The three primary motivations for this new math approach to countability

For hundreds of years, we've accepted the 'uncountability' of the irrational numbers, a proper subset of the reals – our common 'everyday' numbers. We've 'proven' this and accepted it as *fact*.

For math students, you won't find out until graduate school how 'different' the irrationals are from the rationals – the concept of Lebesgue measure is encountered in advanced undergraduate math courses for the first time (again, considering math students).

Math is the *most precise science*. We speak in precise terms, we define things precisely, and we state the exact conditions when a theorem is applicable. It's *the* reason why *all* other sciences use math. In a very real sense, using math automatically gives 'credibility' to ideas. (Abuses of this relationship abound with cranks/crackpots and frauds.)

To make the 'uncountable' countable, make this 'new' knowledge more accessible to younger students thereby encouraging young scientists and engineers to *understand* our common everyday numbers *more deeply* (and thereby encouraging more and better scientists and engineers), and make the unsatisfactory imprecision of 'uncountability' *precise*, we propose the following concepts.

Let's attempt to appropriately modify the notion of infinite countability. We can label infinite countability *any* way we want.. You can call it IC or modify the symbol for infinity, ∞. It almost doesn't matter *what* you label it as long as you're *consistent and explicit* about what it *means*. Let's go with ∞_c for simplicity. Now that we've labeled infinite countability, what are the allowable ways, within mathematics, that we can *modify* that symbol? Immediately obvious are multiplication and exponentiation: $2\infty_c$ and ∞_c^2, but right away, we *must decide* what that *means precisely*!

Let's ignore multiplication and focus on exponentiation. There seems to be more 'power' (to explain) and relevancy in that concept. If we *define* ∞_c^2 to mean the *number of elements* in $\mathbf{Z}^2$, the set of all ordered pairs of integers, we've made 'infinite exponentiation' *precise*. Right away, we have applicability: the rationals, $\mathbf{Q}$. One famous proof of the countability of the rationals actually ties these concepts together. From this proof alone, we can *inspect* the *order* of the rationals: 2. So we've *expanded* the notion of 'size' (of a set) to include countably infinite sets and orders of magnitude of those. This is actually *fundamental* and relevant as we shall soon see.

The size of a set no longer is limited to *finite* sets alone. We can *meaningfully* speak about countably infinite sets as well.. In this 'new framework' (it's not really new; it's 'been there' waiting for us to discover it), we can *precisely identify the number of elements* in countably infinite sets as well. The 'question' from mathematicians (rephrasing line one above) becomes: are the irrational numbers countable in the 'new' sense above? To *assume* “no” is *not* math/science. We *must* attempt to answer this question rationally – based on facts / logical arguments.

The following six brief articles, together, do exactly that. Please forgive any 'sloppiness in notation'.. When appropriate, definitions are precise and explicit. And, we find we give *way* too much 'authority' (associate 'deep meaning') to the transcendentals.. It's my sincere hopes these essays do *not* cause resentment or divisiveness within the math (and math education) community.. It is my hope we can work together formally developing these ideas, integrating them appropriately into mathematics, math education, and above all else – keeping the knowledge *accessible* to 'laypeople' and students..

A conjecture about real density

As mentioned previously, I'm using the mathematical term density rather loosely; here, what I mean by density is positive Lebesgue measure. It's clear the density of the reals is based on the density of its proper subset – the irrationals. But what of those? As with the previous article, let's return to binary representation since that's the simplest form:

1. `sqrt(10)      = 1.011010100000100...`
2. `sqrt(10)/10 =  .1 011010100000100...`
3. `sqrt(p) =     w.uis`

Before we discuss above, I need to cover another related topic: [irrationality measure](). Please visit the link and 'try to make some sense of it'. According to that article, there are basically three kinds of irrational numbers: algebraic, transcendental, and functional. An example of an *algebraic irrational* number is sqrt(2) (which is above on line one in binary format). An example of a *transcendental* irrational number is π (pi, the ratio of circumference to diameter of any circle). Please consult the link for examples of functional irrational numbers. Ready? Okay.. Now let's revisit what Jodie Foster talked about in Contact: the prime numbers: 2, 3, 5, 7, 11,... In binary: 10, 11, 101, 111, 1011,... Why? We'll see.. Prime numbers are unique whole numbers that are only divisible (with whole number result) by themselves and one. So now we're ready to review the statements above. Line one basically states in *binary form* that the square-root of two is equal to the value on the right-hand-side. Line two says what the square-root of two is *divided by two* – again, in binary representation. Notice *the value on line two is simply the value on line one with the 'binary point' moved one* – ***to the left***. Also notice that *both numbers are irrational*. Finally notice that *we can perform this operation forever* – dividing an irrational by two – *to get* ***another unique*** *irrational number*.

