frac{1}{48}\left(n^6 - n^5 - 3n^4 + n^3 + 2n^2\right) \tag{8}$$

$$= \frac{(n-2)(n-1)n^2(n+1)^2}{48} \tag{9}$$

2.3 Multi summations

To calculate similar relations/equations of multi summations we can proceed similar way as the double and triple summation given above. Thus with the help of the mathematical induction it can be shown that

$$\sum_{\substack{i_1,i_2,i_3,\cdots,i_m=1\\(i_1<i_2<i_3<\cdots<i_m)}}^{n} i_1i_2i_3\cdots i_m = \sum_{\substack{i_1,i_2,i_3,\cdots,i_m=1\\(i_1<i_2<i_3<\cdots<i_m)}}^{n-1} i_1i_2i_3\cdots i_m + n\times \sum_{\substack{i_1,i_2,i_3,\cdots,i_{m-1}=1\\(i_1<i_2<i_3<\cdots<i_m)}}^{n-1} i_1i_2i_3\cdots i_{m-1}, \tag{10}$$

where n, $i_1 = 1$ are, $i_2 = 2$, $i_3 = 3$, $\cdots$ are natural numbers and $n > i_m$.

Using the equation (10), all the said double, triple and multiple summations can be evaluated successively. Up to $n = 9$, a table of the values of double, triple and multiple summations has been provided bellow successively.

n	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
1	1									
2	1	1								
3	1	3	2							
4	1	6	11	6						
5	1	10	35	50	24					
6	1	15	85	225	274	120				
7	1	21	175	735	1624	1764	720			
8	1	28	322	1960	6769	13132	13068	5040		
9	1	36	546	4536	22449	67284	118124	109584	40320	
10	1	45	870	9450	63273	269325	723680	1172700	1026576	362880

where $t_0 = constant = 1$, $t_1 = \sum_{i=1}^{n-1} i$, $t_2 = \sum_{\substack{i,j=1\\(i<j)}}^{n-1} ij$, $t_3 = \sum_{\substack{i,j,k=1\\(i<j<k)}}^{n-1} ijk$, $t_4 = \sum_{\substack{i_1,i_2,i_3i_4=1\\(i_1<i_2<i_3<i_4)}}^{n-1} i_1i_2i_3i_4$, $t_5 = \sum_{\substack{i_1,i_2,\cdots,i_5=1\\(i_1<i_2<\cdots<i_5)}}^{n-1} i_1i_2\cdots i_5$, $t_6 = \sum_{\substack{i_1,i_2,\cdots,i_6=1\\(i_1<i_2<\cdots<i_6)}}^{n-1} i_1i_2\cdots i_6$, $t_7 = \sum_{\substack{i_1,i_2,\cdots,i_7=1\\(i_1<i_2<\cdots<i_7)}}^{n-1} i_1i_2\cdots i_7$, $t_8 = \sum_{\substack{i_1,i_2,\cdots,i_8=1\\(i_1<i_2<\cdots<i_8)}}^{n-1} i_1i_2\cdots i_8$,

$$t_9 = \sum_{\substack{i_1, i_2, \cdots, i_9 = 1 \\ (i_1 < i_2 < \cdots < i_9)}}^{n-1} i_1 i_2 \cdots i_9.$$

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