Fractional Circuit Elements: memristors, memcapacitors, meminductors and beyond ⋆

Xiong Wang *

* *Center of Chaos and Complex Network,*
Department of Electronic Engineering, City University of Hong Kong,
Hong Kong SAR, China.(e-mail: wangxiong8686@gmail.com).

Abstract: Memristor was postulated by Chua in 1971 by analyzing mathematical relations between pairs of fundamental circuit variables and realized by HP laboratory in 2008. This relation can be generalized to include any class of two-terminal devices whose properties depend on the state and history of the system. These are called memristive systems, including current-voltage for the memristor, charge-voltage for the memcapacitor, and current-flux for the meminductor. This paper further enlarge the family of elementary circuit elements, in order to model many irregular and exotic nondifferentiable phenomena which are common and dominant to the nonlinear dynamics of many biological, molecular and nanodevices.

Keywords: Memristor,Circuit elements, Memory, Fractional dynamic, Nanoscale device

1. INTRODUCTION

In the classical circuit theory, there are three fundamental two-terminal circuit elements: resistors, capacitors and inductors. The definition of electric current i is the time derivative of electric charge q, and according to Faraday's law voltage v is the time derivative of magnetic flux φ. In 1971 Leon Chua noted that there ought to be six mathematical relations connecting pairs of the four fundamental circuit variables, i, v, q and φ,, seeChua (1971); Strukov et al. (2008); Williams (2008). and Chua and Kang further generalized the concept to a broader class of systems, called memristive systems, Chua and Kang (1976).

2. CLASSICAL NONLINEAR CIRCUIT THEORY

In a mathematical sense, nonlinear circuit theory is mainly about the study of constrained ordinary differential equations involving time and four N-dimensional variables q_n, φ_n, i_n, v_n (standing for charge, flux, current and voltage, respectively), with the following restrictions:

From definition of variables
The definition of electric current i is the time derivative of electric charge q, and according to Faraday's law voltage v is the time derivative of magnetic flux φ. So the $2N$ differential relations $q'_n = i_n$, $\varphi'_n = v_n$ always hold;

From laws of circuit theory
The vectors v, i, satisfy a total amount of N linearly independent relations $Bv = 0$, $Di = 0$ coming from Kirchhoff laws;

From characteristics of device
The characteristics of devices define N more relations among the circuit variables.

This means that nonlinear circuit models can be generally written in the form

$$\frac{dq}{dt} = i \tag{1a}$$
$$\frac{d\varphi}{dt} = v \tag{1b}$$
$$0 = f(q, \varphi, i, v, t), \tag{1c}$$

where f captures Kirchhoff laws but also the constitutive relations of all circuit devices.

3. MEMRISTOR

The three circuit elements connect pairs of the four circuit variables:

(1) *Resistance* R is the rate of change of voltage with current ($R = \frac{dv}{di}$),
(2) *Capacitance* C is the rate of change of charge with voltage ($C = \frac{dq}{dv}$),
(3) *Inductance* L is the rate of change of magnetic flux with current ($L = \frac{d\varphi}{di}$).

From symmetry arguments, Chua reasoned that there should be a fourth fundamental element, which he called a *memristor* (short for memory resistor) M, for a functional relation between charge and magnetic flux, $M = \frac{d\varphi}{dq}$.

In 2008 May, scientists at Hewlett-Packard Laboratories published a paper in *Nature* announcing the invention of the fourth element—memristor.Strukov et al. (2008) In that paper, Strukov et al presented a model to illustrate their invention. Their model provides a simple explanation

⋆ the paper is submitted for consideration of the best student paper award.

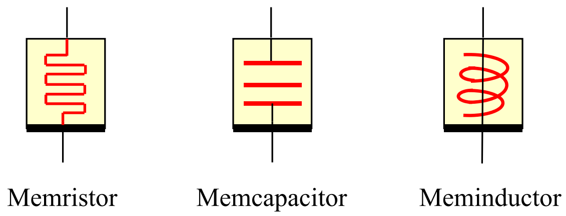

Fig. 1. (Color online) Symbols of memristor, memcapacitor, and meminductor. Figure adapted from Chua and Kang (1976)

for several puzzling phenomena in nanodevices. It shows that the nonlinear effects in electric circuit is ubiquitous in nanoscale electronics.