What about transcendental numbers? Off the top of my head, I can think of only two: π and *e*. That's pretty bad for a math major! One more thing before we state the conjecture. Line three above states the square-root of a prime number is equal to a ***whole*** number and ***u****nique-****i****nfinite-****s****equence* fractional part. We're ready to state the conjecture:

The density of the real numbers is based on the density of the algebraic irrationals which in turn is based on the (countably) infinite primes.

The point of this article is not to impress you with my mathematical fluency .. Allow me to digress.. Many years ago, I realized there was *absolutely nothing* special about me. The ***only*** things that would allow me to 'stand out' in a crowd was my mind and perspectives. I firmly believe I have an average IQ: 100, but I've tested far above.. Why? Because I've *trained my mind* .. More than this, I've come to understand *we all have gifts*; we simply need to discover them. There are mathematicians in the past who seemed connected to an 'infinite fountain of inspiration'. This is a historical fact. They give evidence, as does above, that we are merely *conduits* of inspiration and creativity. And finally, conduits of love. What more evidence of God do you need?

I challenge you to *prove* the conjecture above; I challenge you to find *your* unique gifts and inspirations – *whatever* your interests may be..

in true-love, sam

Open questions about irrational numbers

Is $\pi+e$ irrational? Is sqrt(2)+sqrt(3)? One of the open questions (they don't know either way) dealing with irrational numbers is: *is the sum of of two irrational numbers irrational?* To address this, let's think about the crucial property of irrationals: the unique-infinite-sequence (or uis) following the decimal/binary point representing the unique fractional part of an irrational number (as explained in the previous article). So in decimal format, $3-e$ and $4-\pi$ are what I call the uis-complements of e and π respectively. The reason I choose this term is because under addition, they produce whole numbers. It's not hard to visualize this critical feature in binary format; please try it.

When we fully understand $3-e$, $4-\pi$, 2-sqrt(2),.. are *unique* irrational numbers themselves, this so-called open question becomes almost trivial. From a probability standpoint anyways.. If we randomly choose a number from the real line, chances are it will be irrational is 100%! This is the astounding result from real analysis. What are the chances we choose a *particular* irrational number? 0%. So.. The chances $\pi+e$ or $\pi-e$ is *rational* is zero. When we think about the uniqueness of irrational complements, we're Confident about this claim.

Now we're ready to state another conjecture. Here, we introduce another term: distinct. For purposes of this discussion, I define distinct analogously to linear-independence. Two irrational numbers are distinct if you cannot make a liner combination of one out of the other using rational numbers. More explicitly, two irrational numbers are *distinct*, x and y, if you cannot find *any* non-zero rational numbers, a and b, such that $y = ax + b$ is true. Conjecture 1:

The sum of two distinct irrational numbers is irrational.

Conjecture 2:

The product or quotient of two distinct irrational numbers is irrational.

Conjecture 3:

The exponentiation of two distinct irrational numbers is irrational.

One and two together equate with conjecture 4:

The set of distinct irrational numbers is a field.

Why is this important? In a sense, we don't 'need' rational numbers in math! (Remember, the rationals include the integers – the 'counting numbers'.) Any rational number can be 'approximated' with a nearby irrational. In a very real sense, we can dispense with *all* other numbers! ..Of course, these statements are facetious computationally but illustrate the overwhelming density of the irrationals.

Convince yourself: if you doubt above, try the operations above on your calculator. For instance, try using sqrt(2) and sqrt(3) as components and look at the results you get. Use any two distinct irrational numbers.. If you find two that violate a statement above, *let me know!* We all can make mistakes!

