Matrix Exponential

Pierre-Yves Gaillard

Abstract. Let a be an element of a finite dimensional $\mathbb{C}$-algebra with 1. Then there is a unique polynomial f_a such that $f_a(a) = e^a$ and $\deg f_a < \dim \mathbb{C}[a]$. We give an explicit formula for f_a.

For any element a of any finite dimensional $\mathbb{C}$-algebra with 1, let $f_a \in \mathbb{C}[X]$ be the unique polynomial of degree $< \dim \mathbb{C}[a]$ satisfying

$$f_a(a) = e^a.$$

[Here X is an indeterminate.]

Let $m \in \mathbb{C}[X]$ be the minimal polynomial of a, let λ be a multiplicity $\mu(\lambda)$ root of m, and let $x(\lambda)$ be the image of X in $\mathbb{C}[X]/(X-\lambda)^{\mu(\lambda)}$.

Then f_a can be computed by solving, thanks to Taylor's Formula, the congruences

$$f_a \equiv f_{x(\lambda)} \mod (X-\lambda)^{\mu(\lambda)},$$

where λ runs over the roots of m. Explicitly:

$$f_{x(\lambda)} = e^\lambda \sum_{n<\mu(\lambda)} \frac{(X-\lambda)^n}{n!} \quad ,$$

$$f_a = \sum_\lambda T_\lambda \left(f_{x(\lambda)} \frac{(X-\lambda)^{\mu(\lambda)}}{m} \right) \frac{m}{(X-\lambda)^{\mu(\lambda)}} \quad ,$$

where T_λ means "degree $< \mu(\lambda)$ Taylor polynomial at $X = \lambda$".

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