We can describe the memristor (say, in a charge-controlled setting) not by means of its flux-charge relation $\varphi = \phi(q)$ but, instead, via the differentiated relation $\varphi' = \phi'(q)q'$, that is,

$$v = M(q)i, \tag{2}$$

where $M(q) = \phi'(q)$ is the so-called *memristance.*

4. MEMRISTIVE SYSTEMS

After the original proposal of the memristor, Chua and Kang generalized the concept to a broader class of systems, called memristive systems,Chua and Kang (1976).

The concept of memory device is not necessarily limited to resistances but can in fact be generalized to capacitative and inductive systems.

Quite generally, if x denotes a set of n state variables describing the internal state of the system, $I(t)$ and $O(t)$ are any two complementary constitutive variables (i.e., current, charge, voltage, or flux) denoting input and output of the system, and g is a generalized response, we can define a general class of nth-order u-controlled memory devices as those described by the following relations

$$O(t) = g_{(x,u,t)}[I(t)] \tag{3}$$

$$\dot{x} = f\,(x, u, t) \tag{4}$$

where f is a continuous n-dimensional vector function, and we assume on physical grounds that, given an initial state $u(t = t_0)$ at time t_0, Eq. (4) admits a unique solution.

Memcapacitive and *meminductive* systems are special cases of Eqs. (3) and (4), where the two constitutive variables that define them are charge and voltage for the memcapacitance, and current and flux for the meminductance.

5. FRACTIONAL MEMRISTIVE SYSTEMS

5.1 Fractional circuit variables

Definition 1. Fractional circuit variables are defined from ordinary voltage $v(t)$ and current $i(t)$. Using fractional calculus, for any real number α and β, two fractional circuit variables $v^{\alpha}(t)$ and $i^{\beta}(t)$ are defined as:

$$v^{\alpha}(t) = D^{\alpha}v(t) \tag{5}$$

$$i^{\beta}(t) = D^{\beta}i(t) \tag{6}$$

where the fractional operator D^{α} is

$$f^{(\alpha)}(x) := \begin{cases} \dfrac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}} I^{\lceil\alpha\rceil-\alpha} f(x) & \alpha > 0 \\ f(x) & \alpha = 0 \\ I^{-\alpha} f(x) & \alpha < 0. \end{cases}$$

There are many different specific definitions about fractional calculus. In the case of Riemann-Liouville approach, $I^{-\alpha}f(x) = f^{(-\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt.$

5.2 Fractional circuit elements

Definition 2. A two-terminal (or one-port) circuit element characterized by a constitutive relation $v^{\alpha}(t) = g[i^{\beta}(t)]$ is called (α, β) element.

Here $v^{\alpha}(t), i^{\beta}(t)$ are any two complementary constitutive variables (i.e., current, charge, voltage, or flux) denoting input and output of the system. Noted that here g denote a general functional of $i^{\beta}(t)$.

Take one simplest case for example, $\alpha = 0$, $\beta = 0$ and g is a linear function of $i(t)$: $v(t) = R \cdot i(t)$, where R is the Resistance.

The following are some other simple cases with integer α, β:

- Resistor (v, i), $(\alpha = 0, \beta = 0)$
- Inductor (φ, i), $(\alpha = -1, \beta = 0)$
- Capacitor (v, q), $(\alpha = 0, \beta = -1)$
- Memristor (φ, q), $(\alpha = -1, \beta = -1)$
- Memcapacitor $(\varphi, i^{(-2)})$, $(\alpha = -1, \beta = -2)$
- Meminductor $(v^{(-2)}, q)$, $(\alpha = -2, \beta = -1)$

For other (α, β) pair and more complex response functional g, there could be other very different devise.

6. CONCLUSION

In order to model many irregular and exotic nondifferentiable phenomena which are common and dominant to the nonlinear dynamics of many biological, molecular and nanodevices, this paper further enlarge the family of elementary circuit elements to *Fractional circuit elements.*

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