Counting infinities

This may not be the way they 'do it' in number theory nor may it be 'standard notation' but.. It seems the most intuitive approach toward measuring the actual density of irrationals and rationals. The existing notation in number theory seems to be deliberately obfuscating and I will deliberately avoid it here .. From the previously outlined view, the simplest algebraic irrational described in the simplest way is the square-root of 2 in binary:
sqrt(10) = 1.011010100000100...
This is a *single instance* of an irrational number. The whole portion is immaterial at this moment. The irrationality of the number is *completely defined* by the fractional portion. If we flip the nth bit, this implies with this *one* irrational number, we can create a countably infinite set. Let's notate this with the symbol inf^1-sub-c or simply inf^1-c. Now if we flip the first *and* nth bit, this creates a *whole new* countably infinite set of irrationals with size inf^1-c. Carry this operation on forever implies the size of the set of irrationals we can create from the *single* irrational number sqrt(10) is inf^2-c! This is already quite revealing! From a *single instance*, we've created an order 2 infinite set of irrationals! (If not obvious, inf = infinity – the symbol for infinity, the 'lazy eight'. ^ = super-script or exponentiation. sub-c or -c indicate a sub-script – here, it means 'c' for countable.)

This directly implies the O(sqrt(p)) = 3 or S(sqrt(p)) = inf^3-c where O and S mean order and size respectively. We've made tremendous progress simply by using convenient and intuitive notation! We've begun to actually *count* the irrational numbers! Now, let's use the field properties of the irrationals to count some more! From those, we know O(sqrt(p1)+sqrt(p2)) = 3 and O(sqrt(p1)sqrt(p2)) = 3 which implies S(*,+ o sqrt(p)) ≥ 2inf^3-c. In English, the size of the set with multiplication and addition operating on the set of the root-primes is greater than or equal to two times order-three countably infinite. Before we stop this tack, let's consider one final implication. If we consider sequences of operations on distinct root-primes, they produce order 4 sets:
∑(2,inf)S(∑(i,j)sqrt(p-i)) = inf^4-c and ∑(2,inf)S(∏(i,j)sqrt(p-i)) = inf^4-c
which implies S(**I**) ≥ 2inf^4-c. The irrational numbers are at least order 4.

Let's try another way of looking at this.. Let's explicitly look at addition, subtraction, multiplication, and division as operations on root-primes: +, -, *, 1/ . Let's denote distinct root-primes as a1, a2, a3, and a4. There's only one way to add: +a1+a2+a3+a4. But there are four ways to add: -a1+a2+a3+a4 because we can move the minus sign to four distinct positions. Similarly, -a1-a2+a3+a4 can be added six ways based on the positions of the two minus signs.. And, -a1-a2-a3+a4, 4. Finally, -a1-a2-a3-a4 can only be added one way. That totals 16 ways to add/subtract four distinct numbers. Similarly, we can explore the 16 ways to combine a1a2a3a4. The interesting question is: how many ways can we arithmetically combine four distinct numbers based on the *four* basic operations? The answer is 4^4. Examples are: a1-a2a3/a4 and (1/a1)a2-a3+a4. This implies S(+,-,*,1/ o sqrt(p)) = inf^4-c. So we've shown another way that the irrationals are order 4.

What about the rationals? What's their order? Let's consider the unit-interval. If we look at the whole number 1 then divide by two progressively until that limit is zero, we get the size of that subset as inf^1-c. But we don't have to start with 1; we can start with *any* rational number and move toward another limit rationally. That implies the size of the rationals is inf^2-c. Convention claims they're countably infinite which they are but.. They're clearly not order 1 .. I spent some time trying to convince myself the order of the rationals is greater than 2. I was trying to enumerate all unique complementary sequences. The challenging problem is to find complementary sequences so that their union is the rationals. Try this yourself. Here, what we mean by complementary is disjoint – no common elements. And, the notion relates to completeness so.. We're ready for a new conjecture:

The union of all complementary rational sequences is the set of rational numbers with order 2.

