The Koide formula and its analogues

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The mathematics of analogues to the Koide formula is explored. In this context, a naturally occurring alternative to the Koide formula is shown to fit not only the tau-electron mass ratio, but also the muon-electron mass ratio.

I. THE KOIDE FORMULA

In the 1980s the empirical Koide formula [1] [2] established that

$$\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{a+b+c} \approx \frac{3}{2} \tag{1}$$

when a, b, and c are the experimental electron, muon, and tau masses (see [3] for an historical overview, and [4] for a precursor to this article). By imposing this constraint on the charged lepton masses the Koide formula allows the inference of the less well known tau mass from the better known muon and electron masses. Here we examine an alternative to the Koide formula that allows approximate inference of *both* the tau and muon masses from the precisely known electron mass. This formula, which employs powers of 4.1, is shown to occur naturally when exploring the mathematics of analogues of the Koide formula. In the next section we begin by introducing two instructive identities that will help clarify the behavior of Koide's formula.

II. TWO IDENTITIES

Assume $x > 1$, so that

$$\frac{\left(\sqrt{\frac{1}{x}}+\sqrt{1}+\sqrt{x}\right)^2}{\frac{1}{x}+1+x} = 1 + \frac{2}{\sqrt{\frac{1}{x}}-1+\sqrt{x}} \tag{2a}$$

$$\frac{\left(\sqrt{\frac{1}{x}}+\sqrt{1}+\sqrt{0}\right)^2}{\frac{1}{x}+1+0} = 1 + \frac{2}{\sqrt{\frac{1}{x}}+\sqrt{x}} \quad . \tag{2b}$$

The reader will notice that the left sides of these identities follow the form of the Koide formula, while their right sides both take the form

$$1 + \frac{2}{z} \quad ,$$

where z is an expression in terms of x. By assuming values from infinity to one, z causes the expression $1+2/z$ to assume values from one to three. Conveniently, this can accommodate the fact that

$$f(a,b,c) = \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{a+b+c} \tag{3}$$

also produces values from one to three for positive a, b, and c. It is logical to wonder whether there might exist good approximations of Eq. (3) that follow Eqs. (2a) and (2b) in employing an expression like $1 + 2/z$.

III. THREE APPROXIMATIONS

In fact there at least three such approximations. These all use simple powers of $\sim$4 to produce $\sim$1.5, a point that will prove important later on.

Assume $x \geq 3$, so that

$$q(x) = \frac{\left(\sqrt{\frac{1}{3}}+\sqrt{x^3}+\sqrt{x^5}\right)^2}{\frac{1}{3}+x^3+x^5} \approx 1 + \frac{2}{x-\frac{1}{10}} \tag{4a}$$

$$r(x) = \frac{\left(\sqrt{1}+\sqrt{x^5}+\sqrt{x^7}\right)^2}{1+x^5+x^7} \approx 1 + \frac{2}{x+\frac{1}{11}} \tag{4b}$$

$$s(x) = \frac{\left(\sqrt{3}+\sqrt{x^6}+\sqrt{x^8}\right)^2}{3+x^6+x^8} \approx 1 + \frac{2}{x+\frac{1}{9}} \quad . \tag{4c}$$

Then, for $x = 4 + 1/10$, $4 - 1/11$, and $4 - 1/9$ the above functions give

$$q(4 + 1/10) \approx 1.500\,1087 \tag{5a}$$

$$r(4 - 1/11) \approx 1.500\,1294 \tag{5b}$$

$$s(4 - 1/9) \approx 1.500\,2214 \quad , \tag{5c}$$

where, as required by the right sides of Eqs. (4a)–(4c), all approximate

$$1 + \frac{2}{4} = 1.5 \quad . \tag{6}$$

Note that the above equations are similar in that each employs exponents differing by two, while their first terms form a 1/3, 1, 3 progression. As will be shown later, the above approximations maintain their accuracy over a range of values for x. Equation (4a) will prove especially accurate, where it is just this equation that is linked to the charged lepton masses.

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IV. A FUNCTION USING POWERS OF EXACTLY FOUR

Now define

$$t(n) = \frac{\left(\sqrt{4^{-1}} + \sqrt{4^{n-1}} + \sqrt{4^{n+1}}\right)^2}{4^{-1} + 4^{n-1} + 4^{n+1}} \quad , \tag{7}$$

and let

$$k = 4 + \frac{1}{\pi} \quad , \tag{8}$$

so that

$$t(+k) \approx 1.499\,999\,956\,696\,937 \tag{9a}$$
$$t(-k) \approx 1.499\,999\,293\,883\,009 \quad . \tag{9b}$$

Just as simple powers of ~ 4 produce ~ 1.5 for Eqs. (4a)–(4c), so, above, simple powers of exactly 4 also produce ~ 1.5.