What's the point of all this? To simply befuddle? [wink] No. The purpose of this approach is to get a 'handle' (intuitive idea) of the respective densities of the rationals and irrationals. To say one is countably infinite and sparsely dense while the other is uncountably infinite and dense is *not* adequate. We need new measures that differentiate *exactly why* they're different. The notions introduced above allow us to create an analogy so we can understand the relative densities *completely* .. Here is the analogy we've been looking for.. The density of the irrationals compared to the density of the rationals is *perfectly analogous* to the following scenario. Imagine you pick a random point in **R^3**, what are the chances your number is on the x-axis? Clearly zero. What are the chances it's elsewhere? 100%. **R^3** is two orders of magnitude 'more infinite' than **R^1** (order 12 vs order 4). So we're ready for another conjecture:

The order of the reals is 4; the size is inf^4-c + inf^2-c based on the respective sizes of irrationals and rationals.

This approach gives us a much more intuitive understanding of relative densities. No longer are sets simply 'countably infinite' or 'uncountably dense'.. We have a very precise language now to describe different 'levels of infinities' .. Let's practice our new skills and identify sizes and orders of common sets.. Ready?

What's the size and order of **I^2**? It should be clear the size is *not* 2inf^4-c but inf^8-c. The order is therefore 8.

What's the size and order of **R^2**? By now, the 'technique' should be clear:
S-**R^2** = (inf^4-c + inf^2-c)^2 = (S-**I**)^2 + (S-**Q**)^2 + 2(S-**I**)(S-**Q**)
inf^8-c + inf^4-c + 2inf^6-c
The corresponding order is 8 clearly from the dominating subset **I^2**.

Calculating the size and order of **R^3** is left to the reader.

What about **C**, the set of complex numbers? This may appear confusing at first because a complex number is typically written as a+bi where a is the real part and b is the coefficient of the imaginary part. Don't worry, the complex-plane corresponds to **R^2** so we can 'cheat' by simply inspecting those associated values. S-**C** = S-**R^2** and O-**C** = O-**R^2**.

Size is an appropriate term because it essentially equates with the number of elements in a set. But the old notation typically found in texts on number theory simply makes the knowledge *inaccessible* to many, if not most, students. If the arguments in these last few articles are sound, we have made number theory and real analysis accessible to high-school students. It's a tremendous leap forward for the mathematical sciences and education .. With this perspective, there's not many open questions left, but thank God there are:
Is π an element of {+,-,*,1/ o sqrt(p)}?
Is *e* an element of {+,-,*,1/ o sqrt(p)}?
What's S-**T** where **T** are the transcendentals?
From one perspective, they're inconsequential. From another, 'just' as important as sqrt(2) (possibly more-so). For instance, if they're not elements of the set based on root-primes, they may represent a *subfield* of **I**. As the primes may represent a 'basis' for **I**, π and *e* may be a basis for **T**. It's an interesting problem I pose for the reader.

Pi and e seem important but..

They're not. Before we get into that, I need to provide another argument about the size of the rationals. Let's consider the unit interval. Let's take the midpoint. Now let's take midpoints of *those* two intervals created by taking the first midpoint. Let's keep doing that forever and try to visualize what we just did. Essentially, we're enumerating all the rationals inside the unit interval. Let's try to count all those midpoints we just tried to identify.. 1+2+4+8+16+32.. It's clearly infinity but what order? Most certainly, it's higher than order 1. Let's recall, we identify order 1 with the integers. When we say countable with order 1, we simply mean a one-to-one correspondence with the naturals. Now.. Back to the problem.. How do we relate that infinite sum above to the naturals? They're *both* infinite.. The 'trick' is to think about *rates* of approaching infinity.. The naturals approach infinity in a very regular 'normal' stepwise way: 1, 2, 3, 4, 5,.. - very linear. The terms of the sum above are: 1, 2, 4, 8, 16, 32,.. which, if we think carefully, are on a line with slope 2, y=2x. A sum is a discrete version of an integral so.. If we integrate 2x, we get x^2. Boom: we've just shown another way – the order of the rationals is 2. You might object: “Hey! We only calculated for the unit interval! What about the rest!” At first, I mistakenly thought we must multiply by inf^1-c to get an order 3 set but.. If you think about the start of the problem, we just *arbitrarily* started with the unit interval – we could have chosen *any* large interval in **R**. So.. There we are. :) (This morning, I believed I had made an error in postulating the order of the rationals is 2. I started trying to construct rationals, in binary form, from 'scratch'. I found at least three ways to to make them. If so, then that would imply the rationals are a 3rd order set. It's allowable if we consider the irrationals are 4th order and so the analogy would be, in that case, **R^3** to **R^2**. The probability of choosing a point on the real-plane vs **R^3** is zero but.. With the evidence above, I'm still going with 2nd order rationals.)

Now we're ready to consider the very simple illumination I had today.. Ask yourself the question: where do π and *e come from*? The most natural place π comes from is **R^2**. From an engineering perspective, *e* comes from **C**. Then the 'ahah!' hit me.. We ascribe *so much* importance to those two numbers: π and *e*.. But.. When you think about it carefully, they're 'nothing' but projections from **R^2** onto **R**! They're artifacts from **R^2**! We *do* take them *very* seriously in engineering but.. Now that I think of them in that way – not that important..

You may be able to create an order 2 set from π/*e* alone but.. So you can from *any* algebraic irrational. And, we still don't know those sets are disjoint. Perhaps some/all of those created irrationals belong to the set created by the root-primes. Further, we don't know if π/*e* are *in* the order 4 set created from all root-primes *themselves*. It's interesting from a historical standpoint but in my view, not mathematical.

Countable bases

Conjecture:
Generally, when a set has a countable basis, it has finite order; all finite-order sets have countable bases.

An example of a basis is the natural numbers. These can be used to construct all positive rational numbers. We can inspect the order of this set. The definition of a rational number is an integer divided by another: a/b where a and b are integers, b non-zero. Don't fret about that one point; taking a single point out of a countably infinite set still leaves a countably infinite set. There are an infinite number of choices for a and an infinite number of choices for b. This equates with an *ordered pair* of infinite combinations of integers; the rationals have order 2 .. About the irrationals.. We have previously conjectured that the countable root-primes form a basis for them. We could have chosen the non-perfect-squares as a basis. That's a larger set; wouldn't it be a 'more complete' basis? It's a fair question but I chose the root-primes for simplicity. Admittedly, root-naturals include root-non-perfect-squares include root-primes so the root-primes are a 'fractional subset' of an order 1 set. My point is they're *all* order 1 sets. Again, I chose the primes out of simplicity. With or without the field assumptions, the irrationals have finite order. If the field assumptions are incorrect, some of the operations on irrationals produce rational numbers – producing a *smaller* resultant set. *The real 'trick' for me is to convince mathematicians the irrationals have a* ***countable basis***.. Let's attempt to use (the distasteful) proof-by-contradiction. Suppose the irrationals did not have a countable basis. That would imply you could come up with at least one instance that violated the premise. So you claim to have found a number that is irrational and cannot be created with a countable basis. I say that's impossible. Why? Let's look at the so-called transcendental number *e*. It can be approximated to any desired precision by a finite sum. It has been proven to be *equal* to a (*countably!*) infinite sum. We've created a 'super special' transcendental number from a *countable basis*! That implies *any* irrational number can be created with a countable basis! In other words, the irrationals have finite order.

As you can see, we've moved from number theory to set theory rather quickly. Extending the definition of countability to higher-order sets has allowed this progress. This is the nature of 'fundamental' discoveries such as: the calculus, relativity, and the helical structure of DNA .. I'm sure there are other implications in set theory I cannot 'see right away'.. This is only the beginning .. Back to the irrationals (and their order).. Looking at this problem heuristically, we know it's finite, we know it's greater than 2, we know it's less than or equal to 4 depending on field properties.. That leaves two options: 3 or 4. As mentioned before, 3 is acceptable because it preserves the Lebesgue property. So the real 'final question' (it all depends on) is: do field operations on two distinct irrationals produce another distinct irrational? If we 'try it by hand' with examples and pay particular attention to uises, we find: yes. Based on probability, the order of the irrationals is most likely 4.

Based on some fairly simple assumptions, we've been able to determine (from several views) that the irrationals (and therefore reals) are a *countable* set, that there is *precise meaning* associated with this notion, and that there are *significant implications* (such as in set theory) .. Before we consider one last way to count the irrationals, let's state a final conjecture that relates to above:
Conjecture:
There is only one way to create a truly uncountable set: infinite exponentiation.