A shift of exponents allows Eqs. (9a) and (9b) to be rewritten

$$\begin{aligned} t(+k) &= \frac{\left(\sqrt{4^{-1-\frac{1}{\pi}}} + \sqrt{4^3} + \sqrt{4^5}\right)^2}{4^{-1-\frac{1}{\pi}} + 4^3 + 4^5} \\ &\approx 1.499\,999\,956\,696\,937 \end{aligned} \tag{10a}$$

$$\begin{aligned} t(-k) &= \frac{\left(\sqrt{4^3} + \sqrt{4^5} + \sqrt{4^{7+\frac{1}{\pi}}}\right)^2}{4^3 + 4^5 + 4^{7+\frac{1}{\pi}}} \\ &\approx 1.499\,999\,293\,883\,009 \quad , \end{aligned} \tag{10b}$$

where, for clarity, each equation's terms are now arranged in ascending order of size. Notice that Eqs. (10a) and (10b) each employ the terms 4^3 and 4^5, but whereas Eq. (10a) uses the *small* term $4^{-1-\frac{1}{\pi}}$, Eq. (10b) instead uses the *large* term $4^{7+\frac{1}{\pi}}$ (making it clear that they are achieving a similar result by different means). Equations (10a) and (10b) match Eq. (4a) in using the integer exponents three and five, but Eqs. (10a) and (10b) differ critically in approximating 1.5 by employing powers of 4 *unadjusted* by a constant such as 1/10; hence, Eqs. (10a) and (10b) appear to be more fundamental.

Somehow, with the aid of the reciprocal of π, but without the need of a small constant, both manage to approximate 1.5 much more closely than do Eqs. (4a)–(4c). Leaving aside the issue of whether the reciprocal of π appears coincidentally in Eqs. (10a) and (10b), the key point remains that each produce ~ 1.5 while using powers of precisely 4. Moreover, letting

$$j = 10$$

facilitates the redefinition of k to

$$\begin{aligned} k &= \log_2\left(j \pm \sqrt{j^2-1}\right) \\ &= \pm(4 + 1/3.141\,613\,674\,030\ldots) \\ &= \pm 4.318\,307\,756\,382\ldots \quad , \end{aligned} \tag{11a}$$

so that

$$t(k) = t(\pm 4.318\,307\,756\,382\ldots) = 1.5 \quad , \tag{11b}$$

an *exact* result that establishes a firm connection between the simple powers of 4 used in Eq. (7) and the constant 1.5. Finally, note that if $j \geq 1$, then

$$\left(j + \sqrt{j^2-1}\right)\left(j - \sqrt{j^2-1}\right) = 1 \quad ,$$

so that necessarily

$$\log_2\left(j + \sqrt{j^2-1}\right) = -\log_2\left(j - \sqrt{j^2-1}\right) \quad ,$$

as in Eq. (11a).

V. APPROXIMATION ACCURACY

As noted earlier, the approximations appearing on the right sides of Eqs. (4a)–(4c) are accurate for various x. For example, using Eq. (4a) we get

$$q(4+1/10) \approx 1.500\,1087 \tag{12a}$$
$$q(5+1/10) \approx 1.400\,2429 \tag{12b}$$
$$q(6+1/10) \approx 1.333\,5168 \tag{12c}$$
$$q(7+1/10) \approx 1.285\,8222 \tag{12d}$$
$$q(8+1/10) \approx 1.250\,0473 \tag{12e}$$
$$q(9+1/10) \approx 1.222\,2258 \tag{12f}$$
$$q(10+1/10) \approx 1.199\,9733 \tag{12g}$$
$$q(11+1/10) \approx 1.181\,7709 \tag{12h}$$
$$q(12+1/10) \approx 1.166\,6058 \quad , \tag{12i}$$

values consistently fit by its approximation

$$q(x) \approx 1 + \frac{2}{x - 1/10} \quad . \tag{12j}$$

Moreover, inspection suggests that using the "adjustment constant" $-1/10$ in Eq. (12j) always proves more accurate across the above range than using its "neighboring values" of either $-1/9$ or $-1/11$. The equivalent results for Eq. (4b) give

$$r(4-1/11) \approx 1.500\,1294 \tag{13a}$$
$$r(5-1/11) \approx 1.399\,9883 \tag{13b}$$
$$r(6-1/11) \approx 1.333\,5695 \tag{13c}$$
$$r(7-1/11) \approx 1.286\,1204 \tag{13d}$$
$$r(8-1/11) \approx 1.250\,4882 \tag{13e}$$
$$r(9-1/11) \approx 1.222\,7374 \tag{13f}$$
$$r(10-1/11) \approx 1.200\,5117 \tag{13g}$$
$$r(11-1/11) \approx 1.182\,3106 \tag{13h}$$
$$r(12-1/11) \approx 1.167\,1324 \quad , \tag{13i}$$

values also consistently fit by its approximation

$$r(x) \approx 1 + \frac{2}{x + 1/11} \quad . \tag{13j}$$

And Eq. (4c) gives

$$s(4-1/9) \approx 1.500\,2214 \quad (14a)$$
$$s(5-1/9) \approx 1.399\,6697 \quad (14b)$$
$$s(6-1/9) \approx 1.333\,3783 \quad (14c)$$
$$s(7-1/9) \approx 1.286\,0561 \quad (14d)$$
$$s(8-1/9) \approx 1.250\,5037 \quad (14e)$$
$$s(9-1/9) \approx 1.222\,7975 \quad (14f)$$
$$s(10-1/9) \approx 1.200\,5948 \quad (14g)$$
$$s(11-1/9) \approx 1.182\,4041 \quad (14h)$$
$$s(12-1/9) \approx 1.167\,2292 \quad , \quad (14i)$$