Today I discovered one 'final' argument about the order of the irrationals that should appeal to many mathematicians. It has to do with rational numbers and rational scaling. If some/many are not happy with the root-primes approach, if the primes are not 'dense enough' to convince you they're a sufficient basis for the irrationals, then let's consider positive rational numbers.. If we examine the following: (a/b)sqrt(c/d) where c/d>0, d and b $\neq$ 0, a, b, c, and d are integers, we see that (a/b) is a scaling and translation factor – and – sqrt(c/d) acts as the 'irrational seed' to modify. If we can accept this scenario as dense enough to create *all* irrationals, we don't need to consider operations on them .. If we implement this solution on a computer, we essentially truncate all numbers to a finite sequence. From this perspective, it's easy to see the scenario above is sufficient to create *any* truncated irrational. Once we realize this fact, we can understand the *length* of truncation does not matter. So truncated or not, we need *at most* four integers to uniquely determine an irrational number. This directly implies the order of the irrationals is at most 4 .. If we can show 3 is not enough (from this perspective), we've essentially proven the order of the irrationals.. Let's attempt to do just that.. If we only use three integers to specify a truncated irrational, we only need to miss ***one*** *of all possible* to prove we need more than 3. If we fix one of the four to equal 1, we've lost one degree-of-freedom in the scenario. If it's a, then (a/b) can only be a negative or positive *fraction*. We've missed *all* multiples greater than 1. Therefore, we've just proven we need 4 integers to specify *all possible* (truncated and non-truncated) irrationals. The order of the irrationals is 4.

What this means *exactly* is this.. The size of **I** is the same as the size of $\mathbf{Z}^4$. The number of elements in **I** is the same as the number of elements in $\mathbf{Z}^4$. This is 'uncanny' wisdom .. A few weeks ago, the real numbers were 'uncountable' due to the 'uncountability' of the irrationals.. Now, they're countable and match the size of a known set.. All because a shift in perspective.. So.. Heuristics is absolutely correct: how you *frame* a problem *determines feasibility* of a solution (set) .. If we didn't have heuristics, what *would* we have? *Not science* – that's for sure..

Uncountable sets

Restating the last conjecture:
There is only one way to create a truly uncountable set: infinite exponentiation.

An example of an uncountable basis is $\mathbf{Z}^{\infty}$. That's a truly uncountable set. Other examples of uncountable sets are: $\mathbf{I}^{\infty}$ and $\mathbf{R}^{\infty}$ .. I debated writing/including this brief treatment of 'infinite dimensional spaces' honestly for lack of interest. I've heard of the concept being useful in conventional quantum mechanics .. If you've read any of my physics articles, you know I have no great love of the primary assumptions behind the Standard Model nor convention's *abuse of math* by trying to give credibility to the SM by associating parts of 'advanced math' to it. What does group theory or infinite dimensional spaces have to do with physics? Ask a 'mathematical physicist'.. This critical error in science is much like the axiom: publish or perish. So academicians publish because they *must* in order to keep their jobs.. No matter now inane.. No matter how trivial.. That way, advances in science almost never come in spurts; they're almost always incremental. It takes someone like me who, practically from birth, forcibly if required, *must* think outside the box. How else could an 'average shmuck' come up with a viable deterministic alternative to the SM or above? Divine inspiration? Atheists refuse to accept that.. No.. I sincerely doubt infinite dimensional spaces have anything to do with physical reality and therefore, not much point to study in math.. Perhaps I should have left this section unwritten after all.. [wink] Ah live and learn..

Addendum: after several days checking my own logic, I was *convinced* I was *absolutely wrong*: the reals are uncountable; they're an infinite-order set; period. What convinced me I was wrong *yet again* is the following realization/conjecture:
Any infinite-order set that can be approximated, to <u>any</u> arbitrary precision, by a finite-order set is <u>equivalent</u> to that set.

Because I can show you explicitly why the reals are uncountable and that I can find a particular set to model them with arbitrary precision, the statement above is self-evident.. I challenge you to find a counter-example .. To put the statement above in terms of the reals: because I found an order 4 set to approximate the irrationals (to any arbitrary precision): $\mathbf{Z}^{4}$, the irrationals are *equivalent* to $\mathbf{Z}^{4}$.

“When something is ***really important*** to you, *never* let *anyone* tell you 'it can't be done'.”

Whether or not you can, *you* ***must believe*** *you can!*