values consistently fit by its approximation

$$s(x) \approx 1 + \frac{2}{x+1/9} \quad . \quad (14j)$$

Note, however, that the adjustment constants of $+1/11$ (used in Eq. (13j)) and $+1/9$ (used in Eq. (14j)) are more accurate than their equivalent neighboring values over a smaller range than is covered by $+1/10$ (used in Eq. (12j)). And, finally,

$$q(4+1/\ 9.913\,467\ldots) = 1.5 \quad (15a)$$
$$r(4-1/11.125\,754\ldots) = 1.5 \quad (15b)$$
$$s(4-1/\ 9.143\,166\ldots) = 1.5 \quad (15c)$$

show more precisely just what adjustments must be made to 4 for the above functions to produce exactly 1.5.

VI. THE MUON– AND TAU–ELECTRON MASS RATIOS

Although the use of $1/3$, x^3, and x^5 in Eq. (4a) is empirically inspired by [5] (and to a lesser degree by [6] and [7]), the simplicity and accuracy with which Eq. (4a) can be approximated by the expression $1+\frac{2}{x-1/10}$, as well as its similarity to the more fundamental Eq. (7), make it mathematically interesting in its own right. Hence, Eq. (4a) is *non*-empirical.

This point is important, as with the aid of $4+1/10$ we find that Eq. (4a) gives

$$q(4.1) = 1.500\,1087\ldots \approx 1 + \frac{2}{4.1-0.1} = 1.5 \quad , \quad (16)$$

nearly producing the 1.5 required by the Koide formula, while employing terms giving the proportion

$$\begin{aligned} m_e : m_\mu : m_\tau &\approx 1 \,:\, 3x^3 \,:\, 3x^5 \\ &\approx 1 \,:\, 3\times 4.1^3 \,:\, 3\times 4.1^5 \\ &\approx 1 \,:\, 206.763 \,:\, 3475.686\,03 \quad . \quad (17) \end{aligned}$$

The value 206.763 is calculated in [5] to fit the muon-electron mass ratio to roughly 1 part in 40,000, whereas 3475.686 03 is calculated to fit the tau-electron mass ratio to roughly 1 part in 2000.

VII. ANALYSIS AND CONCLUSION

The Koide formula by itself only imposes a single constraint on the charged lepton masses, and so only facilitates the inference of the less well known tau mass from the better known muon and electron masses. Although this is nontrivial, Eq. (17) gives good approximations for *both* the tau-electron mass ratio and the muon-electron mass ratio while nearly fitting Koide's relation. This addition of the more precisely known muon-electron mass ratio to the list of "mass ratios fit" potentially greatly enhances the Koide formula's credibility.

But how much weight should be given to Eq. (17) and its mass ratios, or, more specifically, Eq. (4a) from which it derives?

That Eq. (4a) is of general mathematical importance is supported by the other approximations and identities that bracket it. Thus, Eqs. (4a)–(4c) are fit by a similar approximation while all possessing a related form (e.g., Eqs. (4a)–(4c) each use exponents differing by two, while their first terms form a 1/3, 1, 3 progression, as noted earlier). More importantly, Eqs. (4a)–(4c), (7), (11a), and especially (11b), establish a firm connection between simple powers of $\sim$4 and the constant 1.5, a key point, as it is just when Eq. (4a) uses integer powers of 4.1 to produce $\sim$1.5 that it generates the mass ratios of Eq. (17). In this way these mass ratios appear as a natural part of the fabric of mathematics.

Moreover, Koide's formula prefigured the mass proportion $m_e : m_\mu : m_\tau \approx 1 : 3\times 4.1^3 : 3\times 4.1^5$ by decades (see [1] and [5], respectively). If the relationship that the formula and the proportion each have to mass is purely a matter of accident, then why should combining them help produce interesting mathematics? It is only if both the proportion and the formula are at least partially valid physically that one would expect combining them to produce interesting mathematical offshoots: e.g., Eqs. (4a) and (7).

All of this lends credence to the general conclusion that the Koide formula is related to the muon- and tau-electron mass ratios non-accidentally. Koide could not have known in advance that such additional mathematics would be forthcoming to support his original conception; that it does suggests that he correctly foresaw the right general direction with his formula, notwithstanding the many uncertainties surrounding the issue of mass.

Of course, it must be conceded that Eq. (4a) is arrived at purely numerically and, moreover, is one of several equations introduced here that have the opportunity to fit the charged lepton mass ratios by chance. But the terms of Eq. (4a) fit the muon- and tau-electron mass ratios to about 1 part in 40,000, and 1 part in 2000, respectively. This is a remarkably precise fit to achieve solely by accident, even given that this article's key equations provide several opportunities to do so